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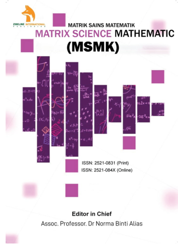

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RESEARCH ARTICLE

CUBIC B-SPLINE SOLUTION FOR A SECOND-ORDER SINGULAR LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

A new approach implementing the cubic B-spline technique is introduced for the numerical solution of a class of singular partial differential equation. Properties of these cubic B-spline functions are first presented. These properties are then used to reduce singular partial differential equation to systems of linear algebraic equations. Numerical examples illustrate the pertinent features of the method and its applicability to a large variety of problems. The examples include regular and singular problems.

KEYWORDS

B-spline, singular Numerical solutions, Error analysis.

1. INTRODUCTION

The singular boundary-value problem (BVP) has arises in many branches of applied mathematics and physics such as gas dynamics, nuclear physics, chemical reactions, atomic structures, atomic calculations, study of positive radial solutions of non-linear elliptic equations etc. Therefore, it has been studied extensively in recent years. There are numerous problems from chemistry, physics and mechanics, which are described by singular value problems (Ciarlet et al., 1970; Russell and Shampine, 1975; Garner and Shivaji, 1990; Mohanty and Swarn, 2006; Jicheng and Xiaonan, 2004). There is a growing literature concerning the numerical solution of singular two-point boundary value problems. For finite difference methods, for collocation methods,for the Ritz-Galerkin method, for the projection methods, for finite element methods, for Cubic spline, for Chebyshev polynomial and B-spline, for an iterative method and for integral methods (Jamet, 1970; Doedel and Reddien, 1984; Reddien and Schumaker, 1976; Ruibin et al., 1997; Jesperson, 1978; Reddien, 1973; Schreiber, 1980; Kanth and Bhattacharya, 2006; Kadalbajoo and Aggarwal, 2005; El-Gebeily and Attili, 1999; Evans, 1999; Salah, 2002). Recently, a group resershers used B-spline method to solve singular boundary value problems (El-Gamel et al., 2018). Very recently, El-Gamel and Adel developed Bernoulli collocation technique for singular boundary value problems in one space dimension (El-Gamel and Adel, 2021).

In this paper, we discuss the use of cubic B-spline for solving the equation which take the following form:

$$u_{xx}(x,y)+\left(\frac{a(x)}{x}\right)u_{xy}(x,y)+u_{yy}(x,y)=f(x,y),\qquad x\in[0,1],\quad y\in[0,1], \tag{1}$$

subject to the boundary conditions

$$u_x(0,y)=0,\quad u_x(1,y)=0, \tag{2}$$

$$u(x,0)=0,\quad u(x,1)=0, \tag{3}$$

where: $a(x)$ is a given function.

Racently, a lot of attention has been devoted to the study of B-spline method to investigate various scientific models. The efficiency of the method has been formally proved by many researchers (Caglar and Caglar, 2008; Chawla et al., 1986; Chawla and Shivkumar, 1987; Chawla and Subramanian, 1988; Mittal and Jain, 2012; Taliaferro, 1979; Usmani and Warsi, 1980). Additional references can be found in the recent papers (El-Gamel and El-Shenawy, 2014; El-Gamel et al., 2014; El-Gamel and El-Shamy, 2016). Spline functions have some attractive properties. Due to the being piecewise polynomial, they can be integrated and differentiated easily. Since they have compact support, numerical methods in which spline functions are used as a basis function lead to matrix systems including band matrices. Such systems have solution algorithms with low computational cost. Numerical examples including regular and singular problems are considered. On the basis of these examples, it will be shown that B-spline method gives slightly better results for the tested examples. The organization of the paper is as follows: in Sect. 2, we describe the basic formulation in terms of B-splines functions required for our subsequent development. Error analysis for the cubic B-spline is presented in Sect. 3. in Sect. 4, we introduce B-splines method and show how the method is used to solve linear singular and nonsingular second-order partial differential equation. Some numerical examples are presented in Sect. 5. Finally, Sect. 6 provides conclusions of the study.

2. THE CUBIC B-SPLINE

Consider equally spaced knots of a partition $\Omega_n : 0 = x_0 < x_1 < \cdots < x_n = 1$ with step $h=\frac{1}{n}$, and $x_i = ih$, for $i = 0,1,2,\cdots,n$. Let $S_3(\Omega_n)$ is the space of continuously differentiable, piecewise, 3-degree polynomials on Ω_n. That

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is $S_3(\Omega_n)$ is the space of the 3-degree B-spline on Ω_n. The i^{th} B-spline basis, $B_{i,3}(x)$, of degree 3, $i \in Z$ is defined recursively as follows (de Boor, 1978):

$$B_{i,0}(x) = \begin{cases} 1, & x < x_{i+1} \\ 0, & otherwise \end{cases}$$

and

$$B_{i,3}(x) = \frac{x - x_i}{x_{i+3} - x_i} B_{i,2}(x) + \frac{x_{i+4} - x}{x_{i+4} - x_{i+1}} B_{i+1,2}(x).$$

The above relations are usually referred to as the Cox-de Boor recursion formula, such that $B_{i,3}$ are compactly supported, $\sum_{i=-\infty}^{\infty} B_{i,3}(x) = 1$, for all $x \in R$ and $B_{i,3} \geq 0$. Table 1 exhibits the coefficients of cubic B-spline and their derivatives, at the knots $x_i, i = 0,1,2,\cdots,n$.

Table 1: The coefficients of cubic B-spline and its derivatives at the knots points

x	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}
B_i	0	1	4	1	0
hB_i'	0	3	0	-3	0
h^2B_i''	0	6	-12	6	0

3. Error Analysis For The Cubic B-spline

The set of B-spline $B_k(x), k = -3,-2,\cdots,n-1$, form a basis for $S_3(\Omega_n)$. Thus we can define our cubic B-spline basis in the form:

$$S(x) = \sum_{k=-3}^{n-1} c_k B_k(x), \qquad x \in [0,1]. \tag{4}$$

Denote by $S_i = S(x_i)$, and $S_i^{(p)} = S^{(p)}(x_i)$ for all p. For any function g evaluated at the nodes x_i, we define Γ by:

$$\Gamma g_i = g_{i-3} + 4g_{i-2} + g_{i-1}, \tag{5}$$

then the following recursive relations can be reduced :

$$\Gamma S_i' = \frac{3}{h}\left[-S_{i-3} + S_{i-1}\right], \tag{6}$$

$$\Gamma S_i'' = \frac{6}{h^2}\left[S_{i-3} - 2S_{i-2} + S_{i-1}\right]. \tag{7}$$

Lemma 3.1. (El-Gamel and El-Shamy, 2016) *Let S be the cubic-spline interpolation of $u \in C^6[0,1]$ defined by equation (4), then the following relations hold for* $i = 0,1,2,\cdots,n$

$$\Gamma S_i' = 6u_{i-2}' + h^2 u_{i-2}^{(3)} + \frac{h^4}{20} u_{i-2}^{(5)} + O\left(h^6\right), \tag{8}$$

and

$$\Gamma S_i'' = 6u_{i-2}'' + \frac{h^2}{2} u_{i-2}^{(4)} + \frac{h^4}{60} u_{i-2}^{(6)} + O\left(h^6\right), \tag{9}$$

Proof. By substituting with Taylor series expansions of u_{i-3}, and u_{i-1} about x_{i-2} in equations (6) and (7), the above relations are obtained.

Theorem 3.1 If $u \in C^6[0,1]$ and S is the cubic B-spline of u defined by (4), then we have (Agarwal and Chow, 1986)

$$S_i'' = u_i'' - \frac{h^2}{12} u_i^{(4)} + \frac{h^4}{360} u_i^{(6)} + O\left(h^6\right), \tag{10}$$

$$S_i' = u_i' - \frac{h^4}{180} u_i^{(5)} + O\left(h^6\right). \tag{11}$$

Proof. Consider any function $g \in C^6[0,1]$, Γg is defined as shown in equation (5). It can be easily proved that (Goh and Abd Majid, 2011).

$$\Gamma g_i = 6g_{i-2} + h^2 g_{i-2}'' + \frac{h^4}{12} g_{i-2}^{(4)} + \frac{h^6}{360} g_{i-2}^{(6)} + O\left(h^8\right). \tag{12}$$

If we assume that

$$g_i = u_i'' - \frac{h^2}{12} u_i^{(4)} + \frac{h^4}{360} u_i^{(6)},$$

then using equation (12) yields:

$$\Gamma g_i = 6u_{i-2}'' + \frac{h^2}{2} u_{i-2}^{(4)} + \frac{h^4}{60} u_{i-2}^{(6)} + O\left(h^6\right). \tag{13}$$

Let

$$d_{2,i} = \left[S_i'' - \left[u_i'' - \frac{h^2}{12} u_i^{(4)} + \frac{h^4}{360} u_i^{(6)}\right]\right],$$

subtracting equation (13) from equation (9) yields:

$$\Gamma d_{2,i} = O\left(h^6 \left\| u^{(6)} \right\|\right). \tag{14}$$

If we assume that

$$g_i = u_i' - \frac{h^4}{180} u_i^{(5)},$$

then using equation (12) yields:

$$\Gamma g_i = 6u_{i-2}' + h^2 u_{i-2}^{(3)} + \frac{h^4}{20} u_{i-2}^{(5)} + O\left(h^6\right). \tag{15}$$

Let

$$d_{1,i} = \left[S_i' - \left[6u_i' + h^2 u_i^{(3)} + \frac{h^4}{20} u_i^{(5)} + O\left(h^6\right)\right]\right],$$

subtracting equation (15) from equation (8) yields:

$$\Gamma d_{1,i} = O\left(h^6 \left\| u^{(8)} \right\|\right). \tag{16}$$

Since the coefficient matrices of the systems of equations (14) and (16) are diagonally dominant and their inverses are bounded then $d_{k,i} = O(h^6), k = 1,2, i = 0,1,2,\cdots,n$, hence; the proof of relations of the above theorem is completed.

Theorem 3.2 The truncation error of equation (1) is given by

$$u_{yy}(x,y_j) + O\left((\Delta y)^2\right) + \frac{a(x)}{x}\left[\frac{d}{dx} u_y(x,y_j) + \frac{(\Delta y)^2}{3!} \frac{d}{dx} u_{yyy}(x,y_j)\right]$$
$$+ O\left((\Delta y)^4\right) = f(x,y_j) - u_{xx}(x,y_j). \tag{17}$$

Proof. Since the finite difference formulas of $u_y(x,y_j)$ and $u_{yy}(x,y_j)$ are known as (Goh and Abd Majid, 2011).

$$u_y(x,y_j) = \frac{u(x,y_{j+1}) - u(x,y_{j-1})}{2(\Delta y)}, \tag{18}$$

and

$$u_{yy}(x,y_j) = \frac{u(x,y_{j+1}) - 2u(x,y_j) + u(x,y_{j-1})}{(\Delta y)^2}, \tag{19}$$

equation (1) will take the form as bellow

$$\left[\frac{u_{j+1}(x) - 2u_j(x) + u_{j-1}(x)}{(\Delta y)^2}\right] + \frac{a(x)}{x} \frac{d}{dx}\left[\frac{u_{j+1}(x) - u_{j-1}(x)}{2(\Delta y)}\right] = f(x,y_j) - u_{xx}(x,y_j),$$

and then applying Taylor series expansions for u_{j+1} and u_{j-1}, the proof required is completed.

Theorem 3.3 If the cubic B-spline method is combined with the finite difference approximation, then the truncation error for equation (1) will be

$O\left((\Delta y)^2 + h^2\right)$.

Proof. By applying equation (11) for the terms $u_y(x, y_j)$ and $u_{yyy}(x, y_j)$, such that $S'_j = \frac{d}{dx}u(x, y_j)$, we can write

$$\frac{d}{dx}u_y(x, y_j) = \frac{d}{dx}u_y(x, y_j) - \frac{h^4}{180}\frac{d^5}{dx^5}u_y(x, y_j) + O\left(h^6\right), \tag{20}$$

and

$$\frac{d}{dx}u_{yyy}(x, y_j) = \frac{d}{dx}u_{yyy}(x, y_j) - \frac{h^4}{180}\frac{d^5}{dx^5}u_{yyy}(x, y_j) + O\left(h^6\right), \tag{21}$$

using equations (10), (20) and (21), then equation (17) will take the form

$$\begin{aligned}
&u_{yy}(x, y_j) + O\left((\Delta y)^2\right) + \frac{a(x)}{x}\left(u_{xy}(x, y_j) - \frac{h^4}{180}\frac{d^5}{dx^5}u_y(x, y_j) + O\left(h^6\right)\right) \\
&+ \frac{a(x)(\Delta y)^2}{(3!)x}\left[\frac{d}{dx}u_{yyy}(x, y_j) - \frac{h^4}{180}\frac{d^5}{dx^5}u_{yyy}(x, y_j) + O\left(h^6\right)\right] + O\left((\Delta y)^4\right) \\
&= f(x, y_j) - \left[u_{xx}(x, y_j) - \frac{h^2}{12}\frac{d^4}{dx^4}u(x, y_j) + \frac{h^4}{360}\frac{d^6}{dx^6}u(x, y_j) + O\left(h^6\right)\right],
\end{aligned}$$

if we define the truncation error e as

$$e = u_{xx}(x, y_j) + \frac{a(x)}{x}u_{xy}(x, y_j) + u_{yy}(x, y_j) - f(x, y_j),$$

then

$$\begin{aligned}
e &= -\frac{(\Delta y)^2}{3!}\left(\frac{a(x)}{x}\right)\frac{d}{dx}u_{yyy}(x, y_j) \\
&+ \frac{h^2}{12}\frac{d^4}{dx^4}u(x, y_j) + \frac{h^4}{180}\left(\frac{a(x)}{x}\right)\frac{d^5}{dx^5}\left[u_y(x, y_j) + \frac{(\Delta y)^2}{3!}u_{yyy}(x, y_j)\right] \\
&- \frac{h^4}{360}\frac{d^6}{dx^6}u(x, y_j) + O\left((\Delta y)^2\right) + O\left((\Delta y)^4\right) + O\left(h^6\right),
\end{aligned}$$

hence; the proof is completed.

4. Scheme of The Method

Let any point in the space can be written as (x, y) such that: $x = x_0, x_1, \cdots, x_n$ be $n+1$ grid points in the interval $[0,1]$, and $y = y_1, y_2, \cdots, y_m$ be m grid points in the interval $[0,1]$. Our attention is focused on transforming the partial differential equation shown in equation (1) to ordinary differential equation. By using the finite difference formulas of both $u_y(x, y_j)$ and $u_{yy}(x, y_j)$ about y_j which are shown in equations (18) and (19), equation (1) will take the form

$$u''_j(x) + \frac{a(x)}{x}\left[\frac{u_{j+1}(x) - u_{j-1}(x)}{2(\Delta y)}\right] + \left[\frac{u_{j+1}(x) - 2u_j(x) + u_{j-1}(x)}{(\Delta y)^2}\right] = f_j(x), \tag{22}$$

where $y_j = j(\Delta y), \left\{j = 1, 2, \cdots, m, for\ m = \frac{1}{\Delta y} - 1\right\}$.

Let the solution $u(x_i, y_j) = u_j(x_i)$ of the problem (22), (2) and (3) be approximated by B-spline function as follow:

$$u_j(x_i) = \sum_{k=-3}^{n-1} c_k^j B_k(x_i), \tag{23}$$

where $\{c_k^j\}_{k=-3}^{n-1}$ are unknown real coefficients and $B_k(x_i)$ are the cubic B-spline functions for x_i such that $x_i = ih, \{i = 0, 1, \cdots, n,\ for\ h = \frac{1}{n}\}$. It is required that the approximate solution satisfies the differential equation at the points x_i, and we can easily deduce the following

$$u'_j(x_i) = \sum_{k=-3}^{n-1} c_k^j B'_k(x_i), \tag{24}$$

and

$$u''_j(x_i) = \sum_{k=-3}^{n-1} c_k^j B''_k(x_i). \tag{25}$$

4.1 Regular Case

If $a(x) = xb(x)$ let $u(x, y_j) = u_j(x), \frac{d}{dx}[u(x, y_j)] = u'_j(x), \frac{d^2}{dx^2}[u(x, y_j)] = u''_j(x)$ and $f\left(x, y_j\right) = f_j\left(x\right)$, then equation (22) takes the form

$$\begin{aligned}
&2(\Delta y)^2 u''_j(x) + b(x)(\Delta y)\left[u_{j+1}(x) - u_{j-1}(x)\right] \\
&+ 2\left[u_{j+1}(x) - 2u_j(x) + u_{j-1}(x)\right] = 2(\Delta y)^2 f_j(x).
\end{aligned} \tag{26}$$

Theorem 4.1 If the approximate solution of the problem (26) is (23) then the discrete collocation system for the determination of the unknown coefficients c_k^j is given by

$$\begin{aligned}
&\sum_{k=-3}^{n-1} 2\left[(\Delta y)^2 B''_k(x_i) - 2B_k(x_i)\right]c_k^1 + \sum_{k=-3}^{n-1}\left[(\Delta y)b(x_i)B'_k(x_i) + 2B_k(x_i)\right]c_k^2 \\
&= 2(\Delta y)^2 f_1(x_i), \qquad j = 1,
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\sum_{k=-3}^{n-1}\left[-(\Delta y)b(x_i)B'_k(x_i) + 2B_k(x_i)\right]c_k^{j-1} + \sum_{k=-3}^{n-1} 2\left[(\Delta y)^2 B''_k(x_i) - 2B_k(x_i)\right]c_k^j \\
&+ \sum_{k=-3}^{n-1}\left[(\Delta y)b(x_i)B'_k(x_i) + 2B_k(x_i)\right]c_k^{j+1} = 2(\Delta y)^2 f_j(x_i), \quad j = 2, 3, \cdots, m-1,
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
&\sum_{k=-3}^{n-1}\left[-(\Delta y)b(x_i)B'_k(x_i) + 2B_k(x_i)\right]c_k^{m-1} + \sum_{k=-3}^{n-1} 2\left[(\Delta y)^2 B''_k(x_i) - 2B_k(x_i)\right]c_k^m \\
&= 2(\Delta y)^2 f_m(x_i), \qquad j = m.
\end{aligned} \tag{29}$$

Proof. Substituting with $j = 1, 2, \cdots, m$, in equation (26) yields

$$\begin{aligned}
&2\left[(\Delta y)^2 u''_1(x) - 2u_1(x)\right] + \left[(\Delta y)b(x)u'_2(x) + 2u_2(x)\right] = 2(\Delta y)^2 f_1(x), \ at\ \ j = 1, \\
&\left[-(\Delta y)b(x)u'_{j-1}(x) + 2u_{j-1}(x)\right] + 2\left[(\Delta y)^2 u''_j(x) - 2u_j(x)\right] \\
&+ \left[(\Delta y)b(x)u'_{j+1}(x) + 2u_{j+1}(x)\right] = 2(\Delta y)^2 f_j(x), \quad at \quad j = 2, 3, \cdots, m-1,
\end{aligned}$$

and

$$\left[-(\Delta y)b(x)u'_{m-1}(x) + 2u_{m-1}(x)\right] + 2\left[(\Delta y)^2 u''_m(x) - 2u_m(x)\right] = 2(\Delta y)^2 f_m(x), \ at\ \ j = m,$$

by substituting $x = x_i$ and then using equations (23)-(25), the proof of the above theorem is completed.

Substituting equation (24) in equation (2) and multiplying by $2(\Delta y)^2$ yields

$$\begin{aligned}
2(\Delta y)^2 \sum_{k=-3}^{n-1} c_k^j B'_k(0) &= 0, \\
2(\Delta y)^2 \sum_{k=-3}^{n-1} c_k^j B'_k(1) &= 0.
\end{aligned} \tag{30}$$

Adding (30) to system of equations (27)-(29), which may be written in matrix-vector form as follow:

$$\begin{bmatrix} A_2 & A_3 & 0 & \cdots & \cdots & 0 \\ A_1 & A_2 & A_3 & 0 & \cdots & 0 \\ 0 & A_1 & A_2 & A_3 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & A_1 & A_2 & A_3 \\ 0 & \cdots & \cdots & 0 & A_1 & A_2 \end{bmatrix} \begin{bmatrix} C^1 \\ C^2 \\ \vdots \\ C^m \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \\ \vdots \\ F^m \end{bmatrix},$$

where:

$$A_1 = \begin{bmatrix} -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 & 0 & \cdots & \cdots & 0 \\ \eta_1 & \eta_2 & \eta_3 & 0 & \cdots & \cdots & 0 \\ 0 & \eta_1 & \eta_2 & \eta_3 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \eta_1 & \eta_2 & \eta_3 \\ 0 & \cdots & \cdots & 0 & -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 & 0 & \cdots & \cdots & 0 \\ \beta_1 & \beta_2 & \beta_1 & 0 & \cdots & \cdots & 0 \\ 0 & \beta_1 & \beta_2 & \beta_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \beta_1 & \beta_2 & \beta_1 \\ 0 & \cdots & \cdots & 0 & -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 & 0 & \cdots & \cdots & 0 \\ \eta_3 & \eta_2 & \eta_1 & 0 & \cdots & \cdots & 0 \\ 0 & \eta_3 & \eta_2 & \eta_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \eta_3 & \eta_2 & \eta_1 \\ 0 & \cdots & \cdots & 0 & -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 \end{bmatrix},$$

and

$$C^j = \begin{bmatrix} c_{-3}^j \\ c_{-2}^j \\ c_{-1}^j \\ \vdots \\ c_{n-1}^j \end{bmatrix}, \qquad F^j = \begin{bmatrix} 0 \\ 2(\Delta y)^2 f_j(x_0) \\ 2(\Delta y)^2 f_j(x_1) \\ \vdots \\ 2(\Delta y)^2 f_j(x_n) \\ 0 \end{bmatrix}, \quad j = 1,2,\cdots,m.$$

Notice that, A_1, A_2, and A_3 are $n+3\times n+3$ dimensional matrices, such that:

$$\eta_1 = -b(x_i)(\Delta y)\left(\frac{-3}{h}\right) + 2(1), \quad \eta_2 = 2(4), \quad \eta_3 = -b(x_i)(\Delta y)\left(\frac{3}{h}\right) + 2(1),$$

and

$$\beta_1 = 2(\Delta y)^2\left(\frac{6}{h^2}\right) - 4(1), \quad \beta_2 = 2(\Delta y)^2\left(\frac{-12}{h^2}\right) - 4(4).$$

Now, we have a linear system of $m(n+3)$ equations of the $m(n+3)$ unknown coefficients, namely, c_k^j. We can obtain the coefficients of the approximate solution by solving this linear system by Q-R method. Using equation (23), the approximate solution of $u(x_i, y_j)$ can be evaluated.

4.2 Singular Case

If $a(x) \neq xb(x)$ then equation (22) will take the form:

$$\left[-(\Delta y)\frac{a(x)}{x}u'_{j-1}(x) + 2u_{j-1}(x)\right] + \left[2(\Delta y)^2 u''_j(x) - 4u_j(x)\right] + \left[(\Delta y)\frac{a(x)}{x}u'_{j+1}(x) + 2u_{j+1}(x)\right] = 2(\Delta y)^2 f_j(x), \tag{31}$$

To overcome the singularity at $x = 0$ in (22), we apply L'Hopital's rule as x approaches zero to the term $\frac{a(x)}{x}\left[u'_{j+1}(x) - u'_{j-1}(x)\right]$ as follows [32]:

$$\lim_{x\to 0}\frac{a(x)}{x}\left[u'_{j+1}(x) - u'_{j-1}(x)\right] = \frac{a(0)}{0}\left[u'_{j+1}(0) - u'_{j-1}(0)\right] = \frac{0}{0},$$

hence

$$\lim_{x\to 0}\left[a'(x)\left[u'_{j+1}(x) - u'_{j-1}(x)\right] + a(x)\left[u''_{j+1}(x) - u''_{j+1}(x)\right]\right] = a(0)\left[u''_{j+1}(0) - u''_{j-1}(0)\right].$$

Then, equation (22) will be written in the following form:

$$\left[-(\Delta y)a(0)u''_{j-1}(0) + 2u_{j-1}(0)\right] + \left[2(\Delta y)^2 u''_j(0) - 4u_j(0)\right] + \left[(\Delta y)a(0)u''_{j+1}(0) + 2u_{j+1}(0)\right] = 2(\Delta y)^2 f_j(0), \quad at \quad x = 0. \tag{32}$$

Theorem 4.2 If the approximate solution of the problem (31) and (32) is (23), then the discrete collocation system for the determination of the unknown coefficients c_k^j is given by at $y = y_1$

$$\sum_{k=-3}^{n-1}\left[2(\Delta y)^2 B''(0) - 4B(0)\right]c_k^1 + \sum_{k=-3}^{n-1}\left[(\Delta y)a(0)B''(0) + 2B(0)\right]c_k^2 = 2(\Delta y)^2 f_1(0), \quad at \quad x_i = 0, \tag{33}$$

and

$$\sum_{k=-3}^{n-1}\left[2(\Delta y)^2 B''(x_i) - 4B(x_i)\right]c_k^1 + \sum_{k=-3}^{n-1}\left[(\Delta y)\frac{a(x_i)}{x_i}B'(x_i) + 2B(x_i)\right]c_k^2 = 2(\Delta y)^2 f_1(x_i), \quad at \quad x_i \neq 0, \tag{34}$$

at $y = y_j$, $j = 2,3,\cdots,m-1$

$$\sum_{k=-3}^{n-1}\left[-(\Delta y)a(0)B''(0) + 2B(0)\right]c_k^{j-1} + \sum_{k=-3}^{n-1}\left[2(\Delta y)^2 B''(0) - 4B(0)\right]c_k^j + \sum_{k=-3}^{n-1}\left[(\Delta y)a(0)B''(0) + 2B(0)\right]c_k^{j+1} = 2(\Delta y)^2 f_j(0), \quad at \quad x_i = 0, \tag{35}$$

and

$$\sum_{k=-3}^{n-1}\left[-(\Delta y)\frac{a(x_i)}{x_i}B'(x_i) + 2B(x_i)\right]c_k^{j-1} + \sum_{k=-3}^{n-1}\left[2(\Delta y)^2 B''(x_i) - 4B(x_i)\right]c_k^j + \sum_{k=-3}^{n-1}\left[(\Delta y)\frac{a(x_i)}{x_i}B'(x_i) + 2B(x_i)\right]c_k^{j+1} = 2(\Delta y)^2 f_j(x_i), \quad at \quad x_i \neq 0, \tag{36}$$

at $y = y_m$

$$\sum_{k=-3}^{n-1}\left[-(\Delta y)a(0)B''(0) + 2B(0)\right]c_k^{m-1} + \sum_{k=-3}^{n-1}\left[2(\Delta y)^2 B''(0) - 4B(0)\right]c_k^m = 2(\Delta y)^2 f_m(0), \quad at \quad x_i = 0, \tag{37}$$

and

$$\sum_{k=-3}^{n-1}\left[-(\Delta y)\frac{a(x_i)}{x_i}B'(x_i) + 2B(x_i)\right]c_k^{m-1} + \sum_{k=-3}^{n-1}\left[2(\Delta y)^2 B''(x_i) - 4B(x_i)\right]c_k^m = 2(\Delta y)^2 f_m(x_i), \quad at \quad x_i \neq 0. \tag{38}$$

Proof. Substituting with $j = 1,2,\cdots,m-1,m$, in equations (31)-(32) yields for $j = 1$

$$\left[2(\Delta y)^2 u''_1(0) - 4u_1(0)\right] + \left[(\Delta y)a(0)u''_2(0) + 2u_2(0)\right] = 2(\Delta y)^2 f_0(0), \quad at \quad x = 0,$$

and

$$\left[2(\Delta y)^2 u_1''(x) - 4u_1(x)\right] + \left[(\Delta y)\frac{a(x)}{x}u_2'(x) + 2u_2(x)\right] = 2(\Delta y)^2 f_0(x), \quad at \quad x \neq 0,$$

for $j = 2,3,\cdots,m-1$;

$$\left[-(\Delta y)a(0)u_{j-1}''(0) + 2u_{j-1}(0)\right] + \left[2(\Delta y)^2 u_j''(0) - 4u_j(0)\right]$$

$$+\left[(\Delta y)a(0)u_{j+1}''(0) + 2u_{j+1}(0)\right] = 2(\Delta y)^2 f_j(0), \quad at \quad x = 0,$$

and

$$\left[-(\Delta y)\frac{a(x)}{x}u_{j-1}'(x) + 2u_{j-1}(x)\right] + \left[2(\Delta y)^2 u_j''(x) - 4u_j(x)\right]$$

$$+\left[(\Delta y)\frac{a(x)}{x}u_{j+1}'(x) + 2u_{j+1}(x)\right] = 2(\Delta y)^2 f_j(x) \quad at \quad x \neq 0,$$

for $j = m$

$$\left[-(\Delta y)a(0)u_{m-1}''(0) + 2u_{m-1}(0)\right] + \left[2(\Delta y)^2 u_m''(0) - 4u_m(0)\right]$$
$$= 2(\Delta y)^2 f_m(0), \quad at \quad x = 0,$$

and

$$\left[-(\Delta y)\frac{a(x)}{x}u_{m-1}'(x) + 2u_{m-1}(x)\right] + \left[2(\Delta y)^2 u_m''(x) - 4u_m(x)\right]$$
$$= 2(\Delta y)^2 f_m(x) \quad at \quad x \neq 0,$$

by substituting $x = x_i$ and then using equations (23)-(25), the proof of the above theorem is completed.

Adding (30) to each equation of (33)-(38) will present m systems, which may be written in matrix-vector form as follow

$$\begin{bmatrix} Q_2 & Q_3 & 0 & \cdots & \cdots & 0 \\ Q_1 & Q_2 & Q_3 & 0 & \cdots & 0 \\ 0 & Q_1 & Q_2 & Q_3 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & Q_1 & Q_2 & Q_3 \\ 0 & \cdots & \cdots & 0 & Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} C^1 \\ C^2 \\ \vdots \\ C^m \end{bmatrix} = \begin{bmatrix} D^1 \\ D^2 \\ \vdots \\ D^m \end{bmatrix},$$

where:

$$Q_1 = \begin{bmatrix} -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 & 0 & \cdots & \cdots & 0 \\ \alpha_{01} & \alpha_{02} & \alpha_{03} & 0 & \cdots & \cdots & 0 \\ 0 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \ddots & & \vdots \\ \vdots & \ddots & \alpha_{21} & \alpha_{22} & \alpha_{23} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \alpha_{n1} & \alpha_{n2} & \alpha_{n3} \\ 0 & \cdots & \cdots & 0 & -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 & 0 & \cdots & \cdots & 0 \\ \mu_{01} & \mu_{02} & \mu_{03} & 0 & \cdots & \cdots & 0 \\ 0 & \mu_{11} & \mu_{12} & \mu_{13} & \ddots & & \vdots \\ \vdots & \ddots & \mu_{21} & \mu_{22} & \mu_{23} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \mu_{n1} & \mu_{n2} & \mu_{n3} \\ 0 & \cdots & \cdots & 0 & -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 & 0 & \cdots & \cdots & 0 \\ \gamma_{01} & \gamma_{02} & \gamma_{03} & 0 & \cdots & \cdots & 0 \\ 0 & \gamma_{11} & \gamma_{12} & \gamma_{13} & \ddots & & \vdots \\ \vdots & \ddots & \gamma_{21} & \gamma_{22} & \gamma_{23} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \gamma_{n1} & \gamma_{n2} & \gamma_{n3} \\ 0 & \cdots & \cdots & 0 & -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 \end{bmatrix},$$

and

$$C^j = \begin{bmatrix} c_{-3}^j \\ c_{-2}^j \\ c_{-1}^j \\ \vdots \\ c_{n-1}^j \end{bmatrix}, \quad D^j = \begin{bmatrix} 0 \\ 2(\Delta y)^2 f_j(x_0) \\ 2(\Delta y)^2 f_j(x_1) \\ \vdots \\ 2(\Delta y)^2 f_j(x_n) \\ 0 \end{bmatrix}, \quad j = 1,2,\cdots,m.$$

Notice that, Q_1, Q_2, and Q_3 are $n+3 \times n+3$ dimensional matrices. Let $l = 4+k$ then the terms α_{il}, μ_{il} and γ_{il} will have the following values for $k = -3,-2,-1$, else they will be zeros:

$$\alpha_{i1} = \begin{cases} -a(0)(\Delta y)\left(\frac{6}{h^2}\right) + 2(1), & i = 0, \\ -\frac{a(x_i)}{x_i}(\Delta y)\left(\frac{-3}{h}\right) + 2(1), & 0 < i \leq n, \end{cases}$$

$$\alpha_{i2} = \begin{cases} -a(0)(\Delta y)\left(\frac{-12}{h^2}\right) + 2(4), & i = 0, \\ 2(4), & 0 < i \leq n, \end{cases}$$

$$\alpha_{i3} = \begin{cases} -a(0)(\Delta y)\left(\frac{6}{h^2}\right) + 2(1), & i = 0, \\ -\frac{a(x_i)}{x_i}(\Delta y)\left(\frac{3}{h}\right) + 2(1), & 0 < i \leq n, \end{cases}$$

$$\mu_{i1} = \mu_{i3} = 2(\Delta y)^2\left(\frac{6}{h^2}\right) - 4(1), \quad 0 \leq i \leq n,$$

$$\mu_{i2} = 2(\Delta y)^2\left(\frac{-12}{h^2}\right) - 4(4), \quad 0 < i \leq n,$$

$$\gamma_{i1} = \begin{cases} a(0)(\Delta y)\left(\frac{6}{h^2}\right) + 2(1), & i = 0, \\ -\frac{a(x_i)}{x_i}(\Delta y)\left(\frac{-3}{h}\right) + 2(1), & 0 < i \leq n, \end{cases}$$

$$\gamma_{i2} = \begin{cases} a(0)(\Delta y)\left(\frac{-12}{h^2}\right) + 2(4), & i = 0, \\ 2(4), & 0 < i \leq n, \end{cases}$$

and

$$\gamma_{i3} = \begin{cases} a(0)(\Delta y)\left(\dfrac{6}{h^2}\right) + 2(1), & i = 0, \\ -\dfrac{a(x_i)}{x_i}(\Delta y)\left(\dfrac{3}{h}\right) + 2(1), & 0 < i \le n. \end{cases}$$

Now, we have a linear system of $m(n+3)$ equations of the $m(n+3)$ unknown coefficients, namely, c_k^j. We can obtain these coefficients of the approximate solution by solving this linear system by Q-R method.

5. Numerical Results

We present some test examples constructed so that the analytical solution was known before-hand. The performance of the B-spline method is measured by the absolute error $E_{B-spline}$ which is defined as

$$E_{B-spline} = | u_{exact} - u_{B-spline} |.$$

All computations were carried out using MATLAB 7.01. For these examples, we use cubic B-spline, the coefficients of $B_{i,3}$ and their derivatives, at the knots $x_i, i = 0,1,2,\cdots,n$ are shown in Table 1.

Example 1: Consider the boundary-value problem

$$u_{xx} + 2u_{xy} + u_{yy} = f(x,y), \qquad x \in [0,1],$$

where

$$f(x,y) = \left[\frac{\pi}{4} y(2x-3) - 6y(2x-1) - 12x\right]\cos\left(\frac{\pi}{2}y\right) + \pi\left[4x^2 - y(x-1)\right]\sin\left(\frac{\pi}{2}y\right),$$

subject to boundary conditions

$$u_x(0,y) = u_x(1,y) = u(x,0) = u(x,1) = 0,$$

whose exact solution is

$$u(x,y) = x^2 y(3-2x)\cos\left(\frac{\pi}{2}y\right).$$

This problem is solved at $n = 20$ and $\Delta y = 0.05$. The maximum absolute error at the points (x, y) are tabulated in Table 2.

Table 2: Maximum absolute error for Example 1.

(x,y)	Maximum absolute error
$(x,0.05)$	1.0 E -4
$(x,0.15)$	2.5 E -4
$(x,0.25)$	3.3 E -4
$(x,0.35)$	3.5 E -4
$(x,0.45)$	3.1 E -4
$(x,0.55)$	2.4 E -4
$(x,0.65)$	1.5 E -4
$(x,0.75)$	7.6 E -5
$(x,0.85)$	4.0 E -5
$(x,0.95)$	1.2 E -5

Example 2: Consider the boundary-value problem

$$u_{xx} + 2u_{xy} + u_{yy} = f(x,y), \qquad x \in [0,1],$$

$$f(x,y) = y(y^2-1)(2x-1) + yx^2(2x-3) + 2x(x-1)(3y^2-1),$$

subject to boundary conditions

$$u_x(0,y) = u_x(1,y) = u(x,0) = u(x,1) = 0,$$

whose exact solution is

$$u(x,y) = \frac{yx^2(2x-3)(y^2-1)}{6}.$$

This problem is solved with $n = 20$ and $\Delta y = 0.05$. The maximum absolute errors at several points (x, y) are tabulated in Table 3.

Table 3: Maximum absolute error for Example 2.

y	Maximum absolute error
.05	2.4E-05
.15	6.6E-05
.25	9.7E-05
.35	1.2E-04
.45	1.3E-04
.55	1.3E-04
.65	1.2E-04
.75	9.7E-05
.85	6.6E-05
.95	2.4E-05

Example 3: Consider the boundary-value problem

$$u_{xx} + \left(\frac{1}{x}\right)u_{xy} + u_{yy} = f(x,y), \qquad x \in [0,1],$$

where

$$f(x,y) = (2x-3)yx^2 + (2x-1)(y^3-y) - (x-1)(3y^2-1),$$

subject to boundary conditions

$$u_x(0,y) = u_x(1,y) = u(x,0) = u(x,1) = 0,$$

whose exact solution is

$$u(x,y) = \frac{x^2(2x-3)(y^3-y)}{6}.$$

This problem is solved with $n = 20$ and $\Delta y = 0.05$. The maximum absolute errors at several points (x, y) are tabulated in Table 4.

Table 4: Maximum absolute error for Example 3.

(x,y)	Maximum absolute error
$(x,0.05)$	3.5 E -5
$(x,0.15)$	8.6 E -5
$(x,0.25)$	1.3 E -4
$(x,0.35)$	1.7 E -4
$(x,0.45)$	2.0 E -4
$(x,0.55)$	2.2 E -4
$(x,0.65)$	2.3 E -4
$(x,0.75)$	2.3 E -4
$(x,0.85)$	2.1 E -4
$(x,0.95)$	1.2 E -4

Example 4: Consider the boundary-value problem

$$u_{xx} + \left(\frac{e^x}{x}\right)u_{xy} + u_{yy} = f(x,y), \qquad x \in [0,1]$$

where

$$f(x,y) = -\frac{\pi^2 x^2(2x-3)\sin(\pi y)}{6} + (2x-1)\sin(\pi y) + \pi e^x(x-1)\cos(\pi y),$$

subject to boundary conditions

$$u_x(0,y) = u_x(1,y) = u(x,0) = u(x,1) = 0,$$

whose exact solution is

$$u(x,y)=\frac{x^2(2x-3)\sin(\pi y)}{6}.$$

This problem is solved with $n=20$ and $\Delta y=0.05$. The maximum absolute errors at several points (x,y) are tabulated in Table 5.

Table 5: Maximum absolute error for Example 4.

y	Maximum absolute error
.05	8.0E-05
.15	2.0E-04
.25	2.8E-04
.35	2.9E-04
.45	2.4E-04
.55	1.4E-04
.65	1.0E-04
.75	1.5E-04
.85	1.9E-04
.95	1.2E-04

6. Conclusion

We presented a method for solving singular second-order partial differential equation. This method is easy to implement and yields the desired accuracy and numerical results demonstrate this. We observed that the method works well for singular differential equations. Thus the proposed method is suggested as an efficient method for solving this problem.

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