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REVIEW ARTICLE

ANALYTIC SOLUTION TO BENJAMIN-BONA-MAHONY EQUATION BY USING LAPLACE ADOMIAN DECOMPOSITION METHOD

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ABSTRACT

In this paper, we want to find the analytic solution of Benjamin-Bona-Mahony (BBM) equation by using Laplace Adomian Decomposition Method. Laplace Adomian Decomposition Method is an excellent mathematical tool for solving linear and nonlinear differential equation. This method is a combination of the famous integral transform known as Laplace transform and the Adomian Decomposition Method (ADM).

KEYWORDS

Laplace transform, Adomian polynomial, Benjamin–Bona–Mahony equation.

1. INTRODUCTION

The Benjamin–Bona–Mahony equation (BBM equation) also known as the regularized long-Wave equation (RLWE) is the partial differential equation is given as [1]:

$$u_t + u_x + uu_x - u_{xxt} = 0\,,\ u(x,0) = g(x)\,.$$

This equation was studied by Benjamin, Bona, and Mahony in 1972 as an improvement of the Korteweg de Vries equation (KdV equation) for modeling long surface gravity waves of small amplitude propagating-unidirectionally in 1+1 dimensions. They show the stability and uniqueness of the solutions to the BBM equation. This contrasts with the KdV equation, which is unstable in its high wavenumber components. Further, while the KdV equation has an infinite number of integrals of motion, the BBM equation only has three [2,3].

Before, in 1966, this equation was introduced by Peregrine, in the study of undular bores [4]. A generalized n-dimensional version is given in [5]. Many nonlinear evolution equations are playing important role in the analysis of some phenomena and including ion acoustic waves in plasmas, dust acoustic solitary structures in magnetized dusty plasmas, and electromagnetic waves in size quantized films [6]. To obtain the traveling wave solutions to these nonlinear evolution equations, many methods were developed, such as the inverse scattering method, Natural transformation, homogeneous balance method, Bäcklund transformation, lie group method, the factorization technique, the Laplace Adomian decomposition method, peseduospectral method, Exp-function method, used to and the Riccati equation expansion method were used to investigate these types equations [3,7,9-12]. The above methods derived many types of solutions from most nonlinear evolution equations.

The BBM equation is well known in physical applications [13]. It describes the model for propagation of long waves which incorporates nonlinear and dissipative effects. It is used in the analysis of the surface waves of long wavelength in liquid, hydro magnetic waves cold plasma, acoustic-gravity waves in compressible fluids, and acoustic waves in harmonic crystals [2]. Many mathematicians paid their attention to the dynamics of the BBM equation [14]. The BBM equation has been investigated as a regularized version of the Kdv equation for shallow water waves. In certain theoretical investigations the equation is superior as a model for long waves; from the standpoint of existence and stability, the equation offers considerable technical advantages over the KdV equation. In addition to shallow water waves, the equation is applicable to the study of drift waves in Plasma or the Rossby waves in rotating fluids. Under certain conditions, it also provides a model of one-dimensional transmitted waves. The main mathematical difference between KdV and BBM models can be most readily appreciated by comparing the dispersion relation for the respective linearized equations. It can be easily seen that these relations are comparable only for small wave numbers and they generate drastically different responses to short waves. This is one of the reasons why, whereas existence regularity theory for the KdV equation is difficult, the theory of the BBM equation is comparatively simple where the BBM equation does not take into account dissipation and nonintegrable [13,14]. The KdV equation describes long nonlinear waves of small amplitude on the surface of inviscid ideal fluid [8]. The KdV equation is integrable by inverse the scattering transform. Solitons exist due to the balance between the weak non-linearity and dispersion of the KdV equation. Soliton is a localized wave that has an infinite support or a localized wave with exponential wings. The solutions of the BBM equation and the KdV equation have been of considerable concern. Zabusky and Kruskal investigated the interaction of solitary waves and the recurrence of initial states [15]. The term soliton is coined to reflect the particle like behavior of the solitary waves under interaction.

The interaction of two solitons emphasized the reality of the preservation of shapes and speeds and of the steady pulse like character of solitons [6,10]. Keeping this importance in mind, we handle the analytical study of the considered BBM Equations via using LADM. We get the required solution in the form of infinite series which has close to that of the exact solution.

2. PRELIMINARIES

Definition 2.1: If $f(t)$ is continuous and real valued function defined for all t, $0 < t < \infty$ and is of exponential order, then the Laplace transform is denoted and defined as

$$L\left[f(t)\right] = F(s) = \int_0^\infty e^{-st} f(t)dt\,.$$

Where s is a parameter of transform, $S > 0$ and L is the operator which transform $f(t)$ into $F(s)$.

2.1 Introduction to Adomian polynomials

Adomian decomposition is a semi analytical method to ordinary differential equations and partial differential equations. The aspect of this method is employment of "Adomian Polynomials" which allow for solution convergence of the linear portion of the equation. The Adomian decomposition method defines the solution $u(t)$ in the form of a series given by

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$$

and replacing the nonlinear term by the given series

$$Qu(x,\ t) = \sum_{n=0}^{\infty} A_n(x,\ t),$$

where A_n represents Adomian polynomials which is computed as

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\left(\sum_{i=0}^{\infty}\lambda^i u_{kx}\right)\right]_{\lambda=0}.$$

3. GENERAL SOLUTION TO BBM EQUATION

First, we find the general solution of the BBM equation by taking its Laplace transform. We use the initial conditions, after simplification, we take the inverse Laplace transform [16-18]. Then assume the infinite series solution of the form $\sum_{n=0}^{\infty} u_n$ to the unknown functions and decompose the nonlinear terms through Adomian polynomials. We get the solution of the general partial differential equation, which is given as:

$$u_t(x,t) + u_x(x,t) + u(x,t)u_x(x,t) - u_{xxt}(x,t) = 0 \tag{1}$$

with initial conditions given by $u(x,0) = g(x)$. From (1), we can write as

$$u_t(x,t) = u_{xxt}(x,t) - u(x,t)u_x(x,t) - u_x(x,t) \tag{2}$$

First, taking the Laplace transform of the given partial differential equation in (2) as:

$L[u_t(x,t)] = L[u_{xxt}(x,t) - u(x,t)u_x(x,t) - u_x(x,t)]$. Which gives

$$sU(x,s) - u(x,0) = L[u_{xxt}(x,t) - u(x,t)u_x(x,t) - u_x(x,t)]$$

$$sU(x,s) = u(x,0) + L[u_{xxt}(x,t) - u(x,t)u_x(x,t) - u_x(x,t)]. \tag{3}$$

After using the initial condition, the above equation implies

$$U(x,s) = \frac{g(x)}{s} + \frac{1}{s}L[u_{xxt}(x,t) - uu_x(x,t) - u_x(x,t)]. \tag{4}$$

Now taking the inverse Laplace transform of (4), we get

$$L^{-1}\{U(x,s)\} = L^{-1}\left[\frac{g(x)}{s}\right] + L^{-1}\left[\frac{1}{s}L[u_{xxt}(x,t) - uu_x(x,t) - u_x(x,t)]\right]$$

$$U(x,t) = g(x) + L^{-1}\left[\frac{1}{s}L[u_{xxt}(x,t) - uu_x(x,t) - u_x(x,t)]\right]. \tag{5}$$

Now let for the unknown function $u(x,t)$ and the infinite series solution is

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

Moreover, the nonlinear term uu_x can easily be decomposed as

$$u(x,t)u_x(x,t) = \sum_{n=0}^{\infty} A_n(x,t),$$

where A_n are Adomian polynomials and given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\left(\sum_{i=0}^{\infty}\lambda^i u_{kx}\right)\right]_{\lambda=0}.$$

First few polynomials can be obtained as

$$A_0 = u_0(x,t)u_{0x}(x,t),$$

$$A_1 = \frac{d}{d\lambda}\left[(u_0(x,t) + \lambda u_1(x,t))(u_{0x}(x,t) + \lambda u_{1x}(x,t))\right]_{\lambda=0}$$
$$= u_0(x,t)u_{1x}(x,t) + u_1(x,t)u_{0x}(x,t).$$

and so on So (5) implies that:

$$\sum_{n=o}^{\infty} u_n(x,t) = g(x) + L^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty} u_{nxxt}(x,t) - \sum_{n=0}^{\infty} A_n(x,t) - \sum_{n=0}^{\infty} u_{nx}(x,t)\right]\right].$$

The recursive relation is given below

$$u_0(x,t) = g(x),$$

$$u_1(x,t) = L^{-1}\left[\frac{1}{s}L[u_{0xxt}(x,t) - A_0(x,t) - u_{0x}(x,t)]\right],$$

$$u_2(x,t) = L^{-1}\left[\frac{1}{s}L[u_{1xxt}(x,t) - A_1(x,t) - u_{1x}(x,t)]\right],$$

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$$u_{n+1}(x,t) = L^{-1}\left[\frac{1}{s}L[u_{nxxt}(x,t) - A_n(x,t) - u_{nx}(x,t)]\right].$$

Now we compute the first term of the solution as:

$$u_0(x,t) = g(x),$$

$$u_1(x,t) = L^{-1}\left[\frac{1}{s}L\left[0 - g(x)g^{'}(x) - g^{'}(x)\right]\right]$$
$$= L^{-1}\left[\frac{1}{s}L\left[-g^{'}(x)\{g(x)+1\}\right]\right]$$
$$= L^{-1}\left[\frac{1}{s}\left[-g^{'}(x)\{g(x)+1\}\right]L[1]\right]$$
$$= L^{-1}\left[\frac{1}{s}\left[-g^{'}(x)\{g(x)+1\}\right]\frac{1}{s}\right]$$
$$= -g^{'}(x)\{g(x)+1\}L^{-1}\left[\frac{1}{s^2}\right]$$
$$= -g^{'}(x)\{g(x)+1\}t.$$

Similarly, the other terms can be computed. Finally, the solution is given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$
$$= u_0(x,t) + u_1(x,t) + u_2(x,t) + ...$$

Example 3.1: Next, we consider the Benjamin-Bona-Mahony equation of nonlinear partial differential equation of the form

$$u_t + u_x + uu_x - u_{xxt} = 0 \qquad (6)$$

with initial conditions given by $u(x,0) = e^x$.

With the help of the above procedure few polynomials can be obtained as

$$A_0 = u_0 u_{0x} = e^x.e^x = e^{2x}$$
$$A_1 = \frac{d}{d\lambda}\left[(u_0 + \lambda u_1)(u_{0x} + \lambda u_{1x})\right]_{\lambda=0}$$
$$= u_0 u_{1x} + u_1 u_{0x}$$
$$= -te^{2x}(3e^x + 2) .$$

and so on. Therefore, by the help of the above procedure we compute few terms of the solution as:

$$u_0(x,t) = e^x .$$

$$u_1(x,t) = -e^x(e^x + 1)t.$$

$$u_2(x,t) = -e^x(4e^x + 1)t + (3e^{3x} + 4e^{2x} + e^x)\frac{t^2}{2} .$$

and so on. Therefore, the solution in the form of infinite series is given by:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$
$$= u_0(x,t) + u_1(x,t) + u_2(x,t) + ...$$
$$= e^x - e^x(e^x + 1)t - e^x(4e^x + 1)t$$
$$+ (3e^{3x} + 4e^{2x} + e^x)\frac{t^2}{2} + ...$$
$$= e^x - (5e^{2x} + 2e^x)t + (3e^{3x} + 4e^{2x} + e^x)\frac{t^2}{2} + ...$$

Example 3.2: Next, we consider the Benjamin-Bona-Mahony equation of nonlinear partial differential equation of the form

$$u_t + u_x + uu_x - u_{xxt} = 0 , \qquad (7)$$

with initial conditions given by $u(x,0) = e^x$.

With the help of the above procedure first few terms of Adomian polynomials can be obtained as

$$A_0 = u_0 u_{0x} = x^2.2x = 2x^3 .$$

$$A_1 = \frac{d}{d\lambda}\left[(u_0 + \lambda u_1)(u_{0x} + \lambda u_{1x})\right]_{\lambda=0}$$
$$= u_0 u_{1x} + u_1 u_{0x}$$
$$= -2x^2(3x^2 + 1)t - 4x^2(x^2 + 1)t$$

and so on. With the help of the above procedure, we compute few terms of the solution as:

$$u_0(x,t) = x^2 .$$

$$u_1(x,t) = -2x(x^2 + 1)t .$$

$$u_2(x,t) = -12xt + t^2(5x^4 + 6x^2 + 1)$$

and so on. So, the solution in the infinite series form is given by:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$
$$= u_0(x,t) + u_1(x,t) + u_2(x,t) + ...$$
$$= x^2 - 2x(x^2 + 1)t - 12xt + t^2(5x^4 + 6x^2 + 1) + ...$$
$$= x^2 - (2x^3 + 14x)t + (5x^4 + 6x^2 + 1)t^2 + ...$$

4. CONCLUSION

We have successfully applied Laplace Adomian decomposition method (LADM) to handle the analytical solution to the BBM equation of PDEs. The respective solution has been obtained in the form of infinite series which is rapidly converging to its exact value.

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