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REVIEW ARTICLE

DECOMPOSITION OF C^m THROUGH Q-PERIODIC DISCRETE EVOLUTION FAMILY

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ABSTRACT

Let $\mho = \{U(m,n): \ m,n \in \mathbf{Z}_+\}_{n \ge m \ge 0}$ be the q-periodic discrete evolution family of square size matrices of order m having complex scalars as entries generated by *L(C^m*-valued, q-periodic sequence of square size matrices $(A_n)_{n \in Z_+}$ where $q \ge 2$ is a natural number. Where the Poincare map $U(q,0)$ is the generator of the discrete evolution family $\mho$. The main objective of this article to decompose *C^m* with the help of discrete evolution family. In fact we decompose $\mathbb{C}^m$ in two sub spaces X_1 and X_2 such that X_1 is due to the stability of the discrete evolution family and the vectors of X_1 will called stable vectors. While X_2 is due to the un-stability of discrete evolution family, and their vectors will be called unstable vectors. More precisely we take the dichotomy of the discrete evolution family with the help of projection P on $\mathbb{C}^m$ and we discuss different results of the spaces X_1 and X_2.

KEYWORDS

Exponential Stability, Strong Stability, Exponential Dichotomy, Discrete Evolution Family

1. INTRODUCTION

Throughout the paper $\mho$ stands for the Discrete evolution family. We denote by $\|.\|$, norm of operators and vectors. 0 and I will denote null and identity operators respectively [1-3]. A family U= {U(m,n): m,n ε **Z** +} of a square size complex valued matrix of order m is called q-periodic discrete evolution family if it satisfies the following properties.

(i) $U(m,r)U(r,n) = U(m,n)$ for all $n \ge r \ge m \ge 0$

(ii) $U(m,m) = I$

(iii) $U(m+q, n+q) = U(m,n)$.

A discrete evolution family $\mho$ is exponentially stable (e-stable) if there exist real constants $M \ge 1$ and $v > 0$ such that

$$\|U(m,n)v\| \le Me^{-v(m-n)}\|v\| \text{ for all } m,n \in Z_+,$$

with $m \ge n$ and every $v \in \mathbb{C}^m$.

(equivalently $\|U(m,n)\| \le Me^{-v(m-n)}$ for all $m,n \in Z+$ with $m \ge n$ for proof see [1].

It is Uniformly stable (u-stable) if

$$\|U(m,n)\| \to 0 \text{ as } (m-n) \to \infty.$$

From above $\mho$ is uniformly stable if and only if it is exponentially stable. A discrete evolution family $\mho$ is strongly stable (s-stable) if

$$\|U(m,n)v\| \to 0 \text{ as } (m-n) \to \infty \text{ for } v \in \mathbb{C}^m.$$

To discuss the exponential dichotomy of $\mho$ it involves an orthogonal projection (often called exponential dichotomic projection), which is such that the discrete evolution family is exponentially stable on its range and non-exponentially stable on its kernel in the sense that all non-zero vectors in it are non-exponentially stable as in forthcoming Definition 1(c). This exponential dichotomic projection is the orthogonal projection onto the exponential stability subspace, and it is dominated by the orthogonal projection onto the strong stability subspace [4].

Exponential dichotomy for $\mathbb{C}^m$ through discrete evolution family is defined in section 2. It is shown that if a discrete evolution family is exponentially dichotomic (i.e. if it has an exponential dichotomy), then the reducing subspace on which it is exponentially stable is maximal (in the sense that no exponential stability includes it) [5]. This subspace clearly is included in the subspace of strong stability associated with the discrete evolution family. If this inclusion is proper, then we consider the characterization of the two types of s-stable vectors associated with discrete evolution family for further detail see [6]. We close the paper in section 3 with a detour towards strong dichotomy of $\mathbb{C}^m$ discrete evolution family.

2. DECOMPOSITION OF $\mathbb{C}^m$ BY DISCRETE EVOLUTION FAMILY

Definition 1. Let $\mho$ be a discrete evolution family defined on $\mathbb{C}^m$. Then a vector $v \in \mathbb{C}^m$.

(a) v is s-stable if

$$\|U(m,n)\| \to 0 \text{ as } (m-n) \to \infty,$$

(b) v is e-stable if there exist real numbers $M(v) \geq 1$ and $\alpha(v) > 0$ such that

$$\|U(m,n)v\| \leq M_{(v)} e^{-\alpha(v)(m-n)} \|v\| \text{ for all } m \geq n,$$

since e-stability obviously implies s-stability from above so an e-stable vector is also referred to as an s-stable-via-e-stable vector. In other words, v is e-stable if and only if v is s-stable-via-e-stable.

(c) v is non-s-stable that is, if

$$\|U(m,n)\| \not\to 0 \text{ as } (m-n) \to \infty,$$

(d) v is non-e-stable that is, if for every $M \geq 1$ and $v > 0$

$$Me^{-v(m-n)} \|v\| < \|U(m,n)v\| \text{ for all } m \geq n,$$

(e) v is s-stable-non-e-stable if it is not e-stable but it is s-stable. So, v is s-stable-non-e-stable if

$$\lim_{(m-n)\to\infty} \|U(m,n)v\| = 0$$

and for every $M \geq 1$ and $v > 0$, we have

$$Me^{-v(m-n)} \|v\| < \|U(m,n)v\|.$$

A subset (in particular, a subspace) of $\mathbb{C}^m$ consisting entirely of s-stable or e-stable vectors will be referred to as an s-stable, or e-stable subset (subspace). A subset (subspace) of $\mathbb{C}^m$ for which all nonzero vectors are s-stable-non-e-stable, or non-s-stable, or non-e-stable, will be referred to as an s-stable-non-e-stable, or non-s-stable, or non-e-stable subset (subspace) respectively [7]. Observe that an s-stable subset is the one that may contain both e-stable (i.e. s-stable-via-e-stable) as well as s-stable-non-e-stable vectors.

If a subspace X_1 of $\mathbb{C}^m$ is $\mathbf{U}$-invariant, then we say that the discrete evolution family $\mathbf{U}$ is e-stable on the subspace X_1 if its restriction $\mathbf{U}|_{X_1}$ is e-stable, which means that there exist a real constant $M \geq 1$ and $\alpha > 0$ such that

$$\|U(m,n)u\| \leq Me^{-\alpha(m-n)} \|u\| \text{ for all } m \geq n \text{ and every } u \in X_1$$

This means that X_1 is a homogeneously e-stable subspace in the sense that each vector u in X_1 is e-stable where $M(v)$ and $\alpha(v)$ of Definition 1 (b) do not depend on v in X_1 (i.e., they are constants not only over $m \geq n$ but also over $v \in X_1$) [8]. On the other hand, if a nonzero subspace X_2 of $\mathbb{C}^m$ is $\mathbf{U}$-invariant, then we say that the discrete evolution family is non-e-stable on the subspace X_2 if its restriction $\mathbf{U}|_{X_2}$ is not only non-e-stable but, more that, if every nonzero vector of X_2 is non-e-stable in the sense of Definition 1 (d). That is, for every $0 \neq v \in X_2$, $M \geq 1$, and $\alpha > 0$

$$e^{-\alpha(m-n)} \|v\| < \frac{1}{M} \|U(m,n)\| \text{ for some } m \geq n.$$

Definition 2. Let $\mathbf{U}$ be a discrete evolution family defined on $\mathbb{C}^m$. Take an orthogonal projection P such that $X_1 = \text{range}(P) = P(\mathbb{C}^m)$ and $X_2 = \text{kernel}(P) = P^{-1}(\{0\})$ are complementary orthogonal subspaces of $\mathbb{C}^m$ that is,

$$\mathbb{C}^m = X_1 \oplus X_2$$

With $X_1^{\perp} = X_2$. If

(a) P commutes with each $\mathbf{U}$ (i.e., $PU(m,n) = U(m,n)P$ for every $m,n \in \mathbb{Z}_+$ or equivalently, X_1 and X_2 are reducing subspaces for $\mathbf{U}$), and

(b) $\mathbf{U}$ is e-stable on X_1 and non-e-stable on X_2 (in the sense that all nonzero vectors in X_2 are non-e-stable),

then we say that P is an e-dichotomic projection for $\mathbf{U}$ and the discrete evolution family is said to e-dichotomic or (P dichotomic).

The generator of a discrete evolution family $\mathbf{U}$ defined on $\mathbb{C}^m$ will be denoted by Poincare map $\{U(q,0)\}$, which is a linear (not necessarily bounded) transformation of a dense linear manifold $\mathbf{D}$ of $\mathbb{C}^m$ the domain of $\{U(q,0)\}$, into $\mathbb{C}^m$.

Remark 1. Consider the setup of Definition 2. Let E be the complementary projection of P (i.e., $E = I - P$ is the orthogonal projection with kernel of $E = range(P)$ and $range(E) = \ker nel(P)$) since the subspaces X_1 = *range (P) and* X_2 = *kern el (P)* reduce each $\mathbf{U}$, it follows that $\mathbf{U}|_{X_1}$ and $\mathbf{U}|_{X_2}$ are also discrete evolution families acting on X_1 and X_2 (i.e., $\mathbf{U}|_{X_1} u = \mathbf{U}Px$ and $\mathbf{U}|_{X_2} v = \mathbf{U}Ex$ for every $x = (u,v)$ in $\mathbb{C}^m = X_1 \oplus X_2$ with $u \in X_1$ and $v \in X_2$). The generators of $\mathbf{U}|_{X_1}$ and $\mathbf{U}|_{X_2}$ are the restrictions, $\{U(q,0)\}|_{X_1}$ and $\{U(q,0)\}|_{X_2}$, of the generator $\{U(q,0)\}$ of $\mathbf{U}$ to their respective domains $\mathbf{D}_{X_1} = \mathbf{D} \cap X_1$ and $\mathbf{D}_{X_2} = \mathbf{D} \cap X_2$ (i.e., $\{U(q,0)\}|_{X_1} u = \{U(q,0)\}u$ for every $u \in \mathbf{D} \cap X_1$ and $\{U(q,0)\}|_{X_2} v = \{U(q,0)\}x$ for every $v \in \mathbf{D} \cap X_2$).

Remark 2. Consider again the setup of *Definition 2*. Observe that $X_2 \setminus \{0\}$ may be empty. Indeed there are discrete evolution family for which the e-dichotomy degenerates in the following sense, there may be no nonzero e-stable vector (i.e., $X_1 = 0$) or there may be no nonzero non-e-stable vector (i.e., $X_2 = 0$). Actually we noticed in section 1 that e-stability may be thought of as (degenerate) e-dichotomy on the whole space $\mathbb{C}^m$ (i.e., $X_1 = \mathbb{C}^m$) which means $P = I$ -also [9]. In the same way, non-e-stability (degenerate) e-dichotomy on the zero space (i.e., $X_2 = \mathbb{C}^m$ which means $X_1 = O$) in the sense that all nonzero vectors are non-e-stable. However, the extra assumption that X_1 is nontrivial (i.e., $O \neq X_1 \neq I$) or, equivalently that the subspaces X_1 and X_2 are nontrivial (i.e., $\{0\} \neq X_2 \neq \mathbb{C}^m$) and $(\{0\} \neq X_1 \neq \mathbb{C}^m)$ ensures that e-dichotomy dose not degenerate. If e-dichotomy degenerates to $X_1 = \mathbb{C}^m$ (i.e., $X_2 = \{0\}$), then the discrete evolution family has surely no s-stable-non-e-stable vectors, if it degenerates to $X_2 = \mathbb{C}^m$ (i.e., to $X_1 = \{0\}$), then we may have still discrete evolution family with no s-stable-non-e-stable vectors [10-12]. We will characterize s-stable vectors associated with discrete evolution family $\mathbf{U}$ on $\mathbb{C}^m$ by using exponential dichotomy. Let $\mathbf{U}$ be a discrete evolution family defined on $\mathbb{C}^m$. Set

$$\mathbf{W} = \{v \in \mathbb{C}^m \ : \ \lim_{(m-n)\to\infty} \|\mathbf{U}(m,n)v\| = 0\}$$

It is already verified that $\mathbf{W}$ is a subspace of $\mathbb{C}^m$, which is referred as the s-stable subspace of $\mathbf{U}$. Thus, consider the decomposition

$$\mathbb{C}^m = \mathbf{W} \oplus \mathbf{W}^{\perp}$$

It is clear that $\mathbf{W}$ is $\mathbf{U}$-invariant (so that $\mathbf{W}^{\perp}$ is $\mathbf{U}^*$-invariant, where $U^*(m,n)$ denotes the adjoint of $U(m,n)$. Moreover if invariant subspace $\mathbf{W}$ is reducing then the restriction of each $U(m,n)$ to $\mathbf{W}^{\perp}$ makes the discrete evolution family $\mathbf{U}|_{\mathbf{W}^{\perp}}$ non-s-stable in the sense that all nonzero vectors in it are non-s-stable. Note that $\mathbf{W}$ is maximal in the sense that every set of s-stable vectors for $\mathbf{U}$ is included in $\mathbf{W}$.

Recall that a projection onto a subspace means a projection whose range is precisely that subspace.

Theorem 2.1. Let $\mathbf{U}$ be a discrete evolution family defined on $\mathbb{C}^m$, and consider its s-stable subspace $\mathbf{W}$. Let S be the orthogonal projection onto the s-stable subspace $\mathbf{W}$. Suppose $\mathbf{U}$ is e-dichotomic and consider the e-dichotomic (orthogonal) projection P of Definition 2 with range X_1 and kernel X_2. Then the following assertions holds true:

(a) $X_1 \subseteq \mathbf{W}$. Equivalently, $P \le S$ (i.e., P is dominated by S)

(b) $\mathbf{W} = X_1 \oplus (\mathbf{W} \cap X_2)$

(c) $X_2 = \mathbf{W}^{\perp} \oplus (\mathbf{W} \cap X_2)$

(d) The decomposition of Definition 2 is referred to $\mathbb{C}^m = X_1 \oplus (\mathbf{W} \cap X_2) \oplus \mathbf{W}^{\perp}$

(e) The subspace X_1 is maximal (in the sense that there is no subspace of $\mathbb{C}^m$ on which $\mathbf{U}$ is e-stable).

(f) $X_1 \subset \mathbf{W}$ if and only if $\mathbf{W} \cap X_2 \neq \{0\}$. In this case $\mathbf{U}$ is e-stable on the nonzero subspace $\mathbf{W} \cap X_2$.

Proof. Suppose that $\mathbf{U}$ is e-dichotomic according to Definition 2. Then there is an orthogonal projection P such that $X_1 = range(P)$ is $\mathbf{U}$-reducing and, for some $M \ge 1$ and some $\alpha > 0$,

$$\|U(m,n)v\| \le Me^{-\alpha(m-n)}\|v\| \text{ for all } m \ge n \text{ and } v \in X_1$$

which implies that $\|U(m,n)v\| \to 0$ as $(m-n) \to \infty$ for every $v \in X_1$, and therefore

$$X_1 \subseteq \mathbf{W},$$

where $\mathbf{W}$ is the s-stable subspace of $\mathbf{U}$. In other words, X_1 is e-stable (for $\mathbf{U}$) and, as it happens with every e-stable subspace, X_1 is tautologically s-stable-via-e-stable. But the above inclusion is equivalent to following inequality,

$$P \le S$$

where S is the orthogonal projection onto $\mathbf{W}$ (i.e., *range* (S) = S (Cᵐ) = $\mathbf{W}$ and $\ker nel(S) = S^{-1}\{0\} = \mathbf{W}^{\perp}$. Moreover, $X_1 \subseteq \mathbf{W}$ yields the decomposition

$$\mathbf{W} = X_1 \oplus (\mathbf{W} \cap X_2),$$

where the subspace $(\mathbf{W} \cap X_2)$ is $\mathbf{U}$-invariant since $\mathbf{W}$ and X_2 are both $\mathbf{U}$-invariant (in fact X_2 and X_1 are both reducing). Since $(X_2 = X_1^{\perp})$, another consequence of the inclusion $X_1 \subseteq \mathbf{W}$ is $\mathbf{W}^{\perp} \subseteq X_2$ and therefore X_2 can be decomposed as

$$X_2 = \mathbf{W}^{\perp} \oplus (\mathbf{W} \cap X_2),$$

and so the decomposition of *Definition 2* is refined to

$$\mathbb{C}^m = X_1 \oplus (\mathbf{W} \cap X_2) \oplus \mathbf{W}^{\perp}$$

If X_1 is not maximal, then there exists a (homogeneously) e-stable subspace $X_1^{'}$ for $\mathbf{U}$ such that $X_1 \subset X_1^{'}$. Then $X_1^{'} \cap X_2 \neq \{0\}$. Take $0 \neq v \in X_1^{'} \cap X_2$. Since $v \in X_2$, for every $M \ge 1$ and every $\alpha > 0$ then we have

$$Me^{-\alpha(m-n)}\|v\| < \|U(m,n)v\|$$

But this contradicts the fact that the vector v is e-stable (which happens because $v \in X_1^{'}$). Therefore, X_1 is maximal. Finally observe that

$$X_1 \subset \mathbf{W}$$

if and only if $\mathbf{W} \cap X_2 \neq \{0\}$ (reason $X_1 = \mathbf{W}$ if and only if $\mathbf{W} \cap X_2 = \{0\}$ since $X_2 = X_1^{\perp}$). Thus the proper inclusion $X_1 \subset \mathbf{W}$ implies that $\mathbf{U}$ is s-stable-non-e-stable on $\mathbf{W} \cap X_2$ because $\mathbf{W} \cap X_2 \neq \{0\}$, $\mathbf{W}$ is s-stable and X_2 is non-e-stable (in the sense that all nonzero vectors in X_2 are non-e-stable).

Remark 3. *Theorem 2.1 (f)* says that, even though X_2 is a non-e-stable subspace (in the sense that all nonzero vectors in X_2 are non-e-stable) it can still contain s-stable (thus s-stable-non-e-stable) vectors whenever the inclusion $X_1 \subset \mathbf{W}$ is proper (Note that the proper inclusion clearly implies that the reducing e-stable subspace X_1 is not the whole space, but it may be zero, and $\mathbf{W}$ may be the whole space). Conversely, if there exist s-stable-non-e-stable vectors, then the inclusion $X_1 \subset \mathbf{W}$ is proper (since, in this case the e-stable X_1 does not contain the s-stable-non-e-stable vectors of $\mathbf{W}$).

Corollary 1. Take a discrete evolution family $\mathbf{U}$ defined on $\mathbb{C}^m$. Let P be the e-dichotomic projection of $\mathbf{U}$ (with range X_1 and kernel X_2) and let S be an orthogonal projection onto the s-stable subspace $\mathbf{W}$ of the $\mathbf{U}$. Then the following assertions are pairwise equivalent

(a) $P = S$

(b) $\mathbf{W} \cap X_2 = \{0\}$

(c) $\mathbf{W} = X_1$

(d) $\mathbf{W} \subseteq X_1$.

Proof. The definitions of P and S ensures that $(a) \Leftrightarrow (c)$ (since the orthogonal projection onto a subspace is unique). Theorem 2.1 (b) ensures that $(c) \Leftrightarrow (b)$. Theorem 2.1 (a) ensures that $(c) \Leftrightarrow (b)$.

Observe that assertions (c) in Corollary 1 (and so any of the above equivalent assertions) implies that $\mathbf{W}$ reduces $\mathbf{U}$ because X_1 reduces $\mathbf{U}$ by *Definition 2*. Moreover, Corollary 1 (d) means that the s-stable subspace $\mathbf{W}$ consists only of e-stable vectors (and so $\mathbf{W}$ is an s-stable-via-e-stable subspace if any of the above equivalent assertions holds true).

Also note from Theorem 2.1 (b) that s-stable-via-e-stable vectors (i.e., X_1) and s-stable-non-e-stable vectors (i.e., $\mathbf{W} \cap X_2$) are complementary orthogonal subspaces of the s-stable space $\mathbf{W}$ once the discrete evolution family $\mathbf{U}$ is e-dichotomic.

3. CONCLUDING REMARK

Suppose the discrete evolution family $\mathbf{U}$ is e-dichotomic and consider the

e-stable subspace X_1 of the e-dichotomic projection P of *Definition 2*, $range(P) = X_1$, since the orthogonal projection onto a subspace is unique, and P commutes with each $\mathbf{U}$ if and only if $range(P)$ and $\ker nel\,(P)$ are reducing subspaces for each $\mathbf{U}$ (and so $\mathbf{U}$ is e-stable on $X_1 = range(P)$ and non-s-stable on $\ker nel\,(P)$), and since X_1 is maximal, we restate *Definition 2* as follows:
*Let X_1 be a maximal e-stable subspace of **U**. Consider the orthogonal projection P onto X_1 so that **U** is e-stable on X_1 and non-s-stable on $\ker nel\,(P)$. If P commutes with each $U(m,n)$, than P is an e-dichotomic projection for **U**, which is said to be e-dichotomic.*

If we replace exponential stability with strong stability in *Definition 2* we come across with s-dichotomic discrete evolution family. Precisely, if $\mathbf{U}$ be a discrete evolution family defined on $\mathbb{C}^m$, and if there exists an orthogonal projection P that commutes with each $U(m,n)$ such that $\mathbf{U}$ is s-stable on its range and non-s-stable on its kernel, then we say that the projection is s-dichotomic for $\mathbf{U}$ and the discrete evolution family is s-dichotomic.

Definition 4. Let $\mathbf{U}$ be a discrete evolution family defined on $\mathbb{C}^m$. Take an orthogonal projection S so that $range(S) = S(\mathbb{C}^m)$ and $\ker nel\,(S) = S^{-1}\{0\}$ are complementary orthogonal subspaces of $\mathbb{C}^m$, that is,

$$\mathbb{C}^m = range(S) \oplus \ker nel\,(S)$$

with $range(S)^{\perp} = \ker nel\,(S)$. If

(a) S commutes with each $\mathbf{U}$ (i.e., $U(m,n)S = SU(m,n)$ or equivalently, $range(S)$ and $\ker nel\,(S)$ are reducing subspaces for $\mathbf{U}$), and

(b) $\mathbf{U}$ is s-stable on $range(S)$ and non-s-stable on $\ker nel\,(S)$ (in the sense that all nonzero vectors in $\ker nel\,(S)$ are non-s-stable).

Then we say that S is a s-dichotomic projection for $\mathbf{U}$ and $\mathbf{U}$ is said to be s-dichotomic.
Consider the s-stable subspace

$$\mathbf{W} = \{x \in \mathbb{C}^m : \lim_{(m-n)\to\infty} \|U(m,n)v\| = 0\}$$

of $\mathbf{U}$. It is clear that the $range(S) \subseteq \mathbf{W}$. On the other hand, suppose that there exists a vector $b = (u,v) \in \mathbb{C}^m$, with $u \in range(S)$ and $v \in \ker nel\,(S)$, such that v lies in $\mathbf{W}/range(S)$. Then

$$\lim_{(m-n)\to\infty} U(m,n)b = U(m,n)uU(m,n)v = (0,0),$$

since $b \in \mathbf{W}$, and so

$$\lim_{(m-n)\to\infty} U(m,n)b = 0$$

which implies that $b = 0$. But b lies in $range(S)$, which is contradiction, therefore $\mathbf{W} \subseteq range(S)$ and so

$$range(S) = \mathbf{W}.$$

Thus by uniqueness of the orthogonal projection onto a subspace, S is in fact the orthogonal projection onto the s-stable subspace of $\mathbf{W}$ of $\mathbf{U}$, which means that this actually is the same orthogonal projection S of Theorem 2.1. Since S commutes with each $U(m,n)$ if and only if $range(S)$ and $\ker nel\,(S)$ are reducing subspaces for $\mathbf{U}$ (which implies that $\mathbf{U}$ must be s-stable on $\mathbf{W} = range(S)$ and non-s-stable on $\ker nel\,(S)$), and since $\mathbf{W}$ is maximal, we may restate *Definition 2* as follows.

*Consider the orthogonal projection S onto **W** so that **U** is e-stable on the **W** and non-e-stable on $\ker nel\,(S)$. If S commutes with each $U(m,n)$, then S is an s-dichotomic projection for **U**, which is said to be s-dichotomic.*

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