Since $A_a \neq \phi$ we have

$$(g_S \cup f_S)(a) = g_S(a) \cup f_S(a)$$

$$\supseteq g_S(ax) \cup f_S(ay)$$

$$\supseteq \bigcap_{(ax,ay)\in A_a} \{g_S(ax) \cup f_S(ay)\}$$

$$\supseteq \bigcap_{(p,q)\in A_a} \{g_S(p) \cup f_S(q)\}$$

$$= (g_S \lozenge f_S)(a).$$

Which means that $g_S \tilde{\cup} f_S \tilde{\supseteq} g_S \lozenge f_S$. And from Proposition 3, we have the reverse inclusion $g_S \lozenge f_S \tilde{\supseteq} g_S \tilde{\cup} f_S$. Hence $g_S \tilde{\cup} f_S = g_S \lozenge f_S$, that is (f_S, S) is a pure uni-soft ideal over U.

$(3) \Rightarrow (1)$: Let $a \in S$. We show that $a \in ((aS)^2]$. Let $I = (a \cup Sa \cup aS \cup SaS]$ be an ideal generated by a. Then by Lemma 3, the soft (χ_I^c, S) over U is a uni-soft right ideal of S over U, hence by (3), (χ_I^c, S) is a pure uni-soft ideal. Thus, by the Theorem 2, I is pure in S. Since $a \in I$ and I is pure in S, therefore there exist $b \in I$ such that $a \leq ab$. This means that

$$a \in (aI] = (a(a \cup Sa \cup aS \cup SaS)]$$
$$= (aa \cup aSa \cup aaS \cup aSaS].$$

Which means that $a \in ((aS)^2]$. Hence S is right weakly regular ordered semigroup.

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