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Study of Nonlocal Boundary Value Problems of Non-Integer Order Hybrid Differential Equations

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ABSTRACT

We study sufficient conditions for existence and uniqueness of solutions to boundary value problems (BVPs) for fractional hybrid differential equations(FHDEs) of the form

$$\begin{cases} \mathbf{D}^{\sigma}[\alpha(t)-\Phi(t,\alpha(t))]=\Psi(t,\beta(t),I^{\theta}(\beta(t))),t\in \mathrm{I}, \\ \mathbf{D}^{\sigma}[\beta(t)-\Phi(t,\beta(t))]=\Psi(t,\alpha(t),I^{\theta}(\alpha(t))),t\in \mathrm{I}, \\ \alpha(0)=0,\ \alpha(1)=\delta\alpha(\omega),\ \beta(0)=0,\ \beta(1)=\delta\beta(\omega), \end{cases}$$

where $\mathrm{I}=[0,1]$, $\sigma,\ \theta\in(1,2]$ and $\delta,\omega\in(0,1)$. We use hybrid fixed point theorem due to Dhage and develop adequate results for existence of solutions to the proposed system of (FHDEs). We also provide a numerical problem to demonstrate our main results.

1. INTRODUCTION

Fractional calculus studied properties of fractional order integrals and derivatives. This area includes the notion and techniques for solving fractional order differential equations. Fractional calculus has recently treated into a hot topic for researchers in various scientific and engineering fields. The Systematic development is available in the books (Miller and Ross, 1993a; Baleanu et al., 2012a). Recently fractional order differential equations of non-integer order have attracted great consideration for their intensive applications in various field of science (Baleanu et al., 2012b; Samko et al., 1993a).

Perturbation methods or techniques are very much skillful for solving dynamical systems. Different types of perturbations of differential equations and their classification are available (Hilfer, 2000). Hybrid differential equations can be treated with hybrid fixed point theory (Herzallah and Baleanu, 2012a; Herzallah, 2012b; Kilbas et al., 2006a). Recently, a scientist developed sufficient condition for existence and uniqueness of solution to the following first order initial value problem for hybrid differential equation with quadratic perturbation (Dhage and Lakshmikantham, 2010).

$$\frac{\mathbf{d}}{\mathbf{dt}}\left[\frac{\alpha(t)}{\Phi(t,\alpha(t))}\right]=\Psi(t,\alpha(t)),t\in\mathrm{I},$$
$$\alpha(t_0)=\alpha_0\in\mathbf{R},$$

Further, they developed some essential differential inequalities and evaluation results. The results were further extended to the class of first order initial value problem for hybrid differential equation with linear perturbation of second type (Dhage and Jadhav, 2013).

$$\frac{\mathbf{d}}{\mathbf{dt}}\left[\alpha(t)-\Phi(t,\alpha(t))\right]=\Psi(t,\alpha(t)),t\in\mathrm{I},$$
$$\alpha(t_0)=\alpha_0\in\mathbf{R}.$$

Recently, extended the studies to hybrid differential equations of non-integer order, and they established appropriate conditions for existence of solutions to the following initial value problem for hybrid differential equations of non-integer order (Lu et al., 2013).

$$\mathbf{D}^{\sigma}[\alpha(t)-\Phi(t,\alpha(t))]=\Psi(t,\alpha(t)),0<\sigma<1,\ a.e\ t\in\mathrm{I},$$
$$\alpha(t_0)=\alpha_0\in\mathbf{R},$$

where $\Phi,\Psi\in C(\mathrm{I}\times\mathbf{R},\mathbf{R})$. Recently, a researcher established enough conditions for existence of solutions to the following coupled system of initial value problem for hybrid differential equations of non-integer order (Bashiri et al., 2016).

$$\begin{aligned} &\mathbf{D}^{\sigma}[\alpha(t)-\Phi(t,\alpha(t))]=\Psi(t,\beta(t),I^{\theta}(\beta(t))),t\in\mathrm{I},\ 0<\sigma\le 1,0<\theta\le 1,\\ &\mathbf{D}^{\sigma}[\beta(t)-\Phi(t,\beta(t))]=\Psi(t,\alpha(t),I^{\theta}(\alpha(t))),t\in\mathrm{I},\ 0<\sigma\le 1,0<\theta\le 1,\\ &\alpha(0)=0,\ \beta(0)=0. \end{aligned} \tag{1.1}$$

Motivated by the above studies, in this paper, we enlarge the outcomes to the case of non-local boundary value problems and investigate sufficient conditions for existence of solutions to the following system of three point boundary value problems for hybrid differential equations of non-integer order

$$\begin{aligned} &\mathbf{D}^{\sigma}[\alpha(t)-\Phi(t,\alpha(t))]=\Psi(t,\beta(t),I^{\theta}(\beta(t))),t\in\mathrm{I},\ 1<\sigma\le 2,\ 1<\theta\le 2,\\ &\mathbf{D}^{\sigma}[\beta(t)-\Phi(t,\beta(t))]=\Psi(t,\alpha(t),I^{\theta}(\alpha(t))),t\in\mathrm{I},\ 1<\sigma\le 2,1<\theta\le 2,\\ &\alpha(0)=0,\ \alpha(1)=\delta\alpha(\omega),\\ &\beta(0)=0,\ \beta(1)=\delta\beta(\omega). \end{aligned} \tag{1.2}$$

We use some classical tools of functional Analysis to develop appropriate conditions for existence of solutions. We also provide a numerical problem to show the applicability of our results.

1. Preliminaries

We denote the set of real numbers by $\mathbf{R}$ and bounded interval by I in $\mathbf{R}$. Let $\Phi\in C(\mathrm{I}\times\mathbf{R}\times\mathbf{R},\mathbf{R})$ and $\Psi\in C(\mathrm{I}\times\mathbf{R}\times\mathbf{R},\mathbf{R})$. Where $C(\mathrm{I}\times\mathbf{R}\times\mathbf{R},\mathbf{R})$ denotes the class of continuous functions and $C(\mathrm{I}\times\mathbf{R}\times\mathbf{R},\mathbf{R})$ denotes the class of functions called the Caratheodory class of function on $\mathrm{I}\times\mathbf{R}\times\mathbf{R}$ such that

(i) the map $t\to\Psi(t,\alpha,\beta)$ is measurable for each $\alpha,\beta\in\mathbf{R}$;

(ii) the map $\alpha\to\Psi(t,\alpha,\beta)$ is measurable for each $\alpha\in\mathbf{R}$;

(iii) the map $\beta \to \Psi(t,\alpha,\beta)$ is continuous for each $\beta \in \mathbf{R}$.

Definition 2.1 *(Kilbas et al., 2006b) The Riemann-Liouville fractional derivative of order ρ of a continuous function $\Phi:(a,+\infty)\to\mathbf{R}$ is defined as*

$$\mathbf{D}^{\rho}\Phi = \frac{1}{\Gamma(m-\rho)}\left(\frac{\mathbf{d}}{\mathbf{dt}}\right)^{m}\int_a^t \frac{\Phi(\vartheta)}{(t-\vartheta)^{\rho-m+1}}\,d\vartheta,$$

where $m=[\rho]+1$, $m=[\rho]$ denotes the integral part of number ρ, provided that the right-hand side is point wise defined on $(a,+\infty)$.

Definition 2.2 *(Kilbas et al., 2006b) The Caputo fractional order derivative of a function Φ on the interval $[a,b]$ is defined by*

$${}^{c}\mathbf{D}^{\rho}\Phi(t) = \frac{1}{\Gamma(m-\rho)}\int_a^t (t-\vartheta)^{m-\rho-1}\Phi^{(m)}(\vartheta)\,d\vartheta,$$

where $m=[\rho]+1$ and $[\rho]$ represents the integer part of ρ.

Note: *Throughout this paper, we use Caputo fractional order derivative.*

Definition 2.3 *(Kilbas et al., 2006b) The Riemann-Liouville fractional integral of order ρ of a function $\Phi:(0,+\infty)\to\mathbf{R}$ is*

$$I^{\rho}\Phi(t) = \frac{1}{\Gamma(\rho)}\int_a^t (t-\vartheta)^{\rho-1}\Phi(\vartheta)\,d\vartheta,$$

provided that the right-hand side is point wise defined on $(0,+\infty)$.

Lemma 2.4 *(Lakshmikantham et al., 2009) The following result holds for fractional differential equations*

$$I^{\rho}\mathbf{D}^{\rho}\alpha(t) = \alpha(t) + e_0 + e_1 t + e_2 t^2 + \ldots + e_{m-1}t^{m-1},$$

for arbitrary $e_i \in \mathbf{R}$, $i = 0,1,2,\ldots,m-1$.

Lemma 2.5 *(Dhage, 2004) Let C be a nonempty, closed convex and bounded subset of a Banach algebra $\mathbf{X}$ and let $S:\mathbf{X}\to\mathbf{X}$ and $T:C\to\mathbf{X}$ be two operators such that*

(i) S is nonlinear contraction;

(ii) T is completely continuous;

(iii) $Sx+Tx\in C$ for all $x\in C$.

Then the operator equation $Sx+Tx=x$ has a solution in C.

Definition 2.6 *(Chang et al., 1996) An element $(\alpha,\beta)\in\mathbf{X}\times\mathbf{X}$ is called a coupled fixed point of a mapping $A:\mathbf{X}\times\mathbf{X}\to\mathbf{X}$ if $A(\alpha,\beta)=\alpha$ and $A(\alpha,\beta)=\beta$.*

Let $\phi:\mathbf{R}^{+}\to\mathbf{R}^{+}$ denote the family of all functions fulfilling $\phi(n)<n$ for $n>0$ and $\phi(n)=0$. By a solution of FHDE, we mean a function $(\alpha,\beta)\in AC(\mathrm{I},\mathbf{R}\times\mathbf{R})$ such that

(i) the function $t\to\alpha-\Phi(t,\alpha)$ is absolutely continuous for each $\alpha\in\mathbf{R}$; and

(ii) (α,β) satisfies the system of Equation in (1.2),

where $AC(\mathrm{I},\mathbf{R}\times\mathbf{R})$ is the space of absolutely continuous real valued functions defined on I.

We place FHDE in (1.2) the space $C(\mathrm{I},\mathbf{R})$ of continuous real-valued functions defined on I. Define a supremum norm $\|.\|$ in $C(\mathrm{I},\mathbf{R})$ by $\|\alpha\| = \sup_{t\in\mathrm{I}}|\alpha(t)|$. Clearly, $X=C(\mathrm{I},\mathbf{R})$ is a Banach algebra with respect to the above norm. The product space $\mathbf{X}\times\mathbf{X}$ is a vector space on $\mathbf{R}$ under the operations of addition and scalar multiplication and is a Banach space under the norm $\|(\alpha,\beta)\| = \|\alpha\|+\|\beta\|$.

The following result is useful for our main results.

Theorem 2.7 *(Bashiri et al., 2016) Let C be a nonempty, closed, convex and bounded subset of the Banach algebra X and $\tilde{S}=C\times C$. Suppose that $T:\mathbf{X}\to\mathbf{X}$ and $S:\tilde{S}\to\mathbf{X}$ are two operators such that*

(c1) there exists $f_{\gamma}>0$ such that for all $x,y\in\mathbf{X}$, we have $\|T(\alpha)-T(\beta)\| \le \nu f_{\gamma}\|\alpha-\beta\|$, for some constant $\nu>0$;

(c2) S is completely continuous;

(c3) $\alpha = T(\alpha)+S(\beta)$ for all $\beta\in C$ implies that $\alpha\in C$.

Then the operator $A(\alpha,\beta)=T(\alpha)+S(\beta)$ has at least a coupled fixed point in $\tilde{S}$ whenever $\nu<1$.

2. Existence results

For the existence of solutions to the system (1.2) we introduce the supposition:

(d_0) the function $t\to\alpha-\Phi(t,\alpha)$ is increasing in $\mathbf{R}$ for all $t\in\mathrm{I}$;

(d_1) $\|\Phi(t,\alpha(t))-\Phi(t,\beta(t))\| \le \eta\|\alpha-\beta\|$ where $\eta \le \frac{1-\delta\omega}{(1-\delta\omega)+(1+\delta)}$;

(d_2) Fix $\Phi_0 = \max_{t\in I}|\phi(t,0)|$;

(d_3) there exists $h\in C(J,\mathbf{R})$ such that $\Psi(t,\alpha(t),\beta(t)) \le h(t), \alpha,\beta\in\mathbf{R}, t\in\mathrm{I}$.

Lemma 3.1 *Suppose that (d_0) holds and $\Phi\in C(\mathrm{I}\times\mathbf{R},\mathbf{R})$ with $\Phi(0,0)=0$. Then for $\beta\in C(\mathrm{I},\mathbf{R})$ and $\theta\ge 0$, the unique solution of the boundary value problem*

$$\mathbf{D}^{\sigma}[\alpha(t)-\Phi(t,\alpha(t))] = \Psi(t,\beta(t),I^{\theta}\beta(t)),\ 1<\sigma,\ \theta\le 2,\ t\in\mathrm{I}$$
$$\alpha(0)=0,\ \alpha(1)=\delta\alpha(\omega),\ \delta\omega<1$$

is given by

$$\alpha(t) = \Phi(t,\alpha(t)) + \frac{t}{(1-\delta\omega)}[\delta\Phi(\omega,\alpha(\omega))-\Phi(1,\alpha(1))] + I^{\sigma}\Psi(t) + \frac{t}{(1-\delta\omega)}[\delta I^{\sigma}\Psi(\omega)-I^{\sigma}\Psi(1)].$$

Let choose $\Phi_0 = \max_{t\in\mathrm{I}}|\Phi(t,0)|$ and $N \ge \left(1+\frac{1+\delta}{1-\delta\omega}\right)\left[L+\Phi_0+\frac{\|h\|}{\Gamma(\sigma+1)}\right]$. Define a subset $\tilde{S}$ of X by $\tilde{S}=\{\alpha\in X:\|\alpha\|\le N\}$. Now it is easy to show that $\tilde{S}$ is convex, bounded and closed subset of the Banach space X. Now, we consider the system (1.2). In view of Lemma 3.1, $(\alpha(t),\beta(t))$ is a solution of the system (1.2) if and only if $(\alpha(t),\beta(t))$ satisfies the following system of integral equations

$$\alpha(t) = \Phi(t,\alpha(t)) + \frac{t}{(1-\delta\omega)}[\delta\Phi(\omega,\alpha(\omega))-\Phi(1,\alpha(1))] + I^{\sigma}\Psi(t) + \frac{t}{(1-\delta\omega)}[\delta I^{\sigma}\Psi(\omega)-I^{\sigma}\Psi(1)],$$
$$\beta(t) = \Phi(t,\beta(t)) + \frac{t}{(1-\delta\omega)}[\delta\Phi(\omega,\beta(\omega))-\Phi(1,\beta(1))] + I^{\sigma}\Psi(t) + \frac{t}{(1-\delta\omega)}[\delta I^{\sigma}\Psi(\omega)-I^{\sigma}\Psi(1)],$$

(3.1)

where $t\in\mathrm{I}$. Define two operators $T:X\to X$ and $S:\tilde{S}\to X$ by

$$T\alpha(t) = \Phi(t,\alpha(t)) + \frac{t}{(1-\delta\omega)}\left[\delta\Phi(\omega,\alpha(\omega))-\Phi(1,\alpha(1))\right], \quad (3.2)$$
$$S\alpha(t) = I^{\sigma}\Psi(t) + \frac{t}{(1-\delta\omega)}\left[\delta I^{\sigma}\Psi(\omega)-I^{\sigma}\Psi(1)\right],$$

then the system of integral equations (3.1) modified into the system of operator equations

$$\alpha(t) = T\alpha(t)+S\beta(t) \quad (3.3)$$
$$\beta(t) = T\beta(t)+S\alpha(t),\ t\in\mathrm{I},$$

and solutions of the system (3.1) are fixed points of the system (3.3).

Lemma 3.2 *By the hypothesis (d_1), the operator T defined by (3.2) is contraction.*

Proof. For $\alpha,\beta\in X$, we have

$$\begin{aligned}\left|T\alpha(t)-T\beta(t)\right| &= \Big|\Phi(t,\alpha(t)) + \frac{t}{(1-\delta\omega)}[\delta\Phi(\omega,\alpha(\omega))-\Phi(1,\alpha(1))] \\ &\quad -\Phi(t,\beta(t)) - \frac{t}{(1-\delta\omega)}[\delta\Phi(\omega,\beta(\omega))-\Phi(1,\beta(1))]\Big| \\ &\le \left|\Phi(t,\alpha(t))-\Phi(t,\beta(t))\right| + \frac{\delta t}{(1-\delta\omega)}\left|\Phi(\omega,\alpha(\omega))-\Phi(\omega,\beta(\omega))\right| \\ &\quad + \frac{t}{(1-\delta\omega)}\left|\Phi(1,\alpha(1))-\Phi(1,\beta(1))\right| \\ &\le \left(1+\frac{\delta}{(1-\delta\omega)}+\frac{1}{(1-\delta\omega)}\right)\left|\Phi(t,\alpha(t))-\Phi(t,\beta(t))\right|\end{aligned}$$

which in view of the assumption (d_1) implies that $|T\alpha-T\beta| \le \left(1+\frac{\delta+1}{(1-\delta\omega)}\right)\eta\|\alpha-\beta\|$. Hence it follows that $\|T\alpha-T\beta\| \le \left(1+\frac{\delta+1}{(1-\delta\omega)}\right)\eta\|\alpha-\beta\|$. This means that T is contraction on X.

Lemma 3.3 *Under the assumption* (d_2), *the operator* S *defined by (3.2) is continuous and compact.*

Proof. For continuity of S, choose a sequence $\alpha_m \in \widetilde{S}$ which converges to a point α in $\widetilde{S}$. Then, by dominated convergence theorem, we have

$$\lim_{m\to\infty} S\alpha_m(t) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-\vartheta)^{\sigma-1} \lim_{m\to\infty} \Psi(\vartheta,\alpha_m(\vartheta), I^{\theta}(\alpha_m(\vartheta)))d\vartheta$$
$$+\frac{t\delta}{(1-\delta\omega)\Gamma(\sigma)}\int_0^{\omega}(\omega-\vartheta)^{\sigma-1}\lim_{m\to\infty}\Psi(\vartheta,\alpha_m(\vartheta), I^{\theta}(\alpha_m(\vartheta)))d\vartheta$$
$$-\frac{t}{(1-\delta\omega)\Gamma(\sigma)}\int_0^1(1-\vartheta)^{\sigma-1}\lim_{m\to\infty}\Psi(\vartheta,\alpha_m(\vartheta), I^{\theta}(\alpha_m(\vartheta)))d\vartheta.$$

Using continuity of Ψ, it follows that

$$\lim_{m\to\infty} S\alpha_m(t) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-\vartheta)^{\sigma-1}\Psi(\vartheta,\alpha(\vartheta), I^{\theta}(\alpha(\vartheta)))d\vartheta$$
$$+\frac{t\delta}{(1-\delta\omega)\Gamma(\sigma)}\int_0^{\omega}(\omega-\vartheta)^{\sigma-1}\Psi(\vartheta,\alpha(\vartheta), I^{\theta}(\alpha(\vartheta)))d\vartheta$$
$$-\frac{t}{(1-\delta\omega)\Gamma(\sigma)}\int_0^1(1-\vartheta)^{\sigma-1}\Psi(\vartheta,\alpha(\vartheta), I^{\theta}(\alpha(\vartheta)))d\vartheta = S\alpha(t),\ t\in I.$$

Hence, S is continuous on $\widetilde{S}$.

For boundedness of the operator S on $\widetilde{S}$, choose $\alpha \in \widetilde{S}$ and using hypothesis (d_2), we have

$$|S\alpha(t)| \le \frac{1}{\Gamma(\sigma)}\left|\int_0^t (t-\vartheta)^{\sigma-1}\Psi(\vartheta,\alpha(\vartheta), I^{\theta}(\alpha(\vartheta)))d\vartheta\right|$$
$$+\frac{t\delta}{(1-\delta\omega)\Gamma(\sigma)}\left|\int_0^{\omega}(\omega-\vartheta)^{\sigma-1}\Psi(\vartheta,\alpha(\vartheta), I^{\theta}(\alpha(\vartheta)))d\vartheta\right|$$
$$+\frac{t}{(1-\delta\omega)\Gamma(\sigma)}\left|\int_0^1(1-\vartheta)^{\sigma-1}\Psi(\vartheta,\alpha(\vartheta), I^{\theta}(\alpha(\vartheta)))\right|d\vartheta$$
$$\le\frac{1}{\Gamma(\sigma)}\Big[\int_0^t (t-\vartheta)^{\sigma-1}|h(\vartheta)|d\vartheta$$
$$+\frac{t\delta}{(1-\delta\omega)}\int_0^{\omega}(\omega-\vartheta)^{\sigma-1}|h(\vartheta)|d\vartheta+\frac{t}{(1-\delta\omega)}\int_0^1(1-\vartheta)^{\sigma-1}|h(\vartheta)|\Big]d\vartheta.$$

Hence, it follows that

$$\|S\alpha\| \le \frac{1}{\Gamma(\sigma+1)}\Big(1+\frac{\delta}{(1-\delta\omega)}+\frac{1}{(1-\delta\omega)}\Big)\|h\|,\ \text{for all}\ \alpha\in S,$$

which implies that S is uniformly bounded on B. For equid-continuity of S, choose $t_1, t_2 \in I$ such that $t_1 < t_2$ and $\alpha \in B$, then, we have

$$\Big|S\alpha(t_1)-S\alpha(t_2)\Big| \le \frac{1}{\Gamma(\sigma)}\Big|\int_0^{t_1}(t_1-\vartheta)^{\sigma-1}\Psi(\vartheta,\alpha(\vartheta), I^{\theta}(\alpha(\vartheta)))d\vartheta$$
$$+\frac{t_1\delta}{1-\delta\omega}\int_0^{\omega}(\omega-\vartheta)^{\sigma-1}\Psi(\vartheta,\alpha(\vartheta), I^{\theta}(\alpha(\vartheta)))d\vartheta$$
$$-\frac{t_1}{1-\delta\omega}\int_0^1(1-\vartheta)^{\sigma-1}\Psi(\vartheta,\alpha(\vartheta), I^{\theta}(\alpha(\vartheta)))d\vartheta$$
$$-\int_0^{t_2}(t_2-\vartheta)^{\sigma-1}\Psi(\vartheta,\alpha(\vartheta), I^{\theta}(\alpha(\vartheta)))d\vartheta$$
$$-\frac{t_2\delta}{1-\delta\omega}\int_0^{\omega}(\omega-\vartheta)^{\sigma-1}\Psi(\vartheta,\alpha(\vartheta), I^{\theta}(\alpha(\vartheta)))d\vartheta$$
$$+\frac{t_2}{1-\delta\omega}\int_0^1(1-\vartheta)^{\sigma-1}\Psi(\vartheta,\alpha(\vartheta), I^{\theta}(\alpha(\vartheta)))d\vartheta\Big|.$$

Which implies that

$$\Big|S\alpha(t_1)-S\alpha(t_2)\Big| \le \frac{1}{\Gamma(\sigma)}\Big|\int_0^{t_1}\Big((t_1-\vartheta)^{\sigma-1}-(t_2-\vartheta)^{\sigma-1}\Big)\Psi(\vartheta,\alpha(\vartheta), I^{\sigma}(\alpha(\vartheta)))d\vartheta\Big|$$
$$+\frac{\delta}{1-\delta\omega}\Big|\int_0^{\omega}(t_1-t_2)(\omega-\vartheta)^{\sigma-1}\Psi(\vartheta,\alpha(\vartheta), I^{\sigma}(\alpha(\vartheta)))d\vartheta\Big|$$
$$+\frac{1}{1-\delta\omega}\Big|\int_0^1(t_1-t_2)(1-\vartheta)^{\sigma-1}\Psi(\vartheta,\alpha(\vartheta), I^{\sigma}(\alpha(\vartheta)))d\vartheta\Big|$$
$$\le\frac{\|h\|}{\Gamma(\sigma+1)}|((t_1)^{\sigma}-(t_2)^{\sigma})|+\frac{\|h\|\delta}{(1-\delta\omega)\Gamma(\sigma+1)}\Big||(t_1-t_2)|$$
$$+\frac{\|h\|}{(1-\delta\omega)\Gamma(\sigma+1)}|(t_1-t_2)|$$
$$=\frac{\|h\|}{\Gamma(\sigma+1)}\Big[|((t_1)^{\sigma}-(t_2)^{\sigma})|+\frac{1+\delta}{(1-\delta\omega)}|(t_1-t_2)|\Big]$$
$$\le\frac{\|h\|}{\Gamma(\sigma+1)}\Big[\sigma|(t_1-t_2)|+\frac{1+\delta}{(1-\delta\omega)}|(t_1-t_2)|\Big]$$
$$=\frac{\|h\|}{\Gamma(\sigma+1)}\Big(\sigma+\frac{1+\delta}{(1-\delta\omega)}\Big)|(t_1-t_2)|\to 0\ \text{as}\ t_1\to t_2,$$

which shows that S is equip-continuous. Hence, S is completely continuous on $\widetilde{S}$.

Theorem 3.4 *Assume that the hypothesis* (d_0)-(d_3) *hold, Then the of system has a solution on* I.

Proof. It is enough to show that system (3.3) has a fixed point. In view of Lemma 3.2, the operator T has a contraction. In view of lemma 3.3, the operator S is continuous and compact.

Now, for $\alpha \in X$ and $\beta \in \widetilde{S}$ such that $\alpha = T\alpha + S\beta$, we have

$$\Big|\alpha(t)\Big| \le \Big|T\alpha(t)\Big| + \Big|S\beta(t)\Big|$$
$$=\Big|\Phi(t,\alpha(t))+\frac{t}{(1-\delta\omega)}\Big[\delta\Phi(\omega,\alpha(\omega))-\Phi(1,\alpha(1))\Big]\Big|$$
$$+\frac{1}{\Gamma(\sigma)}\Big|\int_0^t(t-\vartheta)^{\sigma-1}\Psi(\vartheta,\beta(\vartheta), I^{\theta}(\beta(\vartheta)))d\vartheta$$
$$+\frac{t\delta}{1-\delta\omega}\int_0^{\omega}(\omega-\vartheta)^{\sigma-1}\Psi(\vartheta,\beta(\vartheta), I^{\theta}(\beta(\vartheta)))d\vartheta$$
$$-\frac{t}{1-\delta\omega}\int_0^1(1-\vartheta)^{\sigma-1}\Psi(\vartheta,\beta(\vartheta), I^{\theta}(\beta(\vartheta)))d\vartheta\Big|$$
$$\le\Big|\Phi(t,\alpha(t))\Big|+\frac{\delta}{1-\delta\omega}\Big|\Phi(\omega,\alpha(\omega))\Big|+\frac{1}{1-\delta\omega}\Big|\Phi(1,\alpha(1))\Big|$$
$$+\frac{1}{\Gamma(\sigma)}\|h\|\Big[\frac{t^{\sigma}}{\sigma}+\frac{\delta\omega^{\sigma}}{(1-\delta\omega)\sigma}+\frac{1}{(1-\delta\omega)\sigma}\Big]$$
$$\le\Big(1+\frac{\delta}{1-\delta\omega}+\frac{1}{1-\delta\omega}\Big)\Big|\Phi(t,\alpha(t))\Big|+\frac{1}{\Gamma(\sigma+1)}\|h\|\Big((1+\frac{\delta}{1-\delta\omega}+\frac{1}{1-\delta\omega}\Big)$$
$$=\Big(1+\frac{\delta}{1-\delta\zeta}+\frac{1}{1-\delta\omega}\Big)\Big[|\Phi(t,\alpha(t))-\Phi(t,0)|+|\Phi(t,0)|+\frac{\|h\|}{\Gamma(\sigma+1)}\Big]$$
$$\le\Big(1+\frac{\delta}{1-\delta\omega}+\frac{1}{1-\delta\omega}\Big)\Big[L+\Phi_0+\frac{\|h\|}{\Gamma(\sigma+1)}\Big]$$
$$=\Big(1+\frac{1+\delta}{1-\delta\omega}\Big)\Big[L+\Phi_0+\frac{\|h\|}{\Gamma(\sigma+1)}\Big]\le N.$$

which implies that $\beta \in \widetilde{S}$. Hence by Theorem 2.7, the system (1.2) has a coupled fixed point in $\widetilde{S}$.

Example 3.5 *Consider the following coupled system of fractional hybrid differential equation with boundary conditions.*

$$\begin{cases} {}^{c}\mathbf{D}^{\frac{3}{2}}\Big[\alpha(t)-\Big(e^{-t}+\dfrac{\frac{(t^2+1)}{8}|\alpha(t)|}{20+|\alpha(t)|}\Big)\Big]=\dfrac{t^2}{2}-\dfrac{(t^3+1)}{20}\Big[\sin|\beta(t)|+\cos|I^{\frac{3}{2}}\beta(t)|\Big],\ t\in[0,1],\\ {}^{c}\mathbf{D}^{\frac{3}{2}}\Big[\beta(t)-\Big(e^{-t}+\dfrac{\frac{(t^2+1)}{8}|\beta(t)|}{20+|\beta(t)|}\Big)\Big]=\dfrac{t^2}{2}-\dfrac{(t^3+1)}{20}\Big[\sin|\alpha(t)|+\cos|I^{\frac{3}{2}}\alpha(t)|\Big],\ t\in[0,1],\\ \alpha(0)=0,\ \alpha(1)=\dfrac{1}{2}\alpha(\dfrac{1}{2}),\\ \beta(0)=0,\ \beta(1)=\dfrac{1}{2}\beta(\dfrac{1}{2}). \end{cases} \tag{3.4}$$

From (3.4), we have

$$\Phi(t,\alpha(t)) = e^{-t}+\frac{(t^2+1)}{30}\frac{\frac{(t+1)}{4}|\alpha(t)|}{20+|\alpha(t)|},$$
$$\Psi(t,\alpha(t),\beta(t)) = \frac{t^2}{2}-\frac{(t^3+1)}{20}\Big[\sin|\beta(t)|+\sin|I^{\frac{3}{2}}\beta(t)|\Big]\le\frac{t^2}{2}.$$

Now it is easy to find $L=\frac{1}{4}, M=20, \Phi_0=1, \|h\|=\frac{1}{6}, \sigma=\frac{3}{2}.$

$$\Big(1+\frac{1+\delta}{1-\delta\omega}\Big)\Big[L+\Phi_0+\frac{\|h\|}{\Gamma(\sigma+1)}\Big]=3(\frac{5}{4}+\frac{2}{9\sqrt{\pi}})=12.37<13.$$

Hence all the conditions of Theorem 3.4, are satisfied, which show that the FHDEs system (3.4) has a solution in $\widetilde{S}=\{x\in X\ :\|x\|\le 13\}$.

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