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REVIEW ARTICLE

NUMERICAL SOLUTION OF FRACTIONAL BOUNDARY VALUE PROBLEMS BY USING CHEBYSHEV WAVELET METHOD

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ABSTRACT

In this paper Chebyshev Wavelets Method (CWM) is applied to obtain the numerical solutions of fractional fourth, sixth and eighth order linear and nonlinear boundary value problems. The solutions of the fractional order problems are shown to be convergent to the integer order solution of that problem. The computational work is done successfully with the help of the proposed algorithm and hence this algorithm can be extended to other physical problems. High level of accuracy is obtained by the present method.

KEYWORDS

Chebyshev Wavelets Method, Fractional Boundary Value Problems, Linear and Nonlinear Problems, Exact Solutions.

1. INTRODUCTION

Fractional calculus has a number of applications in science and technology [1-3]. The study of fractional calculus initially started by Gemant and Scot-Blair, they were the first, who proposed a fractional derivative model for Viscoelasticity and anomalous strain and stress [4,5]. Fractional calculus is applied to many other physical phenomena such as frequency dependent damping behavior of many viscoelastic materials, oscillation of earth quakes, fluid-dynamic traffic, control theory and signal processing [6-10].

The different numerical methods are developed for the numerical solutions of different problems in various branches of sciences and engineering. In this regard, a relatively new numerical technique based on Wavelets is being developed. The most common Wavelets schemes are Haar Wavelets (HW), Harmonic Wavelets of successive approximation, Legendre Wavelets and CWM [11-20]. In the present research work, the CWM is fully compatible with the complexity of the problems and has shown extremely accurate results, especially in case of fractional linear and nonlinear boundary problems of fourth, sixth and eighth order [21-27]. Some other well -known methods for the solution of fractional differential equations are given in [28-33].

2.DEFINITIONS AND PRELIMINARIES CONCEPTS

In this section, we give some important definitions and preliminaries concepts about fractional calculus theory, which is the foundation for this paper [28].

Definition 2.1 The Riemann-Liouville fractional integral operator I^{β} of order β on the usual Lebesgue space $L_1[a,b]$ is given by

$$I^{\beta}\left(f(t)\right)=\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s)ds, \beta>0,$$

$$I^{0}\left(f(t)\right)=f(t).$$

This operator has the following properties:

(i) $I^{\beta+\gamma}=I^{\beta}I^{\gamma}$,

(ii) $I^{\beta}L^{\gamma}=I^{\gamma}I^{\beta}$,

(iii) $\left(I^{\beta}(t-a)\right)^{\nu}=\dfrac{\Gamma(\nu+1)}{\Gamma(\beta+\nu+1)}(t-a)^{a+\nu}$,

where $L_1[a,b]$, β, $\gamma\geq 0$ and $\nu>-1$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\beta>0$ is defined as $D^{\beta}\left(f(t)\right)=(\frac{d}{dt})^{n}(I^{n-\beta})\,f(t)$, $n-1<\beta\leq n$, where n is an integer. The derivative of this type has certain disadvantages dealing with the fractional differential equation. There after Caputo proposed a modified fractional differential operator.

Definition 2.3 Caputo proposed fractional differential operator is given by

$$D^{\beta}\left(f(t)\right)=\frac{1}{\Gamma(n-\beta)}\int_0^t (t-s)^{n-\beta-1} f^{(n)}(s)\,ds, n-1<\beta\leq n,$$

where $t>0$, n is any integer.

The Caputo operator has the following two properties:

(i). $(D^{\beta}I^{\beta})\,f(t)=f(t)$,

(ii). $(D^{\beta}I^{\beta})\,f(t)=f(t)-(t+a)^{n}=\sum_{k=0}^{n} f^{(k)}(0^{+})\dfrac{(t-a)^{k}}{k!}$, $t>0$.

3.CHEBYSHEV WAVELET METHOD (CWM)

Wavelets generally constitute a family of functions constructed from dilation and translation of single function $\Psi(x)$ which is called the mother wavelet. For different continuous parameters a and b of dilation and translation respectively, we obtain the following family of continuous wavelet [15].

$$\Psi_{a,b}(x)=|a|^{\frac{1}{2}}\Psi\left(\frac{x-b}{a}\right),\ a,b\in R,\ a\neq 0\ .$$

Similarly, if we restrict the parameter to integer values, that is if $a=a_0^{-k}$, $b=nb_0a_0^{-k}$, $a_0>1$, $b_0>0$, we have the following family of discrete wavelets:

$$\Psi_{k,n}(x)=|a|^{\frac{k}{2}}\Psi(a_0^kx-nb_0)\ ,\ k,n\in Z\ ,$$

where $\Psi_{k,n}$ form a wavelet basis for $L^2(R)$.

For particular values of $a_0=2$ and $b_0=1$, $\Psi_{k,n}(x)$ form an orthogonal basis. The second CW $\Psi_{n,m}(x)=\Psi(k,n,m,x)$ consist of four parameters namely $n=1,2,3,\ldots,2^{k-1}$, where k is assumed any positive integer, m is the degree of the second Chebyshev polynomials, and the normalized time. This CW family is defined on the interval $[0,1)$ as below

$$\Psi_{n,m}(x)=\begin{cases}2^{\frac{1}{2}}T_m(2^kx-2n+1), & \frac{n-1}{2^{k-1}}\le x<\frac{n}{2^{k-1}}\\ 0, & \text{otherwise,}\end{cases} \qquad (3.1)$$

where $T_m(x)=\sqrt{\frac{2}{\pi}}T_m(x)$, $m=0,1,2,\ldots,M-1$.

Here $T_m(x)$ are the second Chebyshev polynomials of degree m with respect to the weight function $w(x)=\sqrt{1-x^2}$ on the interval $[-1,1]$ and satisfy the following recursive formula:

$$T_0(x)=1,$$
$$T_1(x)=2x,$$
$$T_{m+1}(x)=2xT_m(x)-T_{m-1}(x),\ m=1,2,3,\ldots$$

4.CHEBYSHEV WAVELET METHOD (CWM)

In this section, we consider the following fractional boundary value problems

$$D^{\alpha}y(x)=g(x)+f(y)\ ,0<x<b\ ,4<\alpha\le 5 \qquad (4.1)$$

with the boundary conditions

$$y(0)=\alpha_0,\ y'(0)=\alpha_1,\ y''(0)=\alpha_2,\ y(b)=\beta_0,\ y'(b)=\beta_1.$$

The solution of Equation (4.1) can be expressed as a CW series of the form

$$y(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{n,m}\Psi_{n,m}(x)\ ,$$

where $\Psi_{n,m}(x)$ is given in Equation (3.1). We approximate $y(x)$ by the truncated series

$$y_{k,M}(x)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\Psi_{n,m}(x)\cdot \qquad (4.2)$$

To determine $2^{k-1}M$ coefficients, we will use $2^{k-1}M$ conditions. For this, five conditions are given by the following boundary conditions:

$$\left.\begin{aligned}
u_{k,M}(0)&=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\Psi_{n,m}(0)=\alpha_0,\\
\frac{d}{dx}u_{k,M}(0)&=\frac{d}{dx}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\Psi_{n,m}(0)=\alpha_1,\\
\frac{d^2}{dx^2}u_{k,M}(0)&=\frac{d^2}{dx^2}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\Psi_{n,m}(0)=\alpha_2,\\
u_{k,M}(b)&=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\Psi_{n,m}(b)=\beta_0,\\
\frac{d}{dx}u_{k,M}(b)&=\frac{d}{dx}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\Psi_{n,m}(b)=\beta_1\cdot
\end{aligned}\right\} \qquad (4.3)$$

Now using these five boundary conditions, we need $2^{k-1}M-5$ extra conditions to calculate the unknown's coefficients $c_{n,m}$. These conditions can be obtained by putting Equation (4.2) in Equation (4.1) as

$$\frac{d^{\alpha}}{dx^{\alpha}}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-5}c_{n,m}\Psi_{n,m}(x)=g(x)+f\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-5}c_{n,m}\Psi_{n,m}(x)\right)\cdot \qquad (4.4)$$

Assume that Equation (4.4) is exact at $2^{k-1}M-5$ points which we consider as x_i then

$$\begin{aligned}&\frac{d^{\alpha}}{dx^{\alpha}}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-5}c_{n,m}\Psi_{n,m}(x_i)\\ &=g(x_i)+f\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-5}c_{n,m}\Psi_{n,m}(x_i)\right)\cdot\end{aligned} \qquad (4.5)$$

For the choice of x_i, the points are the zeros of the shifted Chebyshev polynomials of degree $2^{k-1}M-5$ in the interval $[0,1]$ that is

$$x_i=\frac{s_i+1}{2},\ \text{where}\ s_i=\cos\left(\frac{(2i-1)\pi}{2^{k-1}M-1}\right),\ i=1,2,\ldots 2^{k-1}M-5\ .$$

Equation (4.3) and Equation (4.5) gives $2^{k-1}M$ linear system or the nonlinear equations as the case may be occur for the problem. Same procedure can be extended to fractional differential equations of order sixth and eight.

5.METHOD IMPLEMENTATION

Problem 1. Consider the following fractional nonlinear boundary value problem of fourth order

$$\frac{d^{\alpha}}{dx^{\alpha}}y=y^2-x^{10}+4x^9-4x^8-4x^7+ \\ 8x^6-4x^4+120x-48,$$

where $3<\alpha\le 4$,

with the following boundary conditions

$$y(0)=0,\ y(1)=1,\ y'(0)=0,\ y'(1)=1\cdot$$

The analytical solution for this problem is

$$y(x)=x^5-2x^4+2x^2\ .$$

Table 1: Numerical results of Problem 1

x	y_{exact}	y_4	$y_{3.75}$	$y_{3.50}$	$y_{3.25}$	Error y_4	OHAM
0.0	0.00000	0.00000000	0.00000000	0.00000000	0.00000000	$1.15E-20$	0.0
0.1	0.01981	0.01981000	0.02194814	0.02939451	0.02365759	$1.20E-20$	7.2710−8
0.2	0.07712	0.07712000	0.08454303	0.11030080	0.09047249	$6.00E-21$	2.4510−7
0.3	0.16623	0.16623000	0.18006704	0.22776959	0.19109495	$0.00E+00$	4.4810−7
0.4	0.27904	0.27904000	0.29835773	0.36436086	0.31370133	$2.00E-20$	6.1810−7
0.5	0.40625	0.40625000	0.42842493	0.50334709	0.44596039	$2.00E-20$	6.9910−7
0.6	0.53856	0.53856000	0.55998891	0.63142968	0.57684394	$4.00E-20$	6.6110−7
0.7	0.66787	0.66787000	0.68493978	0.74096872	0.69828207	$2.00E-20$	5.0710−7
0.8	0.78848	0.78848000	0.79871794	0.83172590	0.80666263	$1.10E-19$	2.8710−7
0.9	0.89829	0.89829000	0.90161568	0.91212073	0.90417530	$1.60E-19$	2.8710−7
1.0	1.00000	1.00000000	1.00000000	1.00000000	1.00000000	$1.00E-19$	0.0

Table 1, shows the solutions given by Chebyshev method when $M=8$ and $k=1$. The analysis of the absolute error between exact solution and approximate solution is done successfully. The numerical solutions obtained by CWM are compared with Optimal Homotopy Asymptotic Method (OHAM). In the table y_{exact} represent the exact solution for Problem 1. The approximate solutions are obtained by Chebyshev Wavelet Method for different order α, that is for $\alpha=3.25$, $\alpha=3.50$, $\alpha=3.75$ and $\alpha=4$. The Error y_4 and OHAM, shows the respective errors given by the CWM and Optimal Homotopy Asymptotic Method.

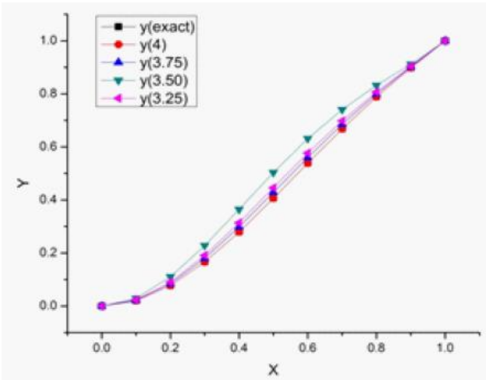

Figure 1: The solution graph, by Chebyshev method for different fractional order α

Problem 2. Given fractional order BVP

$$\frac{d^{\alpha}}{dx^{\alpha}}y=(1+c)\frac{d^{4}}{dx^{4}}y-c\frac{d^{2}}{dx^{2}}y+cx\,,\; 5<\alpha\leq 6,$$

with the following boundary conditions,

$$y(0)=1,$$

$$y(1)=\frac{7}{6}+\sinh(1),$$

$$\frac{d}{dx}y(0)=1,$$

$$\frac{d}{dx}y(1)=1+\cosh(1),$$

$$\frac{d^2}{dx^2}y(0)=0,$$

$$\frac{d^2}{dx^2}y(1)=1+\sinh(1).$$

The exact solution is $y(x)=1+\frac{1}{6}x^3+\sinh(x)$.

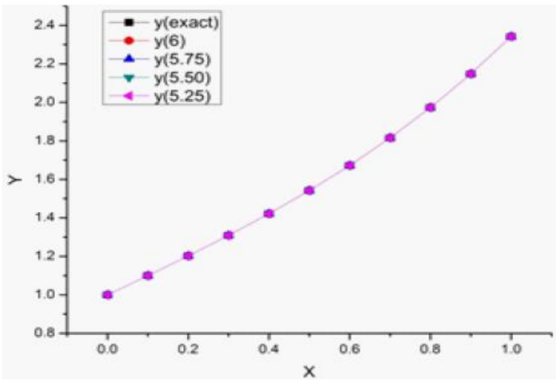

Figure 2: The Chebyshev solutions graph for the fractional differential equations of different order α.

Table 2: The numerical results for Problem 2 for different fractional order α

x	y_{exact}	y_6	$y_{5.75}$	$y_{5.50}$	$y_{5.25}$	Error y_6	OHAM
0.0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	7.0010-20	0.000000
0.1	1.100333416	1.100333416	1.100333270	1.10033311	1.10033296	1.0710-12	5.010-12
0.2	1.202669335	1.202669335	1.202668509	1.20266763	1.20266679	8.2310-12	3.4210-11
0.3	1.309020293	1.309020293	1.309018412	1.30901642	1.30901451	2.1810-11	8.9210-11
0.4	1.421418992	1.421418992	1.421416164	1.42141319	1.42141028	3.3310-11	1.4510-10
0.5	1.541928638	1.541928638	1.541925420	1.54192204	1.54191870	3.4010-11	1.7110-10
0.6	1.672653582	1.672653582	1.672650714	1.67264770	1.67264469	2.4110-11	1.4810-10
0.7	1.815750368	1.815750368	1.815748433	1.81574640	1.81574435	1.1310-11	9.2410-11
0.8	1.973439315	1.973439315	1.973438453	1.97343754	1.97343662	3.0710-12	3.5610-11
0.9	2.148016725	2.148016725	2.148016571	2.14801640	2.14801624	2.9310-13	5.1710-12
1.0	2.341867860	2.341867860	2.341867860	2.34186786	2.34186786	3.0010-19	2.0010^{-17}

In table 2, the Chebyshev Wavelet Methods (CWM) solutions for Problem 2 are given for $M=12$, $k=1$. The Chebyshev Wavelet Methods solutions are given for different fractional orders $\alpha=5.25$, $\alpha=5.50$, $\alpha=5.75$ and for $\alpha=6$. The errors obtained by CWM are compared with error obtained by Optimal Homotopy Asymptotic Method (OHAM). It can be observed from the table that the present method has better accuracy than OHAM.

Problem 3*.* The fractional order differential equation is given by

$$\frac{d^{\alpha}}{dx^{\alpha}}y(x)-e^{-x}y^2(x)\,,\; 0<x<1,\; 7<\alpha\leq 8,$$

with the following boundary conditions

$$\frac{d^{i}}{dx^{i}}y(0)=1,\; i=0,1,2...,7.$$

The analytical solution for this problem is

$$y(x)=e^{x}.$$

Table 3: The numerical results for Problem 3 for different fractional order α.

x	y_{exact}	y_8	$y_{7.75}$	$y_{7.50}$	$y_{7.25}$	Error y_8	OHAM
0.0	1.00	1.00	1.0000	1.000000000	1.000000000	6.00E - 20	0.0
0.1	1.105170918	1.105170918	1.105170918	1.105170918	1.105170918	3.36E - 15	1.4510 - 7
0.2	1.221402758	1.221402758	1.221402758	1.221402758	1.221402758	8.04E - 13	1.020 - 7
0.3	1.349858807	1.349858807	1.349858808	1.349858810	1.349858810	1.92E - 11	2.0410 - 7
0.4	1.491824697	1.491824697	1.491824706	1.491824721	1.491824721	1.80E - 10	1.4910 - 7
0.5	1.648721270	1.648721269	1.648721323	1.648721408	1.648721408	1.01E - 09	1.0610 - 7
0.6	1.822118800	1.822118796	1.822119023	1.822119375	1.822119375	4.08E - 09	1.4410 - 7
0.7	2.013752707	2.013752694	2.013753456	2.013754624	2.013754624	1.32E - 08	1.0510 - 7
0.8	2.225540928	2.225540892	2.225543063	2.225546350	2.225546350	3.63E - 08	1.4510 - 7
0.9	2.459603111	2.459603023	2.459608478	2.459616642	2.459616642	8.81E - 08	1.6210 - 7
1	2.718281828	2.718281634	2.718294048	2.718312421	2.718312421	1.94E - 07	1.6510 - 7

In Table 3, the numerical solutions obtained by CWM are given for $M=13$ and $k=1$. The solutions $y_{7.25}$, $y_{7.50}$, $y_{7.75}$ and y_8 shows the solutions at fractional orders $\alpha=7.25$, $\alpha=7.50$, $\alpha=7.75$ and $\alpha=8$ respectively. The solutions are calculated by the present method for different fractional orders particularly for $\alpha=7.25$, $\alpha=7.50$,

$\alpha = 7.75$ and $\alpha = 8$. The exact solution is represented by y_{exact}. The solutions The error associated with the present method and that of Optimal Homotopy Asymptotic Method (OHAM) method is compared. The table shows that the accuracy of the current method is higher than Optimal Homotopy Asymptotic Method (OHAM).

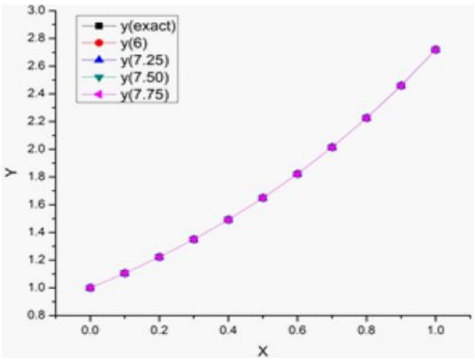

Figure 3: The Chebyshev solutions graph for the fractional differential equations given in Problem 3 of different order α .

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