







where:

$$A_1 = \begin{bmatrix} -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 & 0 & \dots & \dots & 0 \\ \eta_1 & \eta_2 & \eta_3 & 0 & \dots & \dots & 0 \\ 0 & \eta_1 & \eta_2 & \eta_3 & \ddots & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \eta_1 & \eta_2 & \eta_3 \\ 0 & \dots & \dots & 0 & -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 & 0 & \dots & \dots & 0 \\ \beta_1 & \beta_2 & \beta_1 & 0 & \dots & \dots & 0 \\ 0 & \beta_1 & \beta_2 & \beta_1 & \ddots & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \beta_1 & \beta_2 & \beta_1 \\ 0 & \dots & \dots & 0 & -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 & 0 & \dots & \dots & 0 \\ \eta_3 & \eta_2 & \eta_1 & 0 & \dots & \dots & 0 \\ 0 & \eta_3 & \eta_2 & \eta_1 & \ddots & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \eta_3 & \eta_2 & \eta_1 \\ 0 & \dots & \dots & 0 & -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 \end{bmatrix},$$

and

$$C^j = \begin{bmatrix} c_{-3}^j \\ c_{-2}^j \\ c_{-1}^j \\ \vdots \\ c_{n-1}^j \end{bmatrix}, \quad F^j = \begin{bmatrix} 0 \\ 2(\Delta y)^2 f_j(x_0) \\ 2(\Delta y)^2 f_j(x_1) \\ \vdots \\ 2(\Delta y)^2 f_j(x_n) \\ 0 \end{bmatrix}, \quad j = 1, 2, \dots, m.$$

Notice that,  $A_1, A_2,$  and  $A_3$  are  $n+3 \times n+3$  dimensional matrices, such that:

$$\eta_1 = -b(x_i)(\Delta y) \left( \frac{-3}{h} \right) + 2(1), \quad \eta_2 = 2(4), \quad \eta_3 = -b(x_i)(\Delta y) \left( \frac{3}{h} \right) + 2(1),$$

and

$$\beta_1 = 2(\Delta y)^2 \left( \frac{6}{h^2} \right) - 4(1), \quad \beta_2 = 2(\Delta y)^2 \left( \frac{-12}{h^2} \right) - 4(4).$$

Now, we have a linear system of  $m(n+3)$  equations of the  $m(n+3)$  unknown coefficients, namely,  $c_k^j$ . We can obtain the coefficients of the approximate solution by solving this linear system by Q-R method. Using equation (23), the approximate solution of  $u(x_i, y_j)$  can be evaluated.

### 4.2 Singular Case

If  $a(x) \neq xb(x)$  then equation (22) will take the form:

$$\begin{aligned} & \left[ -(\Delta y) \frac{a(x)}{x} u'_{j-1}(x) + 2u_{j-1}(x) \right] + \left[ 2(\Delta y)^2 u''_j(x) - 4u_j(x) \right] \\ & + \left[ (\Delta y) \frac{a(x)}{x} u'_{j+1}(x) + 2u_{j+1}(x) \right] = 2(\Delta y)^2 f_j(x), \end{aligned} \tag{31}$$

To overcome the singularity at  $x = 0$  in (22), we apply L'Hopital's rule as  $x$  approaches zero to the term  $\frac{a(x)}{x} [u'_{j+1}(x) - u'_{j-1}(x)]$  as follows [32]:

$$\lim_{x \rightarrow 0} \frac{a(x)}{x} [u'_{j+1}(x) - u'_{j-1}(x)] = \frac{a(0)}{0} [u'_{j+1}(0) - u'_{j-1}(0)] = \frac{0}{0},$$

hence

$$\begin{aligned} & \lim_{x \rightarrow 0} \left[ a'(x) [u'_{j+1}(x) - u'_{j-1}(x)] + a(x) [u''_{j+1}(x) - u''_{j-1}(x)] \right] \\ & = a(0) [u''_{j+1}(0) - u''_{j-1}(0)] \end{aligned}$$

Then, equation (22) will be written in the following form:

$$\begin{aligned} & \left[ -(\Delta y) a(0) u''_{j-1}(0) + 2u_{j-1}(0) \right] + \left[ 2(\Delta y)^2 u''_j(0) - 4u_j(0) \right] \\ & + \left[ (\Delta y) a(0) u''_{j+1}(0) + 2u_{j+1}(0) \right] = 2(\Delta y)^2 f_j(0), \quad \text{at } x = 0. \end{aligned} \tag{32}$$

**Theorem 4.2** If the approximate solution of the problem (31) and (32) is (23), then the discrete collocation system for the determination of the unknown coefficients  $c_k^j$  is given by at  $y = y_1$

$$\begin{aligned} & \sum_{k=-3}^{n-1} \left[ 2(\Delta y)^2 B''(0) - 4B(0) \right] c_k^1 + \sum_{k=-3}^{n-1} \left[ (\Delta y) a(0) B''(0) + 2B(0) \right] c_k^2 \\ & = 2(\Delta y)^2 f_1(0), \quad \text{at } x_i = 0, \end{aligned} \tag{33}$$

and

$$\begin{aligned} & \sum_{k=-3}^{n-1} \left[ 2(\Delta y)^2 B''(x_i) - 4B(x_i) \right] c_k^1 + \sum_{k=-3}^{n-1} \left[ (\Delta y) \frac{a(x_i)}{x_i} B'(x_i) + 2B(x_i) \right] c_k^2 \\ & = 2(\Delta y)^2 f_1(x_i), \quad \text{at } x_i \neq 0, \end{aligned} \tag{34}$$

at  $y = y_j, j = 2, 3, \dots, m-1$

$$\begin{aligned} & \sum_{k=-3}^{n-1} \left[ -(\Delta y) a(0) B''(0) + 2B(0) \right] c_k^{j-1} + \sum_{k=-3}^{n-1} \left[ 2(\Delta y)^2 B''(0) - 4B(0) \right] c_k^j \\ & + \sum_{k=-3}^{n-1} \left[ (\Delta y) a(0) B''(0) + 2B(0) \right] c_k^{j+1} = 2(\Delta y)^2 f_j(0), \quad \text{at } x_i = 0, \end{aligned} \tag{35}$$

and

$$\begin{aligned} & \sum_{k=-3}^{n-1} \left[ -(\Delta y) \frac{a(x_i)}{x_i} B'(x_i) + 2B(x_i) \right] c_k^{j-1} + \sum_{k=-3}^{n-1} \left[ 2(\Delta y)^2 B''(x_i) - 4B(x_i) \right] c_k^j \\ & + \sum_{k=-3}^{n-1} \left[ (\Delta y) \frac{a(x_i)}{x_i} B'(x_i) + 2B(x_i) \right] c_k^{j+1} = 2(\Delta y)^2 f_j(x_i), \quad \text{at } x_i \neq 0, \end{aligned} \tag{36}$$

at  $y = y_m$

$$\begin{aligned} & \sum_{k=-3}^{n-1} \left[ -(\Delta y) a(0) B''(0) + 2B(0) \right] c_k^{m-1} + \sum_{k=-3}^{n-1} \left[ 2(\Delta y)^2 B''(0) - 4B(0) \right] c_k^m \\ & = 2(\Delta y)^2 f_m(0), \quad \text{at } x_i = 0, \end{aligned} \tag{37}$$

and

$$\begin{aligned} & \sum_{k=-3}^{n-1} \left[ -(\Delta y) \frac{a(x_i)}{x_i} B'(x_i) + 2B(x_i) \right] c_k^{m-1} + \sum_{k=-3}^{n-1} \left[ 2(\Delta y)^2 B''(x_i) - 4B(x_i) \right] c_k^m \\ & = 2(\Delta y)^2 f_m(x_i), \quad \text{at } x_i \neq 0. \end{aligned} \tag{38}$$

*Proof.* Substituting with  $j = 1, 2, \dots, m-1, m$ , in equations (31)-(32) yields

for  $j = 1$

$$\left[ 2(\Delta y)^2 u''_1(0) - 4u_1(0) \right] + \left[ (\Delta y) a(0) u''_2(0) + 2u_2(0) \right] = 2(\Delta y)^2 f_0(0), \quad \text{at } x = 0,$$

and

$$[2(\Delta y)^2 u_1'(x) - 4u_1(x)] + \left[ (\Delta y) \frac{a(x)}{x} u_2'(x) + 2u_2(x) \right] = 2(\Delta y)^2 f_0(x), \quad \text{at } x \neq 0,$$

for  $j = 2, 3, \dots, m-1$ ;

$$[-(\Delta y)a(0)u_{j-1}''(0) + 2u_{j-1}(0)] + [2(\Delta y)^2 u_j''(0) - 4u_j(0)] + [(\Delta y)a(0)u_{j+1}''(0) + 2u_{j+1}(0)] = 2(\Delta y)^2 f_j(0), \quad \text{at } x = 0,$$

and

$$\left[ -(\Delta y) \frac{a(x)}{x} u_{j-1}'(x) + 2u_{j-1}(x) \right] + [2(\Delta y)^2 u_j''(x) - 4u_j(x)] + \left[ (\Delta y) \frac{a(x)}{x} u_{j+1}'(x) + 2u_{j+1}(x) \right] = 2(\Delta y)^2 f_j(x) \quad \text{at } x \neq 0,$$

for  $j = m$

$$[-(\Delta y)a(0)u_{m-1}''(0) + 2u_{m-1}(0)] + [2(\Delta y)^2 u_m''(0) - 4u_m(0)] = 2(\Delta y)^2 f_m(0), \quad \text{at } x = 0,$$

and

$$\left[ -(\Delta y) \frac{a(x)}{x} u_{m-1}'(x) + 2u_{m-1}(x) \right] + [2(\Delta y)^2 u_m''(x) - 4u_m(x)] = 2(\Delta y)^2 f_m(x) \quad \text{at } x \neq 0,$$

by substituting  $x = x_i$  and then using equations (23)-(25), the proof of the above theorem is completed.

Adding (30) to each equation of (33)-(38) will present  $m$  systems, which may be written in matrix-vector form as follow

$$\begin{bmatrix} Q_2 & Q_3 & 0 & \dots & \dots & 0 \\ Q_1 & Q_2 & Q_3 & 0 & \dots & 0 \\ 0 & Q_1 & Q_2 & Q_3 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & Q_1 & Q_2 & Q_3 \\ 0 & \dots & \dots & 0 & Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} C^1 \\ C^2 \\ \vdots \\ C^m \end{bmatrix} = \begin{bmatrix} D^1 \\ D^2 \\ \vdots \\ D^m \end{bmatrix},$$

where:

$$Q_1 = \begin{bmatrix} -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 & 0 & \dots & \dots & 0 \\ \alpha_{01} & \alpha_{02} & \alpha_{03} & 0 & \dots & \dots & 0 \\ 0 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \ddots & & \vdots \\ \vdots & \ddots & \alpha_{21} & \alpha_{22} & \alpha_{23} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \alpha_{n1} & \alpha_{n2} & \alpha_{n3} \\ 0 & \dots & \dots & 0 & -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 & 0 & \dots & \dots & 0 \\ \mu_{01} & \mu_{02} & \mu_{03} & 0 & \dots & \dots & 0 \\ 0 & \mu_{11} & \mu_{12} & \mu_{13} & \ddots & & \vdots \\ \vdots & \ddots & \mu_{21} & \mu_{22} & \mu_{23} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \mu_{n1} & \mu_{n2} & \mu_{n3} \\ 0 & \dots & \dots & 0 & -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 & 0 & \dots & \dots & 0 \\ \gamma_{01} & \gamma_{02} & \gamma_{03} & 0 & \dots & \dots & 0 \\ 0 & \gamma_{11} & \gamma_{12} & \gamma_{13} & \ddots & & \vdots \\ \vdots & \ddots & \gamma_{21} & \gamma_{22} & \gamma_{23} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \gamma_{n1} & \gamma_{n2} & \gamma_{n3} \\ 0 & \dots & \dots & 0 & -\frac{6}{h}(\Delta y)^2 & 0 & \frac{6}{h}(\Delta y)^2 \end{bmatrix},$$

and

$$C^j = \begin{bmatrix} c_{-3}^j \\ c_{-2}^j \\ c_{-1}^j \\ \vdots \\ c_{n-1}^j \end{bmatrix}, \quad D^j = \begin{bmatrix} 0 \\ 2(\Delta y)^2 f_j(x_0) \\ 2(\Delta y)^2 f_j(x_1) \\ \vdots \\ 2(\Delta y)^2 f_j(x_n) \\ 0 \end{bmatrix}, \quad j = 1, 2, \dots, m.$$

Notice that,  $Q_1, Q_2$ , and  $Q_3$  are  $n+3 \times n+3$  dimensional matrices. Let  $l = 4+k$  then the terms  $\alpha_{il}, \mu_{il}$  and  $\gamma_{il}$  will have the following values for  $k = -3, -2, -1$ , else they will be zeros:

$$\alpha_{i1} = \begin{cases} -a(0)(\Delta y) \left( \frac{6}{h^2} \right) + 2(1), & i = 0, \\ -\frac{a(x_i)}{x_i} (\Delta y) \left( \frac{-3}{h} \right) + 2(1), & 0 < i \leq n, \end{cases}$$

$$\alpha_{i2} = \begin{cases} -a(0)(\Delta y) \left( \frac{-12}{h^2} \right) + 2(4), & i = 0, \\ 2(4), & 0 < i \leq n, \end{cases}$$

$$\alpha_{i3} = \begin{cases} -a(0)(\Delta y) \left( \frac{6}{h^2} \right) + 2(1), & i = 0, \\ -\frac{a(x_i)}{x_i} (\Delta y) \left( \frac{3}{h} \right) + 2(1), & 0 < i \leq n, \end{cases}$$

$$\mu_{i1} = \mu_{i3} = 2(\Delta y)^2 \left( \frac{6}{h^2} \right) - 4(1), \quad 0 \leq i \leq n,$$

$$\mu_{i2} = 2(\Delta y)^2 \left( \frac{-12}{h^2} \right) - 4(4), \quad 0 < i \leq n,$$

$$\gamma_{i1} = \begin{cases} a(0)(\Delta y) \left( \frac{6}{h^2} \right) + 2(1), & i = 0, \\ -\frac{a(x_i)}{x_i} (\Delta y) \left( \frac{-3}{h} \right) + 2(1), & 0 < i \leq n, \end{cases}$$

$$\gamma_{i2} = \begin{cases} a(0)(\Delta y) \left( \frac{-12}{h^2} \right) + 2(4), & i = 0, \\ 2(4), & 0 < i \leq n, \end{cases}$$

and

$$Y_{i3} = \begin{cases} a(0)(\Delta y)\left(\frac{6}{h^2}\right) + 2(1), & i = 0, \\ -\frac{a(x_i)}{x_i}(\Delta y)\left(\frac{3}{h}\right) + 2(1), & 0 < i \leq n. \end{cases}$$

Now, we have a linear system of  $m(n+3)$  equations of the  $m(n+3)$  unknown coefficients, namely,  $C_k^j$ . We can obtain these coefficients of the approximate solution by solving this linear system by Q-R method.

**5. NUMERICAL RESULTS**

We present some test examples constructed so that the analytical solution was known before-hand. The performance of the B-spline method is measured by the absolute error  $E_{B-spline}$  which is defined as

$$E_{B-spline} = |u_{exact} - u_{B-spline}|.$$

All computations were carried out using MATLAB 7.01. For these examples, we use cubic B-spline, the coefficients of  $B_{i,3}$  and their derivatives, at the knots  $x_i, i = 0, 1, 2, \dots, n$  are shown in Table 1.

**Example 1:** Consider the boundary-value problem

$$u_{xx} + 2u_{xy} + u_{yy} = f(x, y), \quad x \in [0, 1],$$

where

$$f(x, y) = \left[\frac{\pi}{4}y(2x-3) - 6y(2x-1) - 12x\right] \cos\left(\frac{\pi}{2}y\right) + \pi[4x^2 - y(x-1)] \sin\left(\frac{\pi}{2}y\right),$$

subject to boundary conditions

$$u_x(0, y) = u_x(1, y) = u(x, 0) = u(x, 1) = 0,$$

whose exact solution is

$$u(x, y) = x^2y(3-2x) \cos\left(\frac{\pi}{2}y\right).$$

This problem is solved at  $n = 20$  and  $\Delta y = 0.05$ . The maximum absolute error at the points  $(x, y)$  are tabulated in Table 2.

Table 2: Maximum absolute error for Example 1.	
(x, y)	Maximum absolute error
(x,0.05)	1.0 E -4
(x,0.15)	2.5 E -4
(x,0.25)	3.3 E -4
(x,0.35)	3.5 E -4
(x,0.45)	3.1 E -4
(x,0.55)	2.4 E -4
(x,0.65)	1.5 E -4
(x,0.75)	7.6 E -5
(x,0.85)	4.0 E -5
(x,0.95)	1.2 E -5

**Example 2:** Consider the boundary-value problem

$$u_{xx} + 2u_{xy} + u_{yy} = f(x, y), \quad x \in [0, 1],$$

$$f(x, y) = y(y^2 - 1)(2x - 1) + yx^2(2x - 3) + 2x(x - 1)(3y^2 - 1),$$

subject to boundary conditions

$$u_x(0, y) = u_x(1, y) = u(x, 0) = u(x, 1) = 0,$$

whose exact solution is

$$u(x, y) = \frac{yx^2(2x-3)(y^2-1)}{6}.$$

This problem is solved with  $n = 20$  and  $\Delta y = 0.05$ . The maximum absolute errors at several points  $(x, y)$  are tabulated in Table 3.

Table 3: Maximum absolute error for Example 2.	
y	Maximum absolute error
.05	2.4E-05
.15	6.6E-05
.25	9.7E-05
.35	1.2E-04
.45	1.3E-04
.55	1.3E-04
.65	1.2E-04
.75	9.7E-05
.85	6.6E-05
.95	2.4E-05

**Example 3:** Consider the boundary-value problem

$$u_{xx} + \left(\frac{1}{x}\right)u_{xy} + u_{yy} = f(x, y), \quad x \in [0, 1],$$

where

$$f(x, y) = (2x - 3)yx^2 + (2x - 1)(y^3 - y) - (x - 1)(3y^2 - 1),$$

subject to boundary conditions

$$u_x(0, y) = u_x(1, y) = u(x, 0) = u(x, 1) = 0,$$

whose exact solution is

$$u(x, y) = \frac{x^2(2x-3)(y^3-y)}{6}.$$

This problem is solved with  $n = 20$  and  $\Delta y = 0.05$ . The maximum absolute errors at several points  $(x, y)$  are tabulated in Table 4.

Table 4: Maximum absolute error for Example 3.	
(x, y)	Maximum absolute error
(x,0.05)	3.5 E -5
(x,0.15)	8.6 E -5
(x,0.25)	1.3 E -4
(x,0.35)	1.7 E -4
(x,0.45)	2.0 E -4
(x,0.55)	2.2 E -4
(x,0.65)	2.3 E -4
(x,0.75)	2.3 E -4
(x,0.85)	2.1 E -4
(x,0.95)	1.2 E -4

**Example 4:** Consider the boundary-value problem

$$u_{xx} + \left(\frac{e^x}{x}\right)u_{xy} + u_{yy} = f(x, y), \quad x \in [0, 1]$$

where

$$f(x, y) = -\frac{\pi^2 x^2(2x-3)\sin(\pi y)}{6} + (2x-1)\sin(\pi y) + \pi e^x(x-1)\cos(\pi y),$$

subject to boundary conditions

$$u_x(0, y) = u_x(1, y) = u(x, 0) = u(x, 1) = 0,$$

whose exact solution is

$$u(x, y) = \frac{x^2(2x-3)\sin(\pi y)}{6}.$$

This problem is solved with  $n = 20$  and  $\Delta y = 0.05$ . The maximum absolute errors at several points  $(x, y)$  are tabulated in Table 5.

**Table 5:** Maximum absolute error for Example 4.

$y$	Maximum absolute error
.05	8.0E-05
.15	2.0E-04
.25	2.8E-04
.35	2.9E-04
.45	2.4E-04
.55	1.4E-04
.65	1.0E-04
.75	1.5E-04
.85	1.9E-04
.95	1.2E-04

## 6. CONCLUSION

We presented a method for solving singular second-order partial differential equation. This method is easy to implement and yields the desired accuracy and numerical results demonstrate this. We observed that the method works well for singular differential equations. Thus the proposed method is suggested as an efficient method for solving this problem.

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