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REVIEW ARTICLE

LAGUERRE POLYNOMIALS SOLUTION FOR SOLVING HIGH-ORDER DELAY LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT

The aim of this article is to present an efficient numerical procedure for solving higher-order linear delay differential equations with variable coefficients under the mixed conditions in terms of Laguerre polynomials. Four problems are solved and results are compared with the existing results to show the accuracy and applicability of Laguerre polynomials.

KEYWORDS

Laguerre Polynomials, Delay differential equations, Pantograph type, Numerical solutions.

1. INTRODUCTION

Delay differential equations of constant and variable delays arise in many applications in biology, physics, economy, nonlinear dynamical systems, probability theory and engineering (Ajello et al., 1992; Buhmann and Iserles, 1993; Saaty, 2012). So, they have attracted the attention of many researchers to investigate them. Most of them have no exact solution so, approximate numerical methods are encountered (Bellen and Zennaro, 2013). Many researchers have discussed solutions to delay differential equations like the Collocation methods of various types of polynomials like Chebychev, Hermit, Legender wavelet, Morgan-Voyce, Boubaker and Taylor polynomials (Sedaghat et al., 2012; Yalinba et al., 2011; Hafshejani et al., 2011; Zel et al., 2018; Akkaya et al., 2013; Glsu and Sezer, 2011). Collocation using exponential polynomials was also investigated (Yzba and Sezer, 2013). Other methods as variational iteration method, one-leg θ method, and Runge-Kutta method (Chen and Wang, 2010; Wang and Li, 2007; Liu et al., 2006). A new collocation scheme was developed by Reutskiy (Reutskiy, 2015). Also, Tau method and Jacobi rational-Gauss collocation method were used (Trif, 2012; Doha et al., 2014). Fractional-order delay differential equations were solved using Gegenbauer polynomials in (Usman et al., 2020). Also, fractional delay integro-differential equations were solved using Tau method and Legender wavelet collocation in (Shahmorad et al., 2020; Nemati et al., 2020). We introduce the solution to differential equations with variable delays using Laguerre-collocation method in the form:

$$u^{(m)}(t) = \sum_{i=0}^{m-1} \sum_{j=1}^{J} P_{ij}(t) u(t-\delta_j(t))^{(i)} + f(t), \qquad m \geq 1 \qquad (1)$$

subject to the mixed conditions

$$\sum_{k=0}^{m-1} a_{ik} u^{(k)}(a) + b_{ik} u^{(k)}(b) = \lambda_i, \qquad i = 0,1,\dots,m-1 \qquad (2)$$

Where $P_{ij}(t)$ and $\delta_j(t)$ are given continuous functions on the interval $0 \leq a \leq t \leq b$ and the delays $\delta_j(t)$ are nonnegative that $\delta_j(t) \geq 0$ on that interval.

Orthogonal functions and polynomial series have received considerable attention in dealing with various problems in dynamic systems (Askey, 1975; Nevai, 1994; Marcelln, 2006). The main characteristics of this technique is that it reduces these problems to those for solving a system of algebraic equations, thus greatly simplifying the problems. Many differential equations have their solutions in the form of an orthogonal polynomial such as Legender, Chebyshev, Bessel, Bernoulli and Laguerre differential equations (Fathy et al., 2014; El-Gamel and Abd El-Hady, 2017; El-Gamel and Sameh, 2013; El-Gamel, 2012; Yuzbasi et al., 2011; El-Gamel and Adel, 2019).

Laguerre-collocation method was used to solve various types of equations. It was used to solve high-order Fredholm integro-differential equations, pantograph type volterra integro-differential equation, initial value problems of second order, high-order nonlinear ordinary differential equations, Lane-Emden type functional differential equations, linear delay difference equations, Fredholm integro-differential equations with functional arguments and second-order nonlinear ODE (Savasaneril and Sezer, 2016; Yzba, 2014; Yan and Guo, 2011; Grbz and Sezer, 2016; Grbz and Sezer, 2014; Glsu et al., 2011; Grbz et al., 2014; Burcu, 2020).

This paper is organized as follows: Sect. 2, below briefly references, in which the reader can find an excellent summary of the basic properties of

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Laguerre polynomi-als, along with their proofs. In Sect. 3 we illustrate Laguerre-collocation method using the matrix form of each part of the delay differential equation (1). In Sect. 4 we provide residual error for the proposed method. In Sect. 5 we apply Laguerre method to some numerical examples to show the efficiency of the method. In the last Sect., we give the conclusion of our work.

2. PRELIMINARIES

As was already mentioned in the above introduction, we have excluded the pre-sentation of Laguerre methods, in order to save space, deferring instead to the excellent references in which Laguerre methods along with their proofs are given (Askey, 1975; Marcelln, 2006; Gbrz and Sezer, 2016).

3. MATRIX RELATIONS AND METHODOLOGY

We assume the approximate solution of the problem (1)-(2) in the truncated Laguerre series form:

$$u(t) = \sum_{n=0}^{N} a_n L_n(t) \tag{3}$$

Where $L_n(t)$ denotes the Laguerre polynomials

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(e^{-t}t^n), \quad n = 0,1,2,\ldots$$

And a_n $(n = 0,1,\ldots,N)$ are unknown Laguerre polynomial coefficients, and N is chosen as any positive integer, $N \geq 2$ such that

$$L_0(t) = 1, \qquad L_1(t) = 1 - t$$

$$L_2(t) = 1 - 2t + \frac{1}{2}t^2$$

$\ldots = \cdots$ and so on.

First, we can write (3) in the matrix form

$$[u(t)] = \mathbf{L}(t)\mathbf{A} \tag{4}$$

Where

$\mathbf{L}(t) = [L_0(t) \quad L_1(t) \ldots \quad L_N(t)]$, and $\mathbf{A} = [\boldsymbol{a_0}, \boldsymbol{a_1}, \ldots, \boldsymbol{a_N}]$

then, we use the matrix relation

$$\mathbf{L}(t) = \mathbf{X}(t)\mathbf{H}^\tau \tag{5}$$

where

$\mathbf{X}(t) = [1 \quad t \quad t^2 \ \ldots \ t^N]$

and $$\mathbf{H} = \begin{bmatrix} \frac{(-1)^0}{0!}\binom{0}{0} & 0 & 0 & \ldots & 0 \\ \frac{(-1)^0}{0!}\binom{1}{0} & \frac{(-1)^1}{1!}\binom{1}{1} & 0 & \ldots & 0 \\ \frac{(-1)^0}{0!}\binom{2}{0} & \frac{(-1)^1}{1!}\binom{2}{1} & \frac{(-1)^2}{2!}\binom{2}{2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^0}{0!}\binom{N}{0} & \frac{(-1)^1}{1!}\binom{N}{1} & \frac{(-1)^2}{2!}\binom{N}{2} & \ldots & \frac{(-1)^N}{N!}\binom{N}{N} \end{bmatrix}$$

The matrix relations of $[u(t)]$ and its derivatives are defined by

$[u(t)] = \mathbf{X}(t)\mathbf{H}^\tau\mathbf{A}$

$[u'(t)] = \mathbf{L}'(t)\mathbf{A} = \mathbf{X}(t)\mathbf{B}\mathbf{H}^\tau\mathbf{A}$,

$[u''(t)] = \mathbf{L}''(t)\mathbf{A} = \mathbf{X}(t)\mathbf{B}^2\mathbf{H}^\tau\mathbf{A}$,

$\vdots = \vdots$

$$[u^{(m)}(t)] = \mathbf{L}^{(m)}(t)\mathbf{A} = \mathbf{X}(t)\mathbf{B}^m\mathbf{H}^\tau\mathbf{A} \tag{6}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{7}$$

We need the following lemma

***Lemma* 3. 1**. The following two relations are hold

$$\left[u\left(t - \delta_j(t)\right)\right] = \mathbf{X}(t)\mathbf{G}\mathbf{H}^\tau\mathbf{A} \tag{8}$$

$$\left[u^{(m)}\left(t - \delta_j(t)\right)\right] = \mathbf{X}(t)\mathbf{G}\mathbf{B}^m\mathbf{H}^\tau\mathbf{A}$$

where

$$\mathbf{G} = \begin{bmatrix} \binom{0}{0}(-\delta_j(t))^0 & \binom{1}{0}(-\delta_j(t))^1 & \binom{2}{0}(-\delta_j(t))^2 & \cdots & \binom{N}{0}(-\delta_j(t))^N \\ 0 & \binom{1}{1}(-\delta_j(t))^0 & \binom{2}{1}(-\delta_j(t))^1 & \cdots & \binom{N}{1}(-\delta_j(t))^{N-1} \\ 0 & 0 & \binom{2}{2}(-\delta_j(t))^0 & \cdots & \binom{N}{2}(-\delta_j(t))^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N}(-\delta_j(t))^0 \end{bmatrix}$$

We obtain the following theorem

***Theorem* 3.1.** If the assumed approximate solution of high-order delay linear differential equations (1)-(2) is (3), then the matrix form of the discrete Laguerre- collocation system for the determination of the unknown coefficients **A** is given by:

$$\left(\mathbf{X}\mathbf{B}^m\mathbf{H}^\tau - \sum_{i=0}^{m-1}\sum_{j=1}^{J}\mathbf{P}_{ij}\widetilde{\mathbf{X}}\widetilde{\mathbf{G}}_j\mathbf{B}^i\mathbf{H}^\tau\right)\mathbf{A} = \mathbf{F} \tag{9}$$

where

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}(t_0) \\ \mathbf{X}(t_1) \\ \vdots \\ \mathbf{X}(t_N) \end{bmatrix} = \begin{bmatrix} 1 & t_0 & {t_0}^2 & \cdots & {t_0}^N \\ 1 & t_1 & {t_1}^2 & \cdots & {t_1}^N \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & t_N & {t_N}^2 & \cdots & {t_N}^N \end{bmatrix},$$

$$\mathbf{P}_{ij} = \begin{bmatrix} P_{ij}(t_0) & 0 & \cdots & \cdots & 0 \\ 0 & P_{ij}(t_1) & \cdots & \cdots & 0 \\ 0 & 0 & P_{ij}(t_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & P_{ij}(t_N) \end{bmatrix}$$

$$\widetilde{\mathbf{X}} = \begin{bmatrix} \mathbf{X}(t_0) & 0 & \cdots & 0 \\ 0 & \mathbf{X}(t_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}(t_N) \end{bmatrix}, \ \widetilde{\mathbf{G}}_j = \begin{bmatrix} G_j(t_0) \\ G_j(t_1) \\ \vdots \\ G_j(t_N) \end{bmatrix} \text{ and } \mathbf{F} = \begin{bmatrix} f(t_0) \\ f(t_1) \\ \vdots \\ f(t_N) \end{bmatrix}$$

In equation (9) the dimensions of matrices $\mathbf{X}, \mathbf{B}, \mathbf{H}, \mathbf{P}_{ij}, \widetilde{\mathbf{X}}, \widetilde{\mathbf{G}}_j$ and $\mathbf{A}$ are $(N+1)\times(N+1), (N+1)\times(N+1), (N+1)\times(N+1), (N+1)\times(N+1), (N+1)\times(N+1)^2, (N+1)^2\times(N+1)$ and $(N+1)\times 1$.

Proof. Replacing the terms of (1) with the appropriate representation defined in (3) and (6) and applying the collocation points

$$t_i = a + \frac{b-a}{N}i, \quad i = 0,1,2,\ldots,N \tag{10}$$

to it, we have fundamental matrix equation of equation (9).

Substituting the relation (6) into equation (2) we have the matrix form of mixed conditions

$$\mathbf{U}_i\mathbf{A} = [\lambda_i] \text{ or } [\mathbf{U}_i;\lambda_i], \ i = 0,1,\ldots,m-1 \tag{11}$$

such that:

$$\mathbf{U}_i = \sum_{k=0}^{m-1}[a_{ik}\mathbf{X}(a) + b_{ik}\mathbf{X}(b)]\mathbf{B}^k\mathbf{H}^\tau = [u_{i0} \quad u_{i1} \quad u_{i2} \quad \cdots \quad u_{iN}]$$

$$i = 0,1,\ldots,m-1 \tag{12}$$

Consequently, in order to get the solution of equation (1) under the mixed conditions (2), we replace the last m rows of the matrix (9) by the row matrices (11). Thus we get the new augmented matrix

$$\tilde{\boldsymbol{\varphi}}\mathbf{A} = \tilde{\mathbf{F}} \quad \leftrightarrow \quad [\tilde{\boldsymbol{\varphi}}; \tilde{\mathbf{F}}] \tag{13}$$

$$[\tilde{\boldsymbol{\varphi}}; \tilde{\mathbf{F}}] = \begin{bmatrix} \varphi_{00} & \varphi_{01} & \varphi_{02} & \cdots & \varphi_{0N} & ; & f(t_0) \\ \varphi_{10} & \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1N} & ; & f(t_1) \\ \varphi_{20} & \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2N} & ; & f(t_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ \varphi_{(N-m)0} & \varphi_{(N-m)1} & \varphi_{(N-m)2} & \cdots & \varphi_{(N-m)N} & ; & f(t_{N-m}) \\ u_{00} & u_{01} & u_{02} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & u_{12} & \cdots & u_{1N} & ; & \lambda_1 \\ u_{20} & u_{21} & u_{22} & \cdots & u_{2N} & ; & \lambda_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ u_{(N-m)0} & u_{(N-m)1} & u_{(N-m)2} & \cdots & u_{(N-m)N} & ; & \lambda_{N-m} \end{bmatrix}$$

Solving that linear system of equations (13) results in the values of the unknown Laguerre coefficients $a_0, a_1, \dots, a_N$. Thus, we get the solution of equation (1).

Algorithm

- Input (integer) a, b and N
- Input (double) tol.
- Input (array) $\mathbf{G}$.
- Solve the system $\tilde{\boldsymbol{\varphi}}\mathbf{A} = \tilde{\mathbf{F}}$
- If $|u_N(t_i) - u(t_i)| < tol$ the program ends.
- If else, increase N

4. The Residual Error

The truncated Laguerre series (3) is an approximate solution for (1) under the mixed conditions (2). Error is estimated as the residual function $R_N(t)$ which may be calculated as

$$R_N(t) = u_N{}^{(m)}(t) - \sum_{i=0}^{m-1} \sum_{j=1}^{J} P_{ij}(t) u_N(t - \delta_j(t))^{(i)} - f(t)$$

Substituting the collocation points in (14) and evaluating its absolute, we get the value of the absolute error at each point $R_N(t_i)$. We change N until we reach an acceptable limit of the error that $|R_N(t_i)| \to 0$

5. Numerical Examples

Four numerical examples are given to illustrate the accuracy and effectiveness properties of the method. The absolute errors are given by:

$$\|E_{Laguerre}\| = |u(t) - u_N(t)|$$

Example 1: Consider the following second-order pantograph type delay differential equation:

$$u''(t) + u'(t - e^t) + 2u(t) = 2(t^2 + t + 1) - 2e^t \,, \qquad 0 \le t \le 1$$

Subject to the initial conditions

$$u(0) = 0 \;\; and \;\; u'(0) = 0$$

Using equation (3) and with $N = 3$, solution of the problem may be approximated as

$$u(t) \simeq \sum_{n=0}^{3} a_n L_n(t) \,, \qquad 0 \le t \le 1$$

Collocation points t_i are calculated using (10):

$$t_0 = 0, \quad t_1 = \tfrac{1}{3}, \quad t_2 = \tfrac{2}{3}, \quad t_3 = 1$$

The fundamental matrix equation of the problem is constructed as follows:

$$[\mathbf{X}\mathbf{B}^2\mathbf{H}^\tau - (\mathbf{P}_{01}\mathbf{X}\mathbf{H}^\tau + \mathbf{P}_{12}\tilde{\mathbf{X}}\tilde{\mathbf{G}}_2\mathbf{B}\mathbf{H}^\tau)]\mathbf{A} = \mathbf{F}$$

With: $\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{9} & \frac{1}{27} \\ 1 & \frac{2}{3} & \frac{4}{9} & \frac{8}{27} \\ 1 & 1 & 1 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 0.5 & 0 \\ 1 & -3 & 1.5 & -0.1667 \end{bmatrix}$

$$\mathbf{P_{01}} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \qquad \mathbf{P_{12}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\tilde{\mathbf{X}} = \begin{bmatrix} \mathrm{X}(0) & 0 & 0 & 0 \\ 0 & \mathrm{X}\left(\frac{1}{3}\right) & 0 & 0 \\ 0 & 0 & \mathrm{X}\left(\frac{2}{3}\right) & 0 \\ 0 & 0 & 0 & \mathrm{X}(1) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{9} & \frac{1}{27} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & \frac{4}{9} & \frac{8}{27} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\tilde{\mathbf{G}}_2 = \begin{bmatrix} \mathbf{G}_2(0) \\ \mathbf{G}_2\left(\frac{1}{3}\right) \\ \mathbf{G}_2\left(\frac{2}{3}\right) \\ \mathbf{G}_2(1) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 1 & -1.3956 & 1.9477 & -2.7183 \\ 0 & 1 & -2.7912 & 5.8432 \\ 0 & 0 & 0 & 1 \\ 1 & -1.9477 & 3.7937 & -7.3891 \\ 0 & 1 & -3.8955 & 11.3810 \\ 0 & 0 & 1 & -5.8432 \\ 0 & 0 & 0 & 1 \\ 1 & -2.7183 & 7.3891 & -20.0855 \\ 0 & 1 & -5.4366 & 22.1672 \\ 0 & 0 & 1 & -8.1548 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The augmented matrix of that system of equations is

$$\boldsymbol{\varphi} = \begin{bmatrix} 2 & 1 & 0 & -1.5 & ; & 0 \\ 2 & 0.3333 & -1.2845 & -3.7634 & ; & 0.0977 \\ 2 & -0.3333 & -2.5033 & -6.0959 & ; & 0.3268 \\ 2 & -1 & -3.7183 & -8.9644 & ; & 0.5634 \end{bmatrix}$$

Replacing the last two rows of the augmented matrix with the row matrices of initial conditions, then the new augmented matrix is:

$$\tilde{\boldsymbol{\varphi}} = \begin{bmatrix} 2 & 1 & 0 & -1.5 & ; & 0 \\ 2 & 0.3333 & -1.2845 & -3.7634 & ; & 0.0977 \\ 0 & -1 & -2 & -3 & ; & 0 \\ 1 & 1 & 1 & 1 & ; & 0 \end{bmatrix}$$

Solving that system of equations, the coefficient matrix $\mathbf{A}$ is obtained as

$$\mathbf{A} = \begin{bmatrix} 2 \\ -4 \\ 2 \\ 0 \end{bmatrix}$$

so, the solution $u(t)$ is calculated using equation (6) to be $u(t) = t^2$ which is the exact solution.

Example 2: Consider the following delay differential equation (Savaaneril and Sezer, 2017; Zel et al., 2018):

$$u' + u(t - \ln[t^2 + 1]) + u(t) = (t^2 + 1)e^{-t}. \;\; 0 \le t \le 2$$

with initial condition $u(0) = 1$. The exact solution of that problem $u(t) = e^{-t}$

Using the procedure described earlier with $N = 13$, we compare our results ($\|E_{Laguerre}\|$ and $\|E_{Residual}\|$)

with those of HTL method and Morgan-Voyce method with $N = 6$, in Table 1 (Savaaneril and Sezer, 2017; Zel et al., 2018). The exact solution is shown with Laguerre solution at $N = 13$, in Figure 1. The figure shows that our results are coincident with the exact solution.

Table 1: Error results of Example 2				
t	$\|E_{Laguerre}\|, N = 13$	$\|E_{Residual}\|$	$\|E_{HTL}\|$, $N=6$, (*Savaaneril and Sezer*, 2017)	$\|E_{Morgan-Voyce}\|$, $N=6$ *Zel et al.*, 2018
0	6.1166E-15	1.9525E-11	6.7000E-08	5.5511E-15
0.2	4.9312E-13	3.5144E-13	1.0893E-02	3.1045E-06
0.4	2.8840E-13	3.2742E-15	2.0925E-02	2.0328E-06
0.6	1.5186E-13	1.9964E-13	1.9731E-02	1.0916E-06
0.8	1.1233E-14	2.1176E-13	1.2556E-02	4.4865E-06
1	9.2516E-14	2.8978E-13	6.2690E-03	1.1401E-06
1.2	2.3033E-13	5.7626E-13	2.4134E-03	5.7233E-06
1.4	3.3740E-13	4.2514E-13	2.7172E-03	9.1788E-06
1.6	4.8493E-13	1.0988E-12	1.0988E-02	1.2287E-06
1.8	5.2762E-13	5.6556E-13	8.3206E-03	1.4348E-06
2	9.8100E-13	2.1734E-12	5.6432E-02	1.4414E-06

Example 3: Consider the Pantograph equation of third order (Savaaneril and Sezer, 2017):

$$u'''(t) - u''(t - t^2) + u(t) = t - e^{t^2-1}, \quad 0 \le t \le 1$$

$$u(0) = 1, \;\; u'(0) = 0 \;\text{ and } u''(0) = 1$$

The exact solution is $u(t) = t + e^{-t}$.

Results of our method for $N = 12$ are compared to exact solution and HTL method for $N = 10$ and show in Table 2. The exact and Lageurre solutions are shown in Figure 2 (Savaaneril and Sezer, 2017).

Example 4: Consider the Pantograph equation of third order (Sedaghat et al., 2012; Yalinba et al., 2011; Sezer and Akyz-Dacoglu, 2007):

$$u'''(t) = -u(t) - u(t - 0.3) + e^{-(t-0.3)}, \quad 0 \le t \le 1$$

$$u(0) = u''(0) = 1 \qquad \text{and } u'(0) = -1$$

The exact solution is $u(t) = e^{-t}$.

Maximum absolute errors of our method for $N = 8$ are compared to Taylor method, Chebyshev method and Hermit method in Table 3 (Sezer and Akyz-Dacoglu, 2007; Sedaghat et al., 2012; Yalinba et al., 2011). The exact and Laguerre solutions are shown in Figure 3.

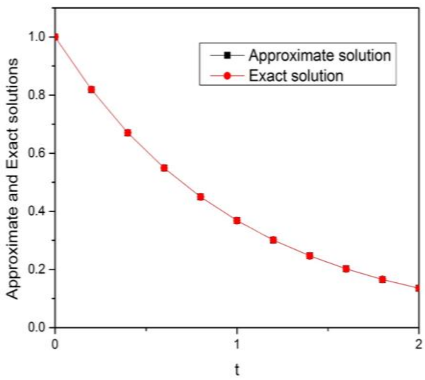

Figure 1: Graphics of the exact solution and Laguerre solution at *N=13* for **Example 2**

Table 2: Results of Example 3				
t	Exact Solution	$\|E_{Laguerre}\|, N = 12$	$\|E_{HTL}\|, N = 10$ (Savaaneril and Sezer, 2017)	$\|E_{Residual}\|$
0	1	0	0	0
0.1	1.004837418	4.6187E-12	1.9000E-08	7.2812E-11
0.2	1.018730753	3.9687E-12	2.1500E-07	1.8639E-11
0.3	1.040818221	3.4833E-12	4.3730E-06	1.6039E-11
0.4	1.070320046	3.1582E-12	2.6149E-05	3.8887E-12
0.5	1.106530660	3.0007E-12	9.8068E-05	2.2760E-12
0.6	1.148811636	3.0272E-12	2.8153E-04	1.0128E-11
0.7	1.196585304	3.2472E-12	6.7721E-04	1.4316E-11
0.8	1.249328964	3.6617E-12	1.4348E-03	3.3693E-12
0.9	1.306569660	4.2782E-12	2.7615E-03	2.9878E-11
1	1.367879441	5.1240E-12	4.9284E-03	5.7968E-12

Table 3: Maximum absolute errors of Example 4				
$\|E_{Laguerre}\|$	Chebychev method (Yalinba et al., 2011)	Hermit method (Yalinba et al., 2011)	Taylor method (Sezer and Akyz-Dacoglu, 2007)	$\|E_{Residual}\|$
9.2238E-14	3.70E-07	6.200E-09	8.54E-08	3.7893E-13

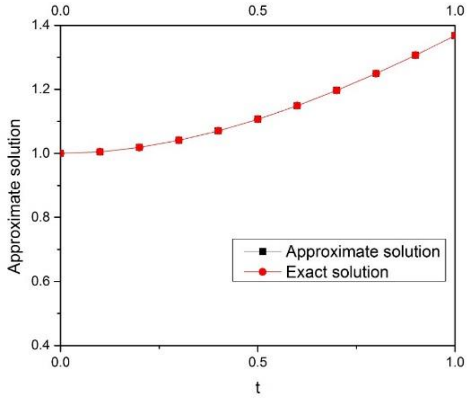

Figure 2: Graphics of the exact solution and Laguerre solution at *N=13* for **Example 3**

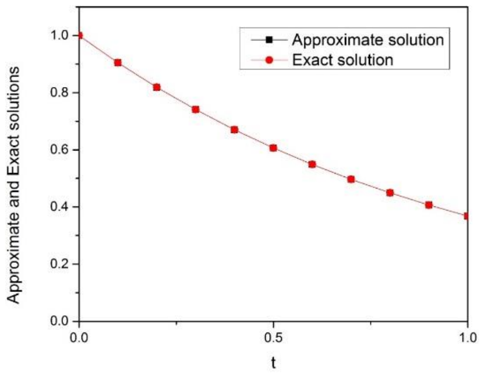

Figure 3: Graphics of the exact solution and Laguerre solution at *N=8* for Example 4

6. Conclusion

Laguerre-collocation method is an efficient method to solve differential equations with constant and variable delays. We have presented numerical examples of equations of various orders and concluded that Laguerre method can solve them with less absolute error than other methods. The obtained values for these examples are shown in different tables and the ability of Laguerre method in solving delay differential equations is presented.

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References

Ajello, W., Freedman, H., Wu, J., 1992. A model of stage structured population growth with density depended time delay. SIAM J. Appl. Math., 52**,** Pp. 855-869.

Akkaya, T., Yalinba, S., Sezer, M., 2013. Numeric solutions for the pantograph type delay differential equation using First Boubaker polynomials, Appl. Math. Comput., 219, Pp. 9484-9492.

Askey, R., 1975. Orthogonal Polynomials and Special Functions, Siam.

Bellen, A., Zennaro, M., 2013. Numerical methods for delay differential equations. Oxford university press.

Buhmann, M., Iserles, A., 1993. Stability of the discretized pantograph differential equation. Math. Comput., 60, Pp. 575-589.

Burcu, G., 2020. A computational approach for solving second-order nonlinear ordinary differential equations by means of Laguerre series. BEU J. Sci., 9, Pp. 78-84.

Chen, X., Wang, L., 2010. The variational iteration method for solving a neutral functional- differential equation with proportional delays. Comput. Math. Appl., 59, Pp. 2696-2702.

Doha, E., Bhrawy, A., Baleanu, D., Hafez, R., 2014. A new Jacobi rational Gauss collocation method for numerical solution of generalized pantograph equations. Appl. Numer. Math., 77, Pp. 43-54.

El-Gamel, M., Abd El-Hady, M., 2017. Numerical solution of the Bagley-Torvik equation by Legendre-collocation method. SeMA Journal, 74, Pp. 371-383.

El-Gamel, M., 2012. An efficient technique for finding the eigenvalues of fourth -order Sturm-Liouville problems. Applied Mathematics, 3, ID:21484.

El-Gamel, M., Adel, W., 2019. Bernoulli polynomial and the numerical solution of high-order boundary value problems. Mathematics in Natural Science, 4, Pp. 14.

El-Gamel, M., Sameh, M., 2013. A Chebychev collocation method for solving Troeschs problem. International Journal of Mathematics and Computer Applications Research (IJMCAR), 3, Pp. 23-32.

Fathy, M., El-Gamel, M., El-Azab, M.S., 2014. Legendre-Galerkin method for the linear Fredholm integro-differential equations, Appl. Math. d Comput., 243, Pp. 789-800.

Glsu, M., Grbz, B., Ztrk, Y., Sezer, M., 2011. Laguerre polynomial approach for solving linear delay difference equations. Appl. Math. Comput., 217, Pp. 6765-6776.

Glsu, M., Sezer, M., 2011. A taylor collocation method for solving highorder linear pantograph equations with linear functional argument. Numer. Meth. Part. D. E., 27, Pp. 1628-1638.

Grbz, B., Sezer, M., 2014. Laguerre polynomial approach for solving Lane-Emden type functional differential equations. Appl. Math. Comput., 242, Pp. 255-264.

Grbz, B., Sezer, M., Gler, C., 2014. Laguerre collocation method for solving Fredholm integro- differential equations with functional arguments, J. Appl. Math.,

Grbz, B., Sezer, M., 2016. Laguerre polynomial solutions of a class of initial and boundary value problems arising in science and engineering fields. Acta Phys. Pol. A., 130, Pp. 194-197.

Hafshejani, M., Vanani, S., Hafshejani, J., 2011. Numerical solution of delay differential equations using Legendre wavelet method. World Appl. Sci. J., 13, Pp. 27-33.

Liu, M., Yang, Z., Xu, Y., 2006. The stability of modified Runge-Kutta methods for the pantograph equation. Math. Comput., 75, Pp. 1201-1215.

Marcelln, F., 2006. Orthogonal Polynomials and Special Functions: Computation and Applica- tions. Springer Science and Business Media.

Nemati, S., Lima, P., Sedaghat, S., 2020. Legendre wavelet collocation method combined with the Gauss-Jacobi quadrature for solving fractional delay-type integro-differential equations. Appl. Numer. Math., 149, Pp. 99-112.

Nevai, P., 1994. Orthogonal Polynomials. In Linear and Complex Analysis Problem Book, Springer, Berlin, Heidelberg.

Reutskiy, S., 2015. A new collocation method for approximate solution of the pantograph functional differential equations with proportional delay. Appl. Math. Comput., 266, Pp. 642-655.

Saaty, T., 2012. Modern nonlinear equations, Courier Corporation.

Savaaneril, N., Sezer, M., 2017. Hybrid Taylor Lucas collocation method for numerical solution of high-order pantograph type delay differential equations with variables delays. Appl. Math. Inf. Sci., 11, Pp. 1795-1801.

Savasaneril, N., Sezer, M., 2016. Laguerre polynomial solution of high-order linear Fredholm integro- differential equations. New Trends Math. Sci., 4, Pp. 273-284.

Sedaghat, S., Ordokhani, Y., Dehghan, M., 2012. Numerical solution of the delay differential equations of pantograph type via Chebyshev polynomials. Commun. Nonlinear Sci., 12, Pp. 4815- 4830.

Sezer, M., Akyz-Dacoglu, A., 2007. A Taylor method for numerical solution of generalized pan- tograph equations with linear functional argument. J. Comput. Appl. Math., Pp. 217-225.

Shahmorad, S., Ostadzad, M., Baleanu, D., 2020. A Tau-like numerical method for solving fractional delay integro-differential equations. Appl. Numer. Math., 151, Pp. 322-336.

Trif, D., 2012. Direct operatorial tau method for pantograph-type equations. Appl. Math. Comput., 219, Pp. 2194-2203.

Usman, M., Hamid, M., Zubair, T., Haq, U., Wang, W., Liu, B., 2020. Novel operational matrices- based method for solving fractional-order delay differential equations via shifted Gegenbauer polynomials. Appl. Math. Comput., 372, Pp. 124985.

Wang, W., Li, S., 2007. On the one-leg θ-methods for solving nonlinear neutral functional differential equations. Appl. Math. Comput., 193, Pp. 285-301.

Yalinba, S., Aynig, M., Sezer, M., 2011. A collocation method using Hermite polynomials for approx- imate solution of pantograph equations. J. Franklin I., 348, Pp. 1128-1139.

Yan, J., Guo, B., 2011. A collocation method for initial value problems of second-order ODEs by using Laguerre functions. Numer. Math. Theory Meth., 4, Pp. 283-295.

Yuzbasi, S., Sahin, N., Sezer, M., 2011. Numerical solutions of system of linear Fredholm integro-differential equations with Bessel polynomial bases. Computers Math. Appl., 61, Pp. 3079-3096.

Yzba, Sezer, M., 2013. An exponential approximation for solutions of generalized pantograph-delay differential equations. Appl. Math. Model., 37, Pp. 9160-9173.

Yzba, 2014. Laguerre approach for solving pantograph-type Volterra integro-differential equations. Appl. Math. Comput., 232, Pp. 1183-199.

Zel, M., Tarak, M., Sezer, M., 2018. A numerical approach for a nonhomogeneous differential equation with variable delays. Appl. Math. Sci., 12, Pp. 145-155.