where $1<\theta,\ \vartheta\le 2,\ 0<\mu,\ \nu<1$ and $\mathtt{H}_1, \mathtt{H}_2 : \mathrm{J}\times\mathrm{R}\times\mathrm{R}\to\mathrm{R}$ are continuous and ${}^c\mathbf{D}$ denotes Caputo fractional derivative. where $\lambda_i, \mu_i \in \mathrm{R}^+$ with $\sum_{i=1}^m \lambda_i\alpha_i<1$ and $\sum_{i=1}^m \mu_i\beta_i<1$. We apply Banach fixed point theorem and Leray-Schauder fixed point theorem to obtain proper conditions for existence and uniqueness results (Ali et al., 2016). We also provide a numerical problem to demonstrate the establish results.

2. Basic material

We recall some fundamental results and definitions (Benchohra eta al., 2008; Ahmad and Sivasaundaram., 2010; Zhang, 2010; Shah et al., 2016).

Definition 2.1. The fractional integral of order $\theta\in\mathrm{R}^+$ of a function $\omega:\mathrm{R}^+\to\mathrm{R}$ is defined by

$$\mathtt{I}^\theta\omega(t)=\frac{1}{\Gamma(\theta)}\int_0^t(t-\zeta)^{\theta-1}\omega(\zeta)\,d\zeta$$

provided the integral converges.

Definition 2.2. The Caputo fractional order derivative of a function $\omega:\mathrm{R}^+\to\mathrm{R}$ is defined by

$${}^c\mathbf{D}^\theta\omega(t)=\frac{1}{\Gamma(n-\theta)}\int_0^t(t-\zeta)^{n-\theta-1}\omega^{(n)}(\zeta)\,d\zeta,$$

where $n=[\theta]+1$ and $[\theta]$ represents the integer part of θ , such that

the right side is point wise defined on R^+

Theorem 2.3. For (FDEs)

$${}^c\mathbf{D}^\theta\omega(t)=\varphi(t),\ \theta\in(n-1,n],$$

the given result holds

$$\omega(t)=\mathtt{I}^\theta\varphi(t)+l_1+l_2t+l_3t^2+\ldots+l_nt^{n-1},$$

for arbitrary $l_i\in\mathrm{R}$, $i=1,2,\ldots,n$.

Lemma 2.4. (Benchohra et al., 2008) Let be a Banach space with closed and convex. Let be a relatively open subset of with and be a continuous and compact (completely continuous) mapping. Then either

1) The mapping $\mathbf{S}$ has a fixed point in $\overline{\mathbf{B}}$ or

2) There exist $u\in\partial\mathbf{B}$ and $\kappa\in(0,1)$ with $\mathbf{B}=\kappa\mathbf{S}u.$

Theorem 2.5. Let $\omega,\sigma\in C[0,1]$ and $D=1-\sum_{i=1}^m\lambda_1\alpha_i$ and $E=1-\sum_{i=1}^m\mu_i\beta_i,$ for a given $u,v\in C(I,\mathrm{R}),$ then the solution of

$$\begin{cases}{}^c\mathbf{D}^\theta u(t)=\omega,\ 1<\theta\le2,\ 0<\mu<1,\ t\in\mathrm{J},\\ {}^c\mathbf{D}^\vartheta v(t)=\sigma,\ 1<\vartheta\le2,\ 0<\nu<1,\ t\in\mathrm{J},\\ u(0)=A,\ u(1)=\sum_{i=1}^m\lambda_iu(\alpha_i),\\ v(0)=B,\ v(1)=\sum_{i=1}^m\mu_iv(\beta_i),\end{cases}$$

(2.1)

is equivalent to the following coupled system of FredHolm integral equations

$$u(t)=a(t)+\int_0^1\mathtt{W}_1(t,\zeta)\omega(\zeta)d\zeta,\qquad \textbf{(2.2)}$$
$$v(t)=b(t)+\int_0^1\mathtt{W}_2(t,\zeta)\sigma(\zeta)d\zeta,$$

where $a(t)=A-\frac{At(1-\sum_{i=1}^m\lambda_i)}{D},\ b(t)=B-\frac{Bt(1-\sum_{i=1}^m\mu_i)}{E}.$ Where $\mathtt{W}_1(t,\zeta),\mathtt{W}_2(t,\zeta)$ are the Green's functions given by

$$\mathtt{W}_1(t,\zeta)=\frac{1}{\Gamma(\theta)}\begin{cases}(t-\zeta)^{\theta-1}-\frac{1}{D}\Big[t(1-\zeta)^{\theta-1}+\sum_{i=1}^m\lambda_i(\alpha_i-\zeta)^{\theta-1}\Big],\ \zeta\le t,\alpha_i<\zeta\le\alpha_{i+1},\\ -\frac{1}{D}\Big[t(1-\zeta)^{\theta-1}+\sum_{i=1}^m\lambda_1(\alpha_i-\zeta)^{\theta-1}\Big],\ t\le\zeta,\alpha_i<\zeta\le\alpha_{i+1},\end{cases}$$

(2.3)

$$\mathtt{W}_2(t,\zeta)=\frac{1}{\Gamma(\vartheta)}\begin{cases}(t-\zeta)^{\vartheta-1}-\frac{1}{E}\Big[t(1-\zeta)^{\vartheta-1}+\sum_{i=1}^m\mu_i(\beta_i-\zeta)^{\vartheta-1}\Big],\ \zeta\le t,\beta_i<\zeta\le\beta_{i+1},\\ -\frac{1}{E}\Big[t(1-\zeta)^{\vartheta-1}+\sum_{i=1}^m\mu_i(\beta_i-\zeta)^{\vartheta-1}\Big],\ t\le\zeta,\beta_i<\zeta\le\beta_{i+1},\end{cases}$$

(2.4)

Proof. By means of Definition 2.2 and Theorem 2.3, the solution of ${}^c\mathbf{D}^\theta u(t)=\omega(t)$ is given by $u(t)=l_1+l_2t+\mathtt{I}^\theta\omega(t).$ **(2.5)**

Now applying the boundary conditions $u(0)=A,\ u(1)=\sum_{i=1}^m\lambda_iu(\alpha_i)$ to **(2.5)** we get after simplification

$$\begin{aligned}u(t)&=\frac{1}{\Gamma(\theta)}\int_0^t(t-\zeta)^{\theta-1}\omega(\zeta)d\zeta-\frac{t}{D\Gamma(\theta)}\int_0^1(1-\zeta)^{\theta-1}\omega(\zeta)d\zeta\qquad\textbf{(2.6)}\\&+A-\frac{At(1-\sum_{i=1}^m\lambda_i)}{D}-\frac{t}{D\Gamma(\theta)}\sum_{i=1}^m\lambda_i\int_0^{\alpha_i}(\alpha_i-\zeta)^{\theta-1}\omega(\zeta)d\zeta\\&=a(t)+\int_0^1\mathtt{W}_1(t,\zeta)\omega(\zeta)d\zeta.\end{aligned}$$

Similarly, we can get second equation of (2.2) by solving second equation of system of BVP (1.1) as

$$\begin{aligned}v(t)&=\frac{1}{\Gamma(\vartheta)}\int_0^1(t-\zeta)^{\vartheta-1}\sigma(\zeta)d\zeta-\frac{t}{E\Gamma\vartheta}\int_0^1(1-\zeta)^{\theta-1}\sigma(\zeta)d\zeta\qquad\textbf{(2.7)}\\&+E-\frac{Bt(1-\sum_{i=1}^m\mu_i)}{E}-\frac{t}{E\Gamma(\vartheta)}\sum_{i=1}^m\mu_i\int_0^{\beta_i}(\beta_i-\zeta)^{\vartheta-1}\sigma(\zeta)d\zeta\\&=b(t)+\int_0^1\mathtt{W}_2(t,\zeta)\sigma(\zeta)d\zeta.\end{aligned}$$

Therefore, in view of Theorem 2.5, the considered coupled system (1.1) is equivalent to the following coupled system of Fred Holm integral equations

$$\begin{cases}v(t)=\frac{1}{\Gamma(\vartheta)}\int_0^1(t-\zeta)^{\vartheta-1}\omega(\zeta)d\zeta-\frac{t}{E\Gamma(\vartheta)}\int_0^1(1-\zeta)^{\theta-1}\omega(\zeta)d\zeta\\ +E-\frac{Bt(1-\sum_{i=1}^m\mu_i)}{E}-\frac{t}{E\Gamma(\vartheta)}\sum_{i=1}^m\mu_i\int_0^{\beta_i}(\beta_i-\zeta)^{\vartheta-1}\omega(\zeta)d\zeta\\ =b(t)+\int_0^1\mathtt{W}_2(t,\zeta)\mathtt{H}_2(\zeta,u(\zeta),{}^c\mathtt{D}^\nu u(\zeta))d\zeta,\\ u(t)=\frac{1}{\Gamma(\theta)}\int_0^t(t-\zeta)^{\theta-1}\omega(\zeta)d\zeta-\frac{t}{D\Gamma(\theta)}\int_0^1(1-\zeta)^{\theta-1}\omega(\zeta)d\zeta\\ +A-\frac{At(1-\sum_{i=1}^m\lambda_i)}{D}-\frac{t}{D\Gamma(\theta)}\sum_{i=1}^m\lambda_i\int_0^{\alpha_i}(\alpha_i-\zeta)^{\theta-1}\omega(\zeta)d\zeta\\ =a(t)+\int_0^1\mathtt{W}_1(t,\zeta)\omega(\zeta)d\zeta.\end{cases}\qquad\textbf{(2.8)}$$

Remark 2.6. We will use

$$\mathbf{W}_1^* = \max_{t\in \mathrm{J}} \int_0^1 |\mathbf{W}_1(t,\zeta)|\, d\zeta,\ \mathbf{W}_2^* = \max_{t\in \mathrm{J}} \int_0^1 |\mathbf{W}_2(t,\zeta)|\, d\zeta,$$

throughout this paper.

3. Method and discussion

Define $\mathbf{U} = \{u(t) \mid u(t) \in C^1(\mathrm{J})\}$ endowed with the norm $\|u\|_{\mathbf{U}} = \max_{t\in\mathrm{J}}[|u(t)| + |{}^{c}\mathbf{D}^{\mu}u(t)|$ and similarly, $\mathbf{V} = \{v(t) \mid v(t) \in C^1(\mathrm{J})\}$ endowed with the norm $\|u\|_{\mathbf{V}} = \max_{t\in\mathrm{J}}[|v(t)| + |{}^{c}\mathbf{D}^{\nu}v(t)|$. Further, define the norm $\|(u,v)\|_{\mathbf{U}\times\mathbf{V}} = \max\{\|u\|_{\mathbf{U}}, \|v\|_{\mathbf{V}}\}$. Then, the product spaces $(\mathbf{U}\times\mathbf{V}, \|\cdot\|_{\mathbf{U}\times\mathbf{V}})$ is Banach space. Define a cone $\mathbf{P} \subset \mathbf{U}\times\mathbf{V}$ by $\mathbf{P} = \{(u,v) \in \mathbf{U}\times\mathbf{V} : u(t), v(t) \geq 0, \text{for all } t \in \mathrm{J}\}$. To proceed further the following assumptions hold:

- $(A_1)\cdot$ $a(t), b(t) : \mathrm{J} \to \mathrm{R}$ are continuous;
- $(A_2)\cdot$ $\mathbf{H}_1, \mathbf{H}_2 : \mathrm{J}\times\mathrm{R}\times\mathrm{R} \to \mathrm{R}$ are continuous.

Define the operator $\mathbf{S} : \mathbf{U}\times\mathbf{V} \to \mathbf{U}\times\mathbf{V}$ by

$$\mathbf{S}(u,v)(t) = \left(a(t) + \int_0^1 \mathbf{W}_1(t,\zeta)\mathbf{H}_1(\zeta, v(\zeta), {}^{c}\mathbf{D}^{\mu}v(\zeta))d\zeta,\ b(t) + \int_0^1 \mathbf{W}_2(t,\zeta)\mathbf{H}_2(\zeta, u(\zeta), {}^{c}\mathbf{D}^{\nu}v(\zeta))d\zeta \right) = (\mathbf{S}_1 v(t), \mathbf{S}_2 u(t)), \quad \textbf{(3.1)}$$

Where

$$\mathbf{S}_1 v(t) = a(t) + \int_0^1 \mathbf{W}_1(t,\zeta)\mathbf{H}_1(\zeta, v(\zeta), {}^{c}\mathbf{D}^{\mu}v(\zeta))d\zeta,\ t \in \mathrm{J},$$

$$\mathbf{S}_2 u(t) = b(t) + \int_0^1 \mathbf{W}_2(t,\zeta)\mathbf{H}_2(\zeta, u(\zeta), {}^{c}\mathbf{D}^{\nu}u(\zeta))d\zeta,\ t \in \mathrm{J}.$$

Theorem 3.1. Assume that (A_1) and (A_2) hold true then the coupled system has at least one fixed point $(u,v) \in C(\mathrm{J})\times C(\mathrm{J})$.

Proof. Define

$$\mathbf{C} = \{(u,v)(t) : (u,v) \in \mathbf{U}\times\mathbf{V} : \|(u,v)\|_{\mathbf{U}\times\mathbf{V}} \leq r\} \subset \mathbf{P}$$

and

$$\max_{t\in\mathrm{J}} |a(t)| = K_1, \qquad \max_{t\in\mathrm{J}} |b(t)| = K_2.$$

Also assume that

$$\max |\mathbf{H}_1(t, v(t), {}^{c}\mathbf{D}^{\mu}v(t))| \leq L,\ \max |\mathbf{H}_2(t, u(t), {}^{c}\mathbf{D}^{\nu}u(t))| \leq M$$

for all $(u,v) \in \mathbf{C}$.

Then

$$\begin{aligned}|\mathbf{S}_1 v(t)| &\leq \max_{t\in\mathrm{J}}\left|\frac{1}{\Gamma(\theta)}\int_0^1 (t-\zeta)^{\theta-1}\mathbf{H}_1(\zeta, v(\zeta), {}^{c}\mathbf{D}^{\mu}v(\zeta))d\zeta - \frac{t}{D\Gamma(\theta)}\int_0^1 (1-\zeta)^{\theta-1}\mathbf{H}_1(\zeta, v(\zeta), {}^{c}\mathbf{D}^{\mu}v(\zeta))d\zeta\right| \\ &+ \max_{t\in\mathrm{J}}\left|A - \frac{At(1-\sum_{i=1}^m \lambda_i)}{D} - \frac{t}{D\Gamma(\theta)}\sum_{i=1}^m \lambda_i \int_0^{\alpha_i}(\alpha_i-\zeta)^{\theta-1}\mathbf{H}_1(\zeta, v(\zeta), {}^{c}\mathbf{D}^{\mu}v(\zeta))d\zeta\right| \\ &\leq K_1 + \frac{L}{\Gamma(\theta+1)}(1+\frac{2}{D}).\end{aligned}$$

By similar way we can do

$$|\mathbf{S}_2 u(t)| \leq K_2 + \frac{M}{\Gamma(\vartheta+1)}(1+\frac{2}{E}).$$

Also we can easily show that

$$\begin{aligned}|{}^{c}\mathbf{D}^{\mu}\mathbf{S}_1 v(t)| &\leq \max_{[t\in\mathrm{J}]}\left|\frac{1}{\Gamma(\theta-\mu)}\int_0^1 (t-\zeta)^{\theta-\mu-1}\mathbf{H}_1(\zeta, v(\zeta), {}^{c}\mathbf{D}^{\mu}v(\zeta))d\zeta\right| \\ &+ \max_{[t\in\mathrm{J}]}\left|\frac{t^{1-\mu}}{D\Gamma(2-\mu)}\int_0^1 (1-\zeta)^{\theta-1}\mathbf{H}_1(\zeta, v(\zeta), {}^{c}\mathbf{D}^{\mu}v(\zeta))d\zeta\right| \\ &+ \max_{[t\in\mathrm{J}]}\left|\frac{A(1-\sum_{i=1}^m \lambda_i)}{D\Gamma(2-\mu)} + \frac{t^{1-\mu}}{D\Gamma(2-\mu)\Gamma(\theta)}\sum_{i=1}^m \lambda_i \int_0^{\alpha_i}(\alpha_i-\zeta)^{\theta-1}d\zeta\right| \\ &\leq \frac{A(1-\sum_{i=1}^m \lambda_i)}{D\Gamma(2-\mu)} + \frac{L}{\Gamma(\theta-\mu+1)}(1+\frac{2}{D\Gamma(2-\mu)}).\end{aligned}$$

Similarly

$$|{}^{c}\mathbf{D}^{\nu}\mathbf{S}_2 u(t)| \leq \frac{B(1-\sum_{i=1}^m \mu_i)}{E\Gamma(2-\nu)} + \frac{M}{\Gamma(\vartheta-\nu+1)}(1+\frac{2}{E\Gamma(2-\nu)}).$$

Thus from the above relations we have

$$\begin{aligned}\|\mathbf{S}_1 v\|_U &\leq [K_1 + \frac{L}{\Gamma(\theta+1)}(1+\frac{2}{D}) + \frac{A(1-\sum_{i=1}^m \lambda_i)}{D\Gamma(2-\mu)} + \frac{L}{\Gamma(\theta-\mu+1)}(1+\frac{2}{D\Gamma(2-\mu)})] \\ \|\mathbf{S}_2 u\|_V &\leq [K_2 + \frac{M}{\Gamma(\vartheta+1)}(1+\frac{2}{E}) + \frac{B(1-\sum_{i=1}^m \mu_i)}{E\Gamma(2-\nu)} + \frac{M}{\Gamma(\vartheta-\nu+1)}(1+\frac{2}{E\Gamma(2-\nu)})].\end{aligned} \quad \textbf{(3.2)}$$

Hence from (3.2) we have

$$\|\mathbf{S}(u,v)\|_{\mathbf{U}\times\mathbf{V}} < \infty. \quad \textbf{(3.3)}$$

Next, we prove that $\mathbf{S}$ is equi-continuous on $\mathbf{J}$. For $(u,v) \in \mathbf{C}$, and $t_1, t_2 \in \mathbf{J}$, such that $t_1 < t_2$. Then we have

$$\begin{aligned}|\mathbf{S}_1 v(t_1) - \mathbf{S}_1 v(t_2)| &\leq \frac{1}{\Gamma\theta}\int_0^{t_1}[(t_1-\zeta)^{\theta-1} - (t_2-\zeta)^{\theta-1}]\,|\mathbf{H}_1(\zeta, v(\zeta), {}^{c}\mathbf{D}^{\mu}v(\zeta)|\,d\zeta \\ &+ \int_{t_1}^{t_2}(t_2-\zeta)^{\theta-1}]\,|\mathbf{H}_1(\zeta, v(\zeta), {}^{c}\mathbf{D}^{\mu}v(\zeta)|\,d\zeta \\ &+ \frac{t_1-t_2}{D\Gamma(\theta)}\int_0^1 (1-\zeta)^{\theta-1}|\mathbf{H}_1(\zeta, v(\zeta), {}^{c}\mathbf{D}^{\mu}v(\zeta)|\,d\zeta \\ &+ \frac{A(1-\sum_{i=1}^m \lambda_i)(t_2-t_1)}{D} + \frac{t_2-t_1}{\Gamma(\theta)}\sum_{i=1}^m \int_0^{\alpha_i}(\alpha_i-\zeta)^{\theta-1}|\mathbf{H}_1(\zeta, v(\zeta), {}^{c}\mathbf{D}^{\mu}v(\zeta)|\,d\zeta \\ &\leq \frac{L}{\Gamma(\theta+1)}[t_1^{\theta} - t_2^{\theta}] + \frac{(t_2-t_1)}{D\Gamma(\theta+1)}L \\ &+ \frac{A(1-\sum_{i=1}^m \lambda_i)(t_2-t_1)}{D} + \frac{(t_2-t_1)}{D\Gamma(\theta+1)}L.\end{aligned} \quad \textbf{(3.4)}$$

Also we have

$$\begin{aligned}|{}^{c}\mathbf{D}^{\mu}\mathbf{S}_1 v(t_1) - {}^{c}\mathbf{D}^{\mu}\mathbf{S}_1 v(t_2)| &\leq \frac{L}{\Gamma(\theta-\mu+1)}[t^{\theta-\mu} - t_2^{\theta-\mu}] + \frac{A(1-\sum_{i=1}^m)}{D\Gamma(2-\mu)}(t_1^{1-\mu} - t_2^{1-\mu}) \\ &+ \frac{2(t_2^{1-\mu} - t_1^{1-\mu})}{D\Gamma(2-\mu)\Gamma(\theta+1)}L.\end{aligned} \quad \textbf{(3.5)}$$

By similar way we can obtain that

$$\begin{aligned}|\mathbf{S}_2 u(t_1) - \mathbf{S}_2 u(t_2)| &\leq \frac{M}{\Gamma(\vartheta+1)}[t_1^{\vartheta} - t_2^{\vartheta}] + \frac{(t_2-t_1)}{E\Gamma(\vartheta+1)}M \\ &+ \frac{B(1-\sum_{i=1}^m \mu_i)(t_2-t_1)}{E} + \frac{(t_2-t_1)}{E\Gamma(\vartheta+1)}M,\end{aligned} \quad \textbf{(3.6)}$$

$$\begin{aligned}|{}^{c}\mathbf{D}^{\nu}\mathbf{S}_2 u(t_1) - {}^{c}\mathbf{D}^{\nu}\mathbf{S}_2 u(t_2)| &\leq \frac{M}{\Gamma(\vartheta-\nu+1)}[t^{\vartheta-\nu} - t_2^{\vartheta-\nu}] + \frac{B(1-\sum_{i=1}^m)}{E\Gamma(2-\nu)}(t_1^{1-\nu} - t_2^{1-\nu}) \\ &+ \frac{2(t_2^{1-\nu} - t_1^{1-\nu})}{E\Gamma(2-\nu)\Gamma(\vartheta+1)}M.\end{aligned} \quad \textbf{(3.7)}$$

Now as $t_1 \to t_2$ in right hand sides of (3.4) (3.7), they tend to zero. Therefore $\left\|\mathbf{S}(u,v)(t_1)-\mathbf{S}(u,v)(t_2)\right\|_{\mathbf{U}\times\mathbf{V}} \to 0$ as $t_1 \to t_2$.

Hence a theorem, we conclude that is completely continuous operator. Finally define $\Omega = \{(u,v) \in \mathbf{U}\times\mathbf{V} : (u,v) = \lambda \mathbf{S}(u,v),\ \lambda \in (0,1)\}$. It is enough to show that Ω is bounded. Let $(u,v)\in\Omega$, then $(u,v)=\lambda\mathbf{S}(u,v)$ for some $0<\lambda<1$. Thus, for every $t\in\mathrm{J}$ we have $u(t)=\lambda\mathbf{S}_1 v(t)$ and $v(t)=\lambda\mathbf{S}_2 u(t)$, then it is easy to show that $\left\|(u,v)\right\|_{\mathbf{U}\times\mathbf{V}}<\infty$.

Which implies that Ω is bounded. Thus, by Schauder fixed point theorem has at least one fixed point which is the solution of (1.1).

Theorem 3.2. If $\mathbf{H}_1,\mathbf{H}_2 : \mathrm{J}\times\mathrm{R}\times\mathrm{R}\to\mathrm{Rz}$ are continuous and if $\max(d_1,d_2)<1$, then Coupled system (1.1) has unique positive solution, where

$$d_1=\max_{t\in\mathrm{J}}\left[2\bar{K}\mathbf{W}_1^*,\ \frac{2\bar{K}}{\Gamma(2-\mu)}\left(1+\frac{1}{2}+\frac{A}{D}+\frac{1}{2D\Gamma(\theta+1)}\right)\right],$$

$$d_2=\max_{t\in\mathrm{J}}\left[2\bar{K}\mathbf{W}_2^*,\ \frac{2\bar{L}}{\Gamma(2-\nu)}\left(1+\frac{1}{2}+\frac{B}{E}+\frac{1}{2E\Gamma(\vartheta+1)}\right)\right].$$

Proof. Let $u,\bar{u},v,\bar{v}\in\mathrm{R}$, then

$$\begin{aligned}|\mathbf{S}_1(v)-\mathbf{S}_1\bar{v}| &= \left|\int_0^1 \mathbf{W}_1(t,\zeta)[\mathbf{H}_1(\zeta,v(\zeta),{}^c\mathbf{D}^{\mu}v(\zeta))-\mathbf{H}_1(\zeta,\bar{v}(\zeta),{}^c\mathbf{D}^{\mu}\bar{v}(\zeta))]d\zeta\right| \\ &\le \bar{K}\int_0^1 \mathbf{W}_1(t,\zeta)[|v-\bar{v}|+|{}^c\mathbf{D}^{\mu}u-{}^c\mathbf{D}^{\mu}\bar{u}|]d\zeta \\ &\le 2\bar{K}\mathbf{W}_1^*\|v-\bar{v}\|_{\mathbf{V}}.\end{aligned}$$

Also, we can write

$$\begin{aligned}|{}^c\mathbf{D}^{\mu}\mathbf{S}_1(v)-{}^c\mathbf{D}^{\mu}\mathbf{S}_1\bar{v}| &\le \max_{t\in\mathrm{J}}\frac{1}{\Gamma(\theta-\mu)}\int_0^1(t-\zeta)^{\theta-\mu-1}|\mathbf{H}_1(\zeta,v(\zeta),{}^c\mathbf{D}^{\mu}v(\zeta))-\mathbf{H}_1(\zeta,\bar{v}(\zeta),{}^c\mathbf{D}^{\mu}\bar{v}(\zeta))|d\zeta \\ &+\frac{1}{D\Gamma(2-\mu)}\max_{\mathrm{J}}\int_0^1(1,\zeta)^{\theta-1}|\mathbf{H}_1(\zeta,v(\zeta),{}^c\mathbf{D}^{\mu}v(\zeta))-\mathbf{H}_1(\zeta,\bar{v}(\zeta),{}^c\mathbf{D}^{\mu}\bar{v}(\zeta))|d\zeta \\ &+\left(\frac{A}{D\Gamma(2-\mu)}+\frac{1}{\Gamma(2-\mu)\Gamma(\theta)}\sum_{i=1}^{n}\lambda_i\int_0^{\alpha}(\alpha-\zeta)^{\theta-1}\right) \\ &|\mathbf{H}_1(\zeta,v(\zeta),{}^c\mathbf{D}^{\mu}v(\zeta))-\mathbf{H}_1(\zeta,\bar{v}(\zeta),{}^c\mathbf{D}^{\mu}\bar{v}(\zeta))|d\zeta \\ &\le\frac{2\bar{K}\|v-\bar{v}\|_{\mathbf{V}}}{\Gamma(\theta-\mu+1)}+\frac{2\bar{K}\|v-\bar{v}\|_{\mathbf{V}}}{2\Gamma(2-\mu)}+\left(\frac{A}{D\Gamma(2-\mu)}+\frac{1}{2D\Gamma(2-\mu)\Gamma(\theta+1)}\right)2\bar{K}\|v-\bar{v}\|_{\mathbf{V}}\end{aligned}$$

implies that $\left\|{}^c\mathbf{D}^{\mu}\mathbf{S}_1(v)-{}^c\mathbf{D}^{\mu}\mathbf{S}_1(\bar{v})\right\|_{\mathbf{V}}\le\frac{2\bar{K}}{\Gamma(2-\mu)}\left(1+\frac{1}{2}+\frac{A}{D}+\frac{1}{2D\Gamma(\theta+1)}\right)\|v-\bar{v}\|_{\mathbf{V}}$.

Now

$$\|\mathbf{S}_1 v-\mathbf{S}_1\bar{v}\|_{\mathbf{V}}\le\max_{t\in\mathrm{J}}\left[2\bar{K}\mathbf{W}_1^*,\frac{2\bar{K}}{\Gamma(2-\mu)}\left(1+\frac{1}{2}+\frac{A}{D}+\frac{1}{2D\Gamma(\theta+1)}\right)\right]\|v-\bar{v}\|_{\mathbf{V}}. \quad \textbf{(3.8)}$$

Similarly

$$\|\mathbf{S}_2 u-\mathbf{S}_2\bar{u}\|_{\mathbf{U}}\le\max_{t\in\mathrm{J}}\left[2\bar{K}\mathbf{W}_2^*,\frac{2\bar{L}}{\Gamma(2-\nu)}\left(1+\frac{1}{2}+\frac{B}{E}+\frac{1}{2E\Gamma(\vartheta+1)}\right)\right]\|u-\bar{u}\|_{\mathbf{V}}. \quad \textbf{(3.9)}$$

Hence, from (3.8) and (3.9), we have

$$\|\mathbf{S}(u,v)-\mathbf{S}(\bar{u},\bar{v})\|_{\mathbf{U}\times\mathbf{V}}\le\max(d_1,d_2)[\|u-\bar{u}\|_{\mathbf{U}}+\|v-\bar{v}\|_{\mathbf{V}}],$$

where $\max(d_1,d_2)<1$. Hence $\mathbf{S}$ has unique fixed point which is the corresponding unique solution of BVP (1.1).

4. Example

Example 4.1.

$$\begin{cases}{}^c\mathbf{D}^{\frac{3}{2}}u(t)=\frac{t}{4}+\frac{\sin|v(t)|+\cos|{}^c\mathbf{D}^{\frac{1}{2}}v(t)|}{40+t^2}, & t\in\mathrm{J}\\ {}^c\mathbf{D}^{\frac{3}{2}}v(t)=\frac{t+1}{10}+\frac{\cos|u(t)|+|{}^c\mathbf{D}^{\frac{1}{2}}u(t)|}{50}, & t\in\mathrm{J}\\ u(0)=1,\ u(0)=2,\ u(1)=\sum_{i=1}^{10}\frac{1}{2^i}u(\frac{1}{2^i}),\ v(1)=\sum_{i=1}^{10}\frac{1}{3^i}v(\frac{1}{3^i}), & i=1,2,,10.\end{cases}$$

Now for coupled system (1.1), $\theta=\vartheta=\frac{3}{2},\mu=\nu=\frac{1}{2}$

$$a(t)=\frac{t}{4},\ b(t)=\frac{t+1}{10},\ \lambda_i=\frac{1}{2^i},\ \mu_i=\frac{1}{3^i},\ \alpha_i=\frac{1}{2^i},\ \beta=\frac{1}{2^i}$$

$$\mathbf{H}_1(t,v(t),{}^cD^{\frac{1}{2}}v(t))=\frac{\sin|v(t)|+\cos|{}^cD^{\frac{1}{2}}v(t)|}{40+t^2},\ \mathbf{H}_2(t,u(t),{}^cD^{\frac{1}{2}}u(t))=\frac{\cos|u(t)|+|{}^cD^{\frac{1}{2}}u(t)|}{50}$$

$$A=1,\ B=2,\ D=1-\sum_{i=1}^{10}\lambda_i\alpha_i=0.667,\ \bar{K}=\frac{1}{40},\ \bar{L}=\frac{1}{50},\ E=1-\sum_{i=1}^{10}\mu_i\beta_i=0.800,$$

$$\mathbf{W}_1^*=0.591,\ \mathbf{W}_2^*=0.508$$

hence $2\bar{K}\mathbf{W}_1^*=0.02956,\ 2\bar{K}\mathbf{W}_2^*=0.02032.$

Now, we computing the values of

$$d_1=\max\left\{2\bar{K}\mathbf{W}_1^*,\frac{2\bar{K}}{\Gamma(2-\mu)}\left(1+\frac{1}{2}+\frac{A}{D}+\frac{1}{2D\Gamma(\theta+1)}\right)\right\}$$

$$=\max\{0.02956,0.1811\}=0.1811$$

and

$$d_2=\max\left\{2\bar{K}\mathbf{W}_2^*,\frac{2\bar{L}}{\Gamma(2-\nu)}\left(1+\frac{1}{2}+\frac{B}{E}+\frac{1}{2E\Gamma(\vartheta+1)}\right)\right\}$$

$$=\max\{0.02030,0.1061\}=0.1061.$$

We see that $\max\{d_1,d_2\}=0.1811<1$. Hence Thank to Theorem 3.2, the given coupled system has a unique solution.

References

[1] Ahmad, B., and Nieto, J.J., (2009). Existence results for a coupled system of nonlinear fractional differential equations with three-point boundaryconditions.*Computers&Mathematics with Applications,* 58(1):1838-1843.

[2] Ahmad, B., and Sivasaundaram, S., (2010). On four-point non-local boundary value problems of non-linear intigro-differential equations of fractional order. *Applied Mathematics and Computation,* 217(2): 480--487.

[3] Ali, A., Shah, K., Khan, R.A., (2016). Existence of positive solutions to a coupled system of nonlinear fractional order differential equations with m-point boundary conditions. *Bulletin of Mathematical Analysis and Applications,* 3 (8): 1-11.

[4] Benchohra, M., Graef, J.R., and Hamani, S., (2008). Existence results for boundary value problems with nonlinear fractional differential equations. *Applied Analysis,* 87: 851-863.

[5] Cui, Z., Yu, P., and Mao, Z., (2012). Existence of solutions for nonlocal boundary value problems of nonlinear fractional differential equations. *Advances in Dynamical Systems and Applications,* **7**(1): 12 pages.

[6] El-Sayed, A.M.A., and Bin-Taher, E.O., (2013). Positive solutions for a nonlocal multi-point boundary-value problem of fractional and second order. Electronic Journal of Differential Equations., 2013(64): 1-8.

[7] El-Shahed, M., and Nieto, J.J., (2010). Nontrivial solutions for a nonlinear multi-point boundary value problem of fractional order. *Computers & Mathematics with Applications,* 59(11): 3438-3443.

[8] Gafiychuk, V., Datsko, B., Meleshko, V., and Blackmore, D., (2009). Analysis of the solutions of coupled nonlinear fractional reaction-diffusion equations. *Chaos, Solitons & Fractals*, 41(3):1095-1104.

[9] Han, X., and Wang, T., (2011). The existence of solutions for a nonlinear fractional multi-point boundary value problem at resonance. *International Journal of Differential Equations*, Article ID 401803 pp.14.

[10] Haq, F., Shah, K., Rahman, G., and Shahzad, M., (2016). Existance and uniquness of positive solution to boundary values problem of fractional differential equations. *Sindh University Research Journal*, 48(2): 451-456.

[11] Kilbas, A.A., Srivastava H.M., and Trujillo, J.J., (2006). Theory and Applications of Fractional Differential Equations. *Elsevier, Amsterdam*, 204.

[12] Lakshmikantham, V., Leela, S., and Vasundhara, J., (2009). Theory of Fractional Dynamic Systems. *Cambridge Academic Publishers, Cambridge*, *UK*.

[13] Miller, K.S., and Ross, B., (1993). An Introduction to the Fractional Calculus and Fractional Differential Equations. *Wiley, New York*.

[14] Perov, A.I., (1964). On the Cauchy problem for a system of ordinary differential equations. *Priblizhen. Metody Reshen. Differ. Uravn.,* 2:115-134.

[15] Rehman, M., and Khan, R.A., (2010). Existence and uniqueness of solutions for multi-point boundary value problems for fractional differential equations. *Applied Mathematics Letters*,23(9):1038-1044.

[16] Salem, H.A.H., (2009). On the fractional order m-point boundary value problem in reflexive Banach spaces and weak topologies. *Journal of Computational and Applied Mathematics,* 224(2)**:** 567-572.

[17] Shah, K., and Khan, R.A., (2015a). Existence and uniqueness of positive solutions to a coupled system of nonlinear fractional order differential equations with anti-periodic boundary conditions. *Journal of Difference Equations and Applications,* **7**(2): 245-262

[18] Shah, K., Ali, A., and Khan, R.A., (2016). Degree theory and existence of positive solutions to coupled systems of multi-point boundary value problems. *Boundary Value Problems*, 43:12.

[19] Shah, K., Khalil, H., and Khan, R.A., (2015b). Investigation of positive solution to a coupled System of impulsive boundary value problems for nonlinear fractional order differential equations. *Chaos, Solitons & Fractals,* 77: 240-246.

[20] Shah, K., Ali, A., and Khan, R.A., (2016). Degree theory and existence of positive solutions to coupled systems of multi-point boundary value problems. *Boundary Value Problems,*12 pages.

[21] Shoab, M., Sarwar, M., Shah, K., Kumum, P., (2016). Fixed point results and its applications to the systems of non-linear integral and differential equations of arbitrary order. *The Journal of Nonlinear Science and its Applications,* 9: 4949-4962.

[22] Yang, A., and Ge, W., (2009). Positive solutions of multi-point boundary value problems of nonlinear fractional differential equation at resonance. *Journal of the Korean Society of Mathematical Education Series B: The Pure & Applied Mathematics,* 16**:**181-193.

[23] Zhang, S., (2010). Positive solutions to singular boundary value problem for nonlinear fractional differential equation. *Computers & Mathematics with Applications,*59: 1300-1309.