$$\begin{cases} D_{0+}^{\alpha}\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad t \in [0,T], \\ a\dfrac{x(0)}{f(0,x(0))} + b\dfrac{x(T)}{f(T,x(T))} = c, \end{cases}$$

where $0 < \alpha < 1$, a,b,c are real constants with $a + b \neq 0$ and $f \in C(J \times R, R_{\setminus\{0\}})$ $g \in C(J \times R, R)$.

In this article, we extend the aforesaid hybrid fractional differential equation to boundary condition involving ordinary derivatives given as

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\right) = h(t,x(t)), \; t \in [0,1], \quad \alpha \in (1,2], \\ \left[\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=0} = 0, \; \left[\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=1} = 0 \end{cases} \tag{1.2}$$

where $g \in C(J \times R, R)$ and $f, h \in C(J \times R, R)$. With the help of theory develop by a researcher, we establish the sufficient conditions under which the considered problem of hybrid fractional differential equations has at least one solution [37]. At the end, we give an example for illustrative purposes.

2. PRELIMINARIES

In this section, we recall some definitions and results of fractional calculus and hybrid fixed point theory, that are necessary for further investigation [28, 29, 37].

Definition 2.1. The Riemann-Liouville fractional derivative of order α > 0 of a function f : (0,∞) → R is given by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_0^t (t-s)^{n-\alpha-1} f(s)\,ds, n-1<\alpha<n,$$

where $n = [\alpha] + 1, [\alpha]$ represents the integer part of a real number α , provided that the right-hand side is point-wise defined on $(0,\infty)$, where Γ is the gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-s} s^{\alpha-1} ds$

Definition 2.2. The Riemann-Liouville fractional integral of order α > 0 of a continuous function f : (0,∞) → R is given by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,ds$$

provided that the right-hand side is point-wise defined on (0,∞) .

Definition 2.3. A map θ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that θ : P → [0,∞) is continuous and

$\theta(tx+(1-t)y) \geq t\theta(x) + (1-t)\theta(y)$,

for all x, y ∈ P and 0 ≤ t ≤1.

The next two lemmas play an important role for obtaining the equivalent integral equation of BVP (1.2).

Lemma 2.4. [1]. If we assume x ∈C (0,1) ∩ L (0,1) , then the fractional differential equation

$D^{\alpha} u(t) = 0$

of order α > 0 has a unique solution of the form

$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \ldots + C_N t^{\alpha-N}$, $C_i \in R$, $i = 1, 2, \ldots, n$ and $n-1<\alpha<n$.

The following law of composition can be easily deduced from Lemma 2.4.

Lemma 2.5. Assume that x ∈C(0,1) ∩ L(0,1) , with a fractional derivative of order α that belongs to C(0,1) ∩ L(0,1) , then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = x(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \ldots + C_N t^{\alpha-N}, \; C_i \in R \text{ and } i = 1, 2, \ldots, N .$$

Lemma 2.6. [1] For x ∈C(0,T) ∩ L(0,T) , the solution of fractional differential equation

$$D^{\alpha} x(t) = y(t), \; n-1<\alpha<n$$

is given by

$$x(t) = I^{\alpha} y(t) + \sum_{i=0}^{n-1} C_i t^i .$$

Let $E = C(J, R)$ be the space of continuous real-valued functions defined on $J = [0,T]$. Define a norm $\|\bullet\|$ and a multiplication in E by $\|x\| = \sup_{t\in J} |x(t)|$ and $(xy)(t) = x(t)y(t)$, for all $t \in J$. Clearly, E is a Banach algebra with respect to the above supremum norm and the multiplication defined on it.

3. HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS

In this section, we consider the Boundary Value Problem (1.2). The following hybrid fixed point theorem for three operators in a Banach algebra E , due to Dhage, will be used to prove the existence result for the Boundary Value Problem (1.2) [37].

Lemma 3.1. Let S be a nonempty, closed convex and bounded subset of a Banach algebra E and let A,C : E → E and B : S → E be three operators satisfying:

a. A and C are Lipschitzian with Lipschitz constants φ_1 and φ_2 , respectively
b. B is completely continuous
c. If $x = AxBy + Cx$, then $x \in S$ for all $y \in S$
d. If $\varphi_1 M + \varphi_2 < 1$ where $M = \|B(S)\|$, then the operator equation $x =$ AxBy + Cx has a solution.

Lemma 3.2. Let z ∈C([0,1], R) , then the solution of BVP of hybrid differential equation of fractional order

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\right) = z(t), \; t \in [0,1], \quad \alpha \in (1,2], \\ \left[\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=0} = 0, \; \left[\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=1} = 0 \end{cases} \tag{3.1}$$

is given by

$$x(t) = f(t,x(t)) + g(t,x(t))\int_0^1 G(t,s)z(s)ds, \tag{3.2}$$

where $\lambda_1 = g(0, x(0)) \neq 0$ and G(t, s) is the Green's function defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} (t-s)^{\alpha-1} - t(1-s)^{\alpha-1}, \; 0 \leq s \leq t \leq 1, \\ -t(1-s)^{\alpha-1}, \; 0 \leq t \leq s \leq 1. \end{cases}$$

Proof. Applying the Riemann-liouville fraction integral operator of order α to both sides of (3.1) and using Lemma (2.6), we have

$$I^{\alpha}\left[D^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right)\right] = C_0 + C_1 t + I^{\alpha} y(t) \tag{3.3}$$

In view of boundary conditions $\left[\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=0} = 0, \left[\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=1} = 0$, (3.3) takes the form

$$x(t) = f(t,x(t)) + g(t,x(t))\int_0^1 G(t,s)z(s)ds,$$

where G(t, s) is the Green's function as given in (3.2).

Thanks to Lemma (3.2), the proposed problem is equivalent to the following integral equation

$$x(t) = f(t,x(t)) + g(t,x(t))\int_0^1 G(t,s)h(s,x(s))ds, \; t \in [0,1] .$$

Here we remarked that

$$\hat{G} = \max_{t\in[0,1]} \int_0^1 |G(t,s)| ds \le \frac{1}{\Gamma(\alpha+1)}.$$

Theorem 3.3. Assume that the functions $g : J \times R \to R \backslash\{0\}$ and $h, f : J \times R \to R$ with f (0, x(0)) = h(0, x(0)) = 0 are continuous and let the following hypothesis hold. (H1). There exist two positive functions θ and ϖ with bound ||θ|| and ||ϖ|| respectively, such that

$$| f(t, x(t)) - f(t, y(t) | \le \theta(t) | x(t) - y(t) |$$

and

$$| g(t, x(t)) - h(t, y(t) | \le \varpi(t) | x(t) - y(t) |$$

for t ∈ J and x, y ∈ R .

(H_2). There exist a function p ∈ C (J, R^+) and a continuous non-decreasing function Ψ :[0,∞) → (0,∞) such that

$$| h(t, x(t)) | \le p(t)\Psi(| x |), (t, x) \in J \times R \quad (3.5)$$

(H_3). There exists a number r > 0 such that

$$r \ge \frac{f_0 + \frac{g_0 \|p\| \Psi(r)}{\Gamma(\alpha+1)}}{1 - (\|\theta\| + \frac{\|\varphi\|\|p\|\Psi(r)}{\Gamma(\alpha+1)})} \quad (3.6)$$

where $f_0 = \sup_{t\in J} | f(t,0) |$ and $g_0 = \sup_{t\in J} | g(t,0) |$ and

$$\|\theta\| + \frac{\|\varphi\|\|p\|\Psi(r)}{\Gamma(\alpha+1)} < 1 \quad (3.7)$$

Then the Problem (1.2) has at least one solution on J .

Proof. Set E = C(J, R) and define a subset S of E as

$$S = \{x \in E : \|x\| \le r\},$$

where r satisfies Inequality (3.6). Clearly, S is closed, convex, and bounded subset of the Banach space E . Due to the integral Equation (3.2), now we define respectively the three operators A: E → E , C : E → E and B : S → E by

$$Ax(t) = f(t, x(t)), \ t \in J \quad (3.8)$$

$$Cx(t) = g(t, x(t)), \ t \in J \quad (3.9)$$

and

$$Bx(t) = \int_0^1 G(t,s) h(s, x(s)) ds, \ t \in J.$$

Then the integral Equation (3.2) can be written in the operator form as

$$x(t) = Ax(t) + Bx(t)Cx(t), \ t \in J \quad (3.10)$$

We shall show that the operators A , B and C satisfy all the conditions of Lemma (3.1). This will be achieved in the following series of steps.

Step 1. We first show that A and C are Lipschitzian on E . Let x, y ∈ E , then by (H1), for t ∈ J , we have

Taking maximum over [0,1] , we get

$$\|Ax - Ay\| \le \|\theta\| \|x - y\| \le \|\theta\| r,$$

where $r = \|x - y\|$, which implies

$$\|Ax - Ay\| \le \|\theta\| r \text{ for all } x, y \in E.$$

Therefore A is a Lipschitzian on E with Lipschitz constant $\|\theta\|$. Now for $C : E \to E, x, y \in E$, we

have

$$| Cx(t) - Cy(t) | = | g(t, x(t) - g(t, y(t) | \le \varpi | x(t) - y(t) |$$

Taking maximum over [0,1], implies

$$\|Cx - Cy\| \le \|\varpi\| \|x - y\| \le \|\phi_2\| r,$$

where $r = \|x - y\|$, which implies

$$\|Cx - Cy\| \le \|\varpi\| r \text{ for all } x, y \in E.$$

Hence, $C : E \to E$ is a Lipschitzian on E with Lipschitz constant $\|\varpi\|$.

Step 2. The operator $B : S \to E$ is completely continuous on S. We first show that the operator B is continuous on E . Let {x_n} be a sequence in S converging to a point x ∈ S . Then by the Lebesgue dominated convergence theorem for all t ∈ J , we obtain

$$\lim_{n\to\infty} Bx_n(t) = \lim_{n\to\infty} \int_0^1 G(t,s) h(s, x_n(s)) ds$$
$$= \int_0^1 G(t,s) \lim_{n\to\infty} h(s, x_n(s)) ds$$
$$= \int_0^1 G(t,s) h(s, x(s)) ds = Bx(t).$$

Hence, $\lim_{n\to\infty} Bx_n(t) = Bx(t)$. So B is continuous on S.

Next we will show that the set B(S) uniformly bounded in S . For any x ∈ S , we have

$$\begin{aligned} | Bx(t) | &\le \int_0 | G(t,s) | | h(s, x(s)) | ds \\ &\le \int_0^1 | G(1,s) || p(s) | \Psi(| x(s) |) ds \\ &\le \frac{\|p\| \Psi(r)}{\Gamma(\alpha+1)} := K, \end{aligned} \quad (3.11)$$

for t ∈ J . Therefore, Bx ≤ K which shows that B is uniformly bounded on S . Now, we will show that B(S) is an equi-continuous set in E . If τ ,t ∈ J with τ < t and x ∈ S , then we have

$$\begin{aligned} | Bx(t) - Bx(\tau) | &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} | h(s, x(s)) | ds \\ &- \int_0^\tau \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} | h(s, x(s)) | ds \\ &+ (t-\tau) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} | h(s, x(s)) | ds \\ &\le \frac{\|p\| \Psi(r)}{\Gamma(\alpha+1)} [t^\alpha - \tau^\alpha + t - \tau]. \end{aligned} \quad (3.12)$$

Now t →τ the right-hand side of the above inequality tends to zero. Therefore, | Bx(t) − Bx(τ) |→ 0 as t →τ .

It follows from the Arzelá-Ascoli theorem that B is a completely continuous operator on S .

Step 3. The hypothesis (c) of Lemma (3.1) is satisfied. Let x ∈ E and y ∈ S be arbitrary elements such that x = AxBx + Cx . Then we have

$$| x(t) | = | Ax(t) + Bx(t)Cx(t) |$$
$$\le | Ax(t) | + | Bx(t)Cx(t) |$$