A NOTE ON COMMUTATIVITY OF PRIME NEAR RING WITH GENERALIZED \( \beta \)-DERIVATION

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ABSTRACT

In this paper, we prove commutativity of prime near rings by using the notion of \( \beta \)-derivations. Let \( M \) be a prime near ring. If there exist \( u_1, u_2 \in M \) and two sided generalized \( \beta \)-derivation \( G \) associated with the non-zero two sided \( \beta \)-derivation \( g \) on \( M \), where \( \beta : M \to M \) is a homomorphism, satisfying the following conditions:

i. \( G([p_1, q_1]) = p_1 \cdot G(p_1) \cdot G(q_1) \cdot p_1 \in M \) \( \forall p_1, q_1 \in M \)

ii. \( G([p_1, q_1]) = p_1 \cdot G(p_1) \cdot G(q_1) \cdot p_1 \in M \) \( \forall p_1, q_1 \in M \)

Then \( M \) is a commutative ring.

KEYWORDS

\( \beta \)-derivations, commutativity, homomorphism.

1. INTRODUCTION

A non-empty set \( M \) with two binary operations namely addition and multiplication is said to be a right near ring if \( M \) under addition is a group, \( M \) under multiplication is a semi group and \([M, +, \cdot] \) satisfies right distributive law. If \( M_1 = \{ p \in M : p_1 = 0 \} \) then \( M_2 \) is called zero symmetric near ring. For all \( p_1, q_1 \in M, p_1 M q_1 = 0 \), this implies that \( p_1 = 0 \) or \( q_1 = 0 \), then \( M \) is called prime near ring (Bell and Mason, 1987). A mapping \( G : M \to M \) is known as two sided \( \beta \)-derivation if there is a \( \beta : M \to M \) such that \( G(p_1 q_1) = G(p_1) \beta(q_1) + \beta(p_1) q_1 \) and \( G(q_1 p_1) = \beta(p_1) \beta(q_1) + G(q_1) \) for all \( p_1, q_1 \in M \). A mapping \( G : M \to M \) is said to be two sided generalized \( \beta \)-derivation if there is a \( \beta : M \to M \) such that \( G(p_1 q_1) = G(p_1) \beta(q_1) + \beta(p_1) q_1 \) or equivalently \( G(p_1 q_1) = \beta(p_1) \beta(q_1) + G(q_1) \) for all \( p_1, q_1 \in M \). Let \( M \) be a near ring whose center \( Z(M) \) is defined as: \( Z(M) = \{ c \in M : c m = mc, \forall m \in M \} \).

For any \( p_1, q_1 \in M \), \( p_1 q_1, q_1 p_1 \) and \( [p_1, q_1] = p_1 q_1 - q_1 p_1 \) is known as Jordan and lie product, respectively. In recent years, various mathematicians have studied derivation and generalized derivation for commutativity of prime, semi prime ring and \( \Gamma \) rings (Albas and Argac, 2004; Basudeb, 2010; Daif and Bell, 1992; De Filippis and Rehman, 2010; Golbasi and Koc, 2011; Golbasi and Koc, 2009; Khan et al., 2013; Khan et al., 2013; Quadri et al., 2003). In near ring, some comparable result has also been derived (Asharf and Shakir, 2008; Ashraf et al., 2004; Beidar et al., 1996; Bell and Mason, 1992; Boua and Oukhtite, 2011; Kamal, 2001; Raina et al., 2009; Yukman, 2007; Khan et al., 2021).

In this paper, we investigate results for prime near ring involving two sided generalized \( \beta \)-derivation.

2. MAIN RESULTS

Theorem 2.1. Let \( M \) be a prime near ring, If there exist \( u_1, u_2 \in M \) and two sided generalized \( \beta \)-derivation \( G \) associated with the non-zero two sided \( \beta \)-derivation \( g \) on \( M \), where \( \beta : M \to M \) is a homomorphism, satisfying the following conditions:

i. \( G([p_1, q_1]) = p_1 \cdot G(p_1) \cdot G(q_1) \cdot p_1 \in M \) \( \forall p_1, q_1 \in M \)

ii. \( G([p_1, q_1]) = p_1 \cdot G(p_1) \cdot G(q_1) \cdot p_1 \in M \) \( \forall p_1, q_1 \in M \)

Then \( M \) is a commutative ring.

Proof. i. Since \( G([p_1, q_1]) = p_1 \cdot G(p_1) \cdot G(q_1) \cdot p_1 \in M \) \( \forall p_1, q_1 \in M \)

Also \( [p_1, q_1] = [p_1, q_1] \cdot p_1, \) replacing \( q_1 \) by \( q_1 p_1 \) in equation (1), we obtain \( G([p_1, q_1]) = G([p_1, q_1]) \).

This gives \( G([p_1, q_1]) = p_1 \cdot G(p_1) \cdot G(q_1) \cdot p_1 \in M \).

Since \( \beta \) is a homomorphism, so the last relation implies...
\[ G(p_0 q_0) = p_0^{u_0} | \beta(p_0) | \beta(q_0) | \beta(p_0) | p_0^{r_0}. \]

This gives \[ G([p_0, q_0]) = p_0^{u_0} | \beta(p_0) | \beta(q_0) | \beta(p_0) | p_0^{r_0} \quad \forall \quad p_0, q_0 \in M \] (2)

By using definition of generalized \( \beta \)-derivation, we have
\[ G([p_0, q_0]) = G(p_0, q_0) \beta(p_0) + \beta([p_0, q_0]) \beta(p_0) \]

By using equation (1) and equation (2), we obtain
\[ p_0^{u_0} | \beta(p_0) | \beta(q_0) | \beta(p_0) | p_0^{r_0} = p_0^{u_0} | \beta(p_0) | \beta(q_0) | p_0^{r_0} \beta(p_0) + | \beta(p_0) | \beta(q_0) | \beta(p_0) | p_0^{r_0} \]

From this we arrive at
\[ | \beta(p_0) | \beta(q_0) | \beta(p_0) | p_0^{r_0} = 0 \]

Replacing \( q_1 \) by \( r_1 \), we get
\[ | \beta(p_0) | \beta(r_1 q_1) | \beta(p_0) | p_0^{r_0} = 0 \]

Since \( \beta \) is a homomorphism, we have
\[ | \beta(p_0) | \beta(r_1 q_1) | \beta(p_0) | p_0^{r_0} = 0 \]

This implies
\[ | \beta(p_0) | \beta(r_1) | \beta(p_0) | q_0 | p_0^{r_0} = 0 \quad \forall \quad p_0, q_0, r_1 \in M \]

The last relation gives
\[ | \beta(p_0) | \beta(r_1) | \beta(p_0) | q_0 | p_0^{r_0} = 0 \quad \forall \quad p_0, r_1 \in M \]

Since \( M \) is prime then for each \( p_0 \in M \), we get \( g(p_0) = 0 \) or \( | \beta(p_0) | \beta(r_1) | = 0 \). Since \( g \) is a nonzero two sided \( \beta \)-derivation, we have
\[ | \beta(p_0) | \beta(r_1) | = 0 \quad \forall \quad p_0 \in Z(M) \]

This along with Lemma [3, khan et.al., 2021], we get \( g(M) = Z(M) \)

Hence \( M \) is a commutative ring.

Let \( X \) be a commutative ring and
\[ M = \begin{pmatrix} 0 & p_1 & q_1 \\ 0 & 0 & r_1 \\ 0 & 0 & 0 \end{pmatrix} \]

We define the following mappings on \( M \).

Let \( \beta: M \rightarrow M \) is a mapping defined by:
\[ \beta \begin{pmatrix} 0 & p_1 & q_1 \\ 0 & 0 & r_1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

and \( g: M \rightarrow M \) is a mapping defined by:
\[ g \begin{pmatrix} 0 & p_1 & q_1 \\ 0 & 0 & r_1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

Then \( g \) is nonzero \( \beta \)-derivation on \( M \). If \( A = \begin{pmatrix} 0 & 0 & -r_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), then \( AM = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)

which proves that \( M \) is not prime. Moreover, if \( G \begin{pmatrix} 0 & p_1 & q_1 \\ 0 & 0 & r_1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), then \( G \) is a generalized \( \beta \)-derivation on \( M \) which satisfies the condition \( G([A, B]) = [A, B] \) \quad \forall \quad A, B \in M \).

**Theorem 2.2.** Let \( M \) be a prime near ring and \( G \) a two-sided generalized \( \beta \)-derivation associated with the non-zero two sided \( \beta \)-derivation \( g \) on \( M \), where \( \beta: M \rightarrow M \) is homomorphism. If there exist \( u_0, v_0 \in M \), then the following hold:

\[ iG(p_0 q_1) = p_0^{u_0} (| \beta(p_0) | \beta(q_1)) p_0^{r_0} \quad \forall \quad p_0, q_1 \in M \]

\[ iiG(p_0 q_1) = -p_0^{u_0} (| \beta(p_0) | \beta(q_1)) p_0^{r_0} \quad \forall \quad p_0, q_1 \in M \]

Then \( M \) is a commutative ring.

**Proof.** i. Since
\[ G(p_0 q_0) = p_0^{u_0} (| \beta(p_0) | \beta(q_0)) p_0^{r_0} \]

Also \( p_0 q_0 = \begin{pmatrix} p_1 q_0 \\ 0 & 0 & r_1 \end{pmatrix} \), replacing \( q_1 \) by \( v_0 q_1 \) in equation (7), we obtain
\[ G(p_0 q_1) = G(p_0 q_1) p_0 \]

This gives
\[ G(p_0 q_1) = p_0^{u_0} (| \beta(p_0) | \beta(q_0)) p_0^{r_0} \quad \forall \quad p_0, q_1 \in M \]

By using definition of generalized \( \beta \)-derivation, we have
\[ G([p_0, q_0]) = G(p_0, q_0) \beta(p_0) + \beta([p_0, q_0]) \beta(p_0) \]

By using equation (4) and equation (5), we obtain
\[ -p_0^{u_0} (| \beta(p_0) | \beta(q_0)) p_0^{r_0} \beta(p_0) = -p_0^{u_0} (| \beta(p_0) | \beta(q_0)) p_0^{r_0} \beta(p_0) + | \beta(p_0) | \beta(q_0) | \beta(p_0) | p_0^{r_0} \]

From this we arrive at
\[ | \beta(p_0) | \beta(q_0) | \beta(p_0) | p_0^{r_0} = 0 \]

Replacing \( q_1 \) by \( r_1 \), we get
\[ | \beta(p_0) | \beta(r_1 q_1) | \beta(p_0) | p_0^{r_0} = 0 \]

Since \( \beta \) is a homomorphism, we have
\[ | \beta(p_0) | \beta(r_1) | \beta(p_0) | q_0 | p_0^{r_0} = 0 \]

This implies
\[ | \beta(p_0) | \beta(r_1) | \beta(q_0) | p_0^{r_0} = 0 \quad \forall \quad p_0, q_0, r_1 \in M \]

The last relation gives
\[ | \beta(p_0) | \beta(r_1) | \beta(p_0) | q_0 | p_0^{r_0} = 0 \quad \forall \quad p_0, r_1 \in M \]

Since \( M \) is prime then for each \( p_0 \in M \), we get \( g(p_0) = 0 \) or \( | \beta(p_0) | \beta(r_1) | = 0 \).

Since \( g \) is a nonzero two sided \( \beta \)-derivation, we have \( | \beta(p_0) | \beta(r_1) | = 0 \) or \( r_1 \in Z(M) \). By using Lemma [2, khan et.al., 2021], we have \( g(p_0) \in Z(M) \). This along with Lemma [3, khan et.al., 2021], we get
\[ g(M) = Z(M) \]

Hence \( M \) is a commutative ring.
\[p_0 \cdot \beta(p_1 \cdot q_0 \cdot q_1) = p_0 \cdot (\beta(p_1 \cdot q_1) \cdot p_0) = p_0 \cdot \beta(p_1) \cdot q_1 \]

From this we arrive at
\[(\beta(p_1) \cdot q_1) \cdot p_0 = 0\]

This implies
\[(\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

From this we get
\[(\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

This gives
\[(\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

Replacing \(q_1\) by \(q_1 q_1\) in last equation, we get
\[(\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

Since \(\beta\) is a homomorphism, we have
\[(\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

By using equation (29), we obtain
\[(\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

Since \(\beta\) is a homomorphism, so the last relation implies
\[p_0 \cdot (\beta(p_1) \cdot q_1) = 0\]

This gives
\[(\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

This implies
\[(\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

Replacing \(p_1\) by \(-p_1\), we get
\[(\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

Since \(\beta\) is a homomorphism, we have
\[(-\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

From this we get
\[(-\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

This gives
\[(-\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

This implies
\[(-\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

The last relation gives
\[(\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

Hence \(M\) is a commutative ring.

\[G(p_0 \cdot q_1) = \beta(p_1) \cdot q_1 \]

This gives
\[G(p_0 \cdot q_1) = \beta(p_1) \cdot q_1 \]

Since \(\beta\) is a homomorphism, so we have
\[G(p_0 \cdot q_1) = \beta(p_1) \cdot q_1 \]

By using equation (11) and equation (12), we obtain
\[G(p_0 \cdot q_1) = \beta(p_1) \cdot q_1 \]

This gives
\[G(p_0 \cdot q_1) = \beta(p_1) \cdot q_1 \]

From this we get
\[G(p_0 \cdot q_1) = \beta(p_1) \cdot q_1 \]

This implies
\[G(p_0 \cdot q_1) = \beta(p_1) \cdot q_1 \]

Since \(\beta\) is a homomorphism, so we have
\[G(p_0 \cdot q_1) = \beta(p_1) \cdot q_1 \]

By using definition of generalized \(\beta\)-derivation, we have
\[G(p_0 \cdot q_1) = G(p_0) \cdot q_1 \]

By using equation (11) and equation (12), we obtain
\[G(p_0 \cdot q_1) = G(p_0) \cdot q_1 \]

From this we arrive at
\[(\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

This implies
\[(\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

From this we get
\[(\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

This implies
\[(\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

Replacing \(p_1\) by \(-p_1\), we get
\[(\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

Since \(\beta\) is a homomorphism, so we have
\[(-\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

From this we get
\[(-\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]

This implies
\[(-\beta(p_1) \cdot q_1) \cdot (\beta(p_1) \cdot q_1) = 0\]
This gives
\[-(β(p₁), β(r₁)) β(q₁)g(−p₁) = 0\]
This implies
\[\{β(p₁), β(r₁)\}β(q₁)g(−p₁) = 0 \quad \forall \ p₁, q₁, r₁ \in M\]
The last relation gives
\[\{β(p₁), β(r₁)\}Mg(−p₁) = 0 \quad \forall \ p₁, r₁ \in M\]
(14)
Since M is prime then for each \(p₁ \in M\), we get \(g(−p₁) = 0\) or \(\{β(p₁), β(r₁)\} = 0\). Since \(g\) is a nonzero two sided \(β\)-derivation, so we have \(\{β(p₁), β(r₁)\} = 0\) or \(p₁ \in Z(M)\), by using Lemma [2, khan et al., 2021], we have \(g(−p₁) = Z(M)\). This along with Lemma [3, khan et al., 2021], we get \(g(M) = Z(M)\).

Hence M is a commutative ring.

**Example 2.** Let X be a commutative ring and
\[M = \begin{pmatrix} p₁ & q₁ & r₁ \\ 0 & 0 & 0 \end{pmatrix} : p₁, q₁, r₁ \in X\].
We define the following mappings on M.

Let \(β: M \to M\) is a mapping defined by:
\[β = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : p₁, q₁, r₁ \in X\].
and \(g: M \to M\) is a mapping defined by:
\[g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : p₁, q₁, r₁ \in X\].

Then g is nonzero \(β\)-derivation on M. If \(A = \begin{pmatrix} 0 & 0 & r₁ - q₁ \\ 0 & 0 & 0 \end{pmatrix}\), then \(AMA = 0\) which proves that M is not prime. Moreover, if \(G = \begin{pmatrix} 0 & p₁ & q₁ \\ 0 & 0 & r₁ \end{pmatrix}\), then G is a generalized \(β\)-derivation on M which satisfies the condition \(G(AoB) = (AoB) \quad \forall \ A, B \in M\).

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