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RESEARCH ARTICLE

A NOTE ON β-DERIVATIONS IN PRIME NEAR RING

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ABSTRACT

In this paper, we prove commutativity of prime near rings by using the notion of β-derivations. Let M be a prime near ring. If there exist $p, q \in M$ and two sided nonzero β-derivation f on M, where $\beta: M \to M$ is a homomorphism, satisfying the following conditions:

i. $f([s,t]) = s^p[\beta(s),\beta(t)]s^q \quad \forall \ s,t \in M$
ii. $f([s,t]) = -s^p[\beta(s),\beta(t)]s^q \quad \forall \ s,t \in M$

KEYWORDS

β-derivations, commutativity, homomorphism.

1. INTRODUCTION

Throughout this paper M is a zero symmetric right near ring. If $M = \{s\epsilon M : 0s = 0\}$ then M is called zero symmetric. For all $s,t \in M$, $sMt = 0$, this implies that $s = 0$ or $t = 0$, then M is called prime near ring (Bell and Mason, 1987). A mapping $f: M \to M$ is known as two sided β-derivation if there is a function $\beta: M \to M$ such that $f(st) = f(s)\beta(t) + \beta(s)f(t)$ and $f(st) = \beta(s)f(t) + f(s)\beta(t)$ for all $s,t \in M$. Let M be a near ring whose center $Z(M)$ is defined as: $Z(M) = \{c\epsilon M : cm = mc, \ for\ all\ m\epsilon M\}$. For any $s,t \in M$, $sot = st + ts$ and $[s,t] = st - ts$ is known as Jordan and lie product, respectively.

In recent years, many authors contributed in the theory of rings, near rings and Γ-rings (Basudeb, 2010; Bell and Mason, 1992; Golbasi and Koc, 2009; Golbasi and Koc, 2011; Khan et al., 2013; Khan et al., 2013; Khan et al., 2016; Raina et al., 2009). Later they have investigated the properties for prime and semi prime near rings involving derivation (Ashraf et al., 2004; Ashraf and Shakir, 2008; Beidar et al., 1996; Daif and Bell, 1992; Kamal, 2001; Quadri et al., 2003; Wang, 1994). In this paper, we investigated results for prime near ring involving two sided β-derivation.

2. MAIN RESULTS

Theorem 2.1. Let M be a prime near ring. If there exist $p, q \in M$ and two sided nonzero β-derivation f on M, where $\beta: M \to M$ is a homomorphism, satisfying the following conditions:

i. $f([s,t]) = s^p[\beta(s),\beta(t)]s^q \quad \forall \ s,t \in M$
ii. $f([s,t]) = -s^p[\beta(s),\beta(t)]s^q \quad \forall \ s,t \in M$

Then M is a commutative ring.

Proof. i. Since

$$f([s,t]) = s^p[\beta(s),\beta(t)]s^q \quad \forall \ s,t \in M \qquad (1)$$

Also $[s,ts] = [s,t]s$, replacing t by ts in equation (1), we obtain

$$f([s,t]s) = f([s,ts]).$$

This gives

$$f([s,t]s) = s^p[\beta(s),\beta(ts)]s^q.$$

Since β is a homomorphism, so the last relation implies

$$f([s,t]s) = s^p[\beta(s),\beta(t)\beta(s)]s^q.$$

This gives

$$f([s,t]s) = s^p[\beta(s),\beta(t)]s^q\ \beta(s) \quad \forall \ s,t \in M \qquad (2)$$

By using definition of β-derivation, we have

$$f([s,t]s) = f[s,t]\beta(s) + \beta([s,t])f(s)$$

By using equation (1) and equation (2), we obtain

$$s^p[\beta(s),\beta(t)]s^q\ \beta(s) = s^p[\beta(s),\beta(t)]s^q\beta(s) + [\beta(s),\beta(t)]f(s)$$

From this we arrive at

$$[\beta(s),\beta(t)]f(s) = 0$$

Replacing t by ut, we get

$$[\beta(s),\beta(ut)]f(s) = 0$$

Since β is a homomorphism, we have

$$[\beta(s),\beta(u)\beta(t)]f(s) = 0$$

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This implies

$$[\beta(s),\beta(u)]\beta(t)f(s)=0 \qquad \forall \ \ s,t,u\,\epsilon\, M$$

The last relation gives

$$[\beta(s),\beta(u)]Mf(s)=0 \qquad \forall \ \ s,u\,\epsilon\, M \qquad (3)$$

Since M is prime then for each $s\,\epsilon\, M$, we get $f(s)=0$ or $[\beta(s),\beta(u)]=0$. Since f is a nonzero two sided β-derivation, so we have $[\beta(s),\beta(u)]=0$ or $s\,\epsilon\, Z(M)$, by using Lemma [2, [14]], we have $f(s)\,\epsilon\, Z(M)$. This along with Lemma [3, [14]], we get

$$f(M)\subset Z(M)$$

Hence M is a commutative ring.

ii. Since

$$f([s,t])=-s^p[\beta(s),\beta(t)]s^q \ \ \forall \ \ s,t\,\epsilon\, M \qquad (4)$$

Also $[s,ts]=[s,t]s$, replacing t by ts in equation (1), we obtain

$$f([s,t]s)=f([s,ts]).$$

This gives

$$f([s,t]s)=-s^p[\beta(s),\beta(ts)]s^q.$$

Since β is a homomorphism, so the last relation implies

$$f([s,t]s)=-s^p[\beta(s),\beta(t)\beta(s)]s^q.$$

This gives

$$f([s,t]s)=-s^p[\beta(s),\beta(t)]s^q\,\beta(s) \ \ \forall \ \ s,t\,\epsilon\, M \qquad (5)$$

By using definition of β-derivation, we have

$$f([s,t]s)= f[s,t]\beta(s)+\beta([s,t])f(s)$$

By using equation (4) and equation (5), we obtain

$$-s^p[\beta(s),\beta(t)]s^q\,\beta(s)=-s^p[\beta(s),\beta(t)]s^q\beta(s)+[\beta(s),\beta(t)]f(s)$$

From this we arrive at

$$[\beta(s),\beta(t)]f(s)=0$$

Replacing t by ut, we get

$$[\beta(s),\beta(ut)]f(s)=0$$

Since β is a homomorphism, we have

$$[\beta(s),\beta(u)\beta(t)]f(s)=0$$

This implies

$$[\beta(s),\beta(u)]\beta(t)f(s)=0 \qquad \forall \ \ s,t,u\,\epsilon\, M$$

The last relation gives

$$[\beta(s),\beta(u)]Mf(s)=0 \qquad \forall \ \ s,u\,\epsilon\, M \qquad (6)$$

Since M is prime then for each $s\,\epsilon\, M$, we get $f(s)=0$ or $[\beta(s),\beta(u)]=0$. Since f is a nonzero two sided β-derivation, so we have $[\beta(s),\beta(u)]=0$ or $s\,\epsilon\, Z(M)$, by using Lemma [2, [14]], we have $f(s)\,\epsilon\, Z(M)$. This along with Lemma [3, [14]], we get

$$f(M)\subset Z(M)$$

Hence M is a commutative ring.

Example 1. Let X be a commutative ring and $M=\{\begin{pmatrix}0&0&s\\0&0&t\\0&0&0\end{pmatrix}: s,t\,\epsilon\, X\}$. We define the following mappings on M:
Let $\beta: M\to M$ is a mapping defined by:

$$\beta\begin{pmatrix}0&0&s\\0&0&t\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&0\\t&0&0\\0&0&0\end{pmatrix}$$

and $f: M\to M$ is a mapping defined by:

$$f\begin{pmatrix}0&0&s\\0&0&t\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&s\\0&0&0\\0&0&0\end{pmatrix},$$

Then f is nonzero β-derivation on M. If A= $\begin{pmatrix}0&0&s\\0&0&0\\0&0&0\end{pmatrix}$, then $AMA=0$ which proves that M is not prime. Moreover, f satisfies the condition $f([A,B])=[A,B] \ \ \forall \ A,B\,\epsilon\, M$.

Theorem 2.2. Let M be a prime near ring. If there exist $p,q\,\epsilon\, M$ and two sided nonzero β-derivation f on M, where $\beta: M\to M$ is a homomorphism, satisfying the following conditions:

i. $f(sot)=s^p\big(\beta(s)o\beta(t)\big)s^q \qquad \forall \ \ s,t\,\epsilon\, M$
ii. $f(sot)=-s^p\big(\beta(s)o\beta(t)\big)s^q \qquad \forall \ \ s,t\,\epsilon\, M$

Then M is a commutative ring.

Proof. i. Since

$$f(sot)=s^p\big(\beta(s)o\beta(t)\big)s^q \qquad \forall \ \ s,t\,\epsilon\, M \qquad (7)$$

Also $so(ts)=(sot)s$, replacing t by ts in equation (7), we obtain

$$f\big((sot)s\big)=f(so(ts))$$

This gives

$$f\big((sot)s\big)=s^p\big(\beta(s)o\beta(ts)\big)s^q$$

Since β is a homomorphism, so the last relation implies

$$f\big((sot)s\big)=s^p(\beta(s)o\beta(t)\beta(s))s^q$$

This gives

$$f\big((sot)s\big)=s^p\big(\beta(s)o\beta(t)\big)s^q\,\beta(s) \quad \forall \ \ s,t\,\epsilon\, M \qquad (8)$$

By using definition of β-derivation, we have

$$f\big((sot)s\big)=f(sot)\beta(s)+\beta(sot)f(s)$$

By using equation (7) and equation (8), we obtain

$$s^p\big(\beta(s)o\beta(t)\big)s^q\,\beta(s)=s^p\big(\beta(s)o\beta(t)\big)s^q\,\beta(s)+\big(\beta(s)o\beta(t)\big)f(s)$$

From this we arrive at

$$\big(\beta(s)o\beta(t)\big)f(s)=0$$

This implies

$$\big(\beta(s)\beta(t)+\beta(t)\,\beta(s)\big)f(s)=0$$

From this we get

$$\beta(s)\beta(t)f(s)+\beta(t)\beta(s)f(s)=0$$

This gives

$$\beta(s)\beta(t)f(s)=-\beta(t)\beta(s)f(s) \ \ \forall \ \ s,t\,\epsilon\, M \qquad (9)$$

Replacing t by ut in last equation, we get

$$\beta(s)\beta(ut)f(s)=-\beta(ut)\beta(s)f(s)$$

Since β is a homomorphism, so we have

$$\beta(s)\beta(u)\beta(t)f(s)=-\beta(u)\beta(t)\beta(s)f(s)$$

By using equation (9), we obtain

$$\beta(s)\beta(u)\beta(t)f(s)=-\beta(u)(-\beta(s)\,\beta(t)f(s))$$

Since β is a homomorphism, so the last relation implies

$$\beta(s)\beta(u)\beta(t)f(s)=\beta(-u)\beta(-s)\,\beta(t)f(s)$$

This gives

$$\beta(s)\beta(u)\beta(t)f(s) - \beta(-u)\beta(-s)\,\beta(t)f(s) = 0$$

This implies

$$(\beta(s)\beta(u) + \beta(u)\beta(-s))\,\beta(t)f(s) = 0$$

Replacing s by –s, we get

$$(\beta(-s)\beta(u) + \beta(u)\beta(s))\,\beta(t)f(-s) = 0$$

Since β is a homomorphism, so we have

$$(-\beta(s)\beta(u) + \beta(u)\beta(s))\,\beta(t)f(-s) = 0$$

From this we get

$$-(\beta(s)\beta(u) - \beta(u)\beta(s))\,\beta(t)f(-s) = 0$$

This gives

$$-[\,\beta(s), \beta(u)]\,\beta(t)f(-s) = 0$$

This implies

$$[\,\beta(s), \beta(u)]\beta(t)f(-s) = 0 \qquad \forall s,t,u \,\epsilon\, M$$

The last relation gives

$$[\,\beta(s), \beta(u)]Mf(-s) = 0 \qquad \forall s,u \,\epsilon\, M \qquad (10)$$

Since M is prime then for each $s \,\epsilon\, M$, we get $f(-s) = 0$ or $[\beta(s), \beta(u)] = 0$. Since f is a nonzero two sided β-derivation, so we have $[\beta(s), \beta(u)] = 0$ or $s \,\epsilon\, Z(M)$, by using Lemma [2, [14]], we have $f(s) \,\epsilon\, Z(M)$. This along with Lemma [3, [14]], we get

$$f(M) \subset Z(M)$$

Hence M is a commutative ring.

ii. . Since

$$f(sot) = -s^p\big(\beta(s)o\beta(t)\big)s^q \qquad \forall\; s,t \,\epsilon\, M \qquad (11)$$

Also $so(ts) = (sot)s$, replacing t by ts in equation (11), we obtain

$$f\big((sot)s\big) = f\big(so(ts)\big)$$

This gives

$$f\big((sot)s\big) = -s^p\big(\beta(s)o\beta(ts)\big)s^q$$

Since β is a homomorphism, so the last relation implies

$$f\big((sot)s\big) = -s^p(\beta(s)o\beta(t)\beta(s))s^q$$

This gives

$$f\big((sot)s\big) = -s^p\big(\beta(s)o\beta(t)\big)s^q\,\beta(s) \quad \forall\; s,t \,\epsilon\, M \qquad (12)$$

By using definition of β-derivation, we have

$$f\big((sot)s\big) = f(sot)\beta(s) + \beta(sot)f(s)$$

By using equation (11) and equation (12), we obtain

$$-s^p\big(\beta(s)o\beta(t)\big)s^q\,\beta(s) = -s^p\big(\beta(s)o\beta(t)\big)s^q\,\beta(s) + \big(\beta(s)o\beta(t)\big)f(s)$$

From this we arrive at

$$\big(\beta(s)o\beta(t)\big)f(s) = 0$$

This implies

$$\big(\beta(s)\beta(t) + \beta(t)\,\beta(s)\big)f(s) = 0$$

From this we get

$$\beta(s)\beta(t)f(s) + \beta(t)\beta(s)f(s) = 0$$

This gives

$$\beta(s)\beta(t)f(s) = -\beta(t)\beta(s)f(s) \;\; \forall\;\; s,t \,\epsilon\, M \qquad (13)$$

Replacing t by ut in last equation, we get

$$\beta(s)\beta(ut)f(s) = -\beta(ut)\beta(s)f(s)$$

Since β is a homomorphism, so we have

$$\beta(s)\beta(u)\beta(t)f(s) = -\beta(u)\beta(t)\beta(s)f(s)$$

By using equation (13), we obtain

$$\beta(s)\beta(u)\beta(t)f(s) = -\beta(u)(-\beta(s)\,\beta(t)f(s))$$

Since β is a homomorphism, so the last relation implies

$$\beta(s)\beta(u)\beta(t)f(s) = \beta(-u)\beta(-s)\,\beta(t)f(s)$$

This gives

$$\beta(s)\beta(u)\beta(t)f(s) - \beta(-u)\beta(-s)\,\beta(t)f(s) = 0$$

This implies

$$(\beta(s)\beta(u) + \beta(u)\beta(-s))\,\beta(t)f(s) = 0$$

Replacing s by –s, we get

$$(\beta(-s)\beta(u) + \beta(u)\beta(s))\,\beta(t)f(-s) = 0$$

Since β is a homomorphism, so we have

$$(-\beta(s)\beta(u) + \beta(u)\beta(s))\,\beta(t)f(-s) = 0$$

From this we get

$$-(\beta(s)\beta(u) - \beta(u)\beta(s))\,\beta(t)f(-s) = 0$$

This gives

$$-[\,\beta(s), \beta(u)]\,\beta(t)f(-s) = 0$$

This implies

$$[\,\beta(s), \beta(u)]\beta(t)f(-s) = 0 \qquad \forall s,t,u \,\epsilon\, M$$

The last relation gives

$$[\,\beta(s), \beta(u)]Mf(-s) = 0 \qquad \forall s,u \,\epsilon\, M \qquad (14)$$

Since M is prime then for each $s \,\epsilon\, M$, we get $f(-s) = 0$ or $[\beta(s), \beta(u)] = 0$. Since f is a nonzero two sided β-derivation, so we have $[\beta(s), \beta(u)] = 0$ or $s \,\epsilon\, Z(M)$, by using Lemma [2, [14]], we have $f(s) \,\epsilon\, Z(M)$. This along with Lemma [3,[14]], we get

$$f(M) \subset Z(M)$$

Hence M is a commutative ring.

Example 2. Let X be a commutative ring and $M = \{\begin{pmatrix} 0 & 0 & s \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} : s,t,u \,\epsilon\, X\}$. We define the following mappings on M:

Let $\beta: M \rightarrow M$ is a mapping defined by:

$$\beta\begin{pmatrix} 0 & 0 & s \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & s \\ t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $f: M \rightarrow M$ is a mapping defined by:

$$f\begin{pmatrix} 0 & 0 & s \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Then f is nonzero β-derivation on M. If A= $\begin{pmatrix} 0 & 0 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $AMA = 0$ which proves that M is not prime. Moreover, f satisfies the condition $f(AoB) = (AoB) \quad \forall\; A,B \,\epsilon\, M$.

References

Ashraf, M., Ali, A., Ali, S. 2004. $(\sigma;\tau)$-Derivations on prime near rings, *Arch. Math. (Brno)* 40, 281-286.

Ashraf, M., Shakir, A. 2008. On $(\sigma;\tau)$-derivations of prime near-rings II, *Sarajevo J. Math.,* 4, 23-30.

Basudeb, D. 2010. Remarks on generalized derivations in prime and semi prime rings, *Int. J. Math. Math. Sci.*, Art. ID 646587, 6.

Beidar, K.I., Fong, Y., Wang, X.K. 1996. Posner and Herstein theorems for derivations of 3-prime near-rings, *Comm. Algebra*, 24, 1581-1589.

Bell, H.E., Mason, G. 1987. On derivations in near-rings, in: Near-Rings and Near-Fields (G. Betsch, ed.), North-Holland, Amsterdam, 31-35.

Bell, H.E., Mason, G. 1992. On derivations in near-rings and rings, *Math. J. Okayama Univ.,* 34, 135-144.

Daif, M.N., Bell, H.E. 1992. Remarks on derivations on semi prime rings, Internet. *J. Math. Math. Sci.*, 15, 205-206.

Golbasi, O., Koc, E. 2009. Notes on commutativity of prime rings with generalized derivation, Commun. *Fac. Sci. Univ., Ank. (Ser. A1*), 58, 39-46.

Golbasi, O., Koc, E. 2011. Some commutativity theorems of prime rings with generalized $(\sigma;\tau)$-derivation, Commun. *Korean Math. Soc.,* 26, 445-454.

Kamal, A.A.M. 2001. σ-Derivations on prime near-rings, Tamkang *J. Math.,* 32, 89-93.

Khan, A.R., Chaudhry, M. A., Javaid, I. 2013. Generalized derivations on prime-rings, *World Applied Sciences Journal,* 23(12), 59-63.

Khan, R., Chaudhry, M. A., and Javaid, I. 2016. Some results on prime and semiprime Γ -rings, ANALELE Universitatii Oradea Fasc. *Mathematica*, Tom XXIII, 1, 177-187.

Khan, R., Javaid, I., and Chaudhry, M. A. 2013. Derivations on semiprime Γ-rings, *Utilitas Mathematica*, 90, 171-185.

Khan, R., Mumtaz, K. and Waqas, M. M. 2021. On Commutativity of Prime Near Rings, *Matrix Science Mathematics*, 5(1), 6-15.

Quadri, M. A., Khan, M.S., Rehman, N. 2003. Generalized derivations and commutativity of prime rings, Indian J. *Pure Appl. Math.*, 34, 1393-1396.

Raina, R., Bhat, V.K., Kumari, N. 2009. Commutativity of prime Γ-near rings with Γ-$(\sigma;\tau)$- derivation, *Acta Math. Acad. Paedagog. Nyhazi*, 25, 165-173.

Wang, X. K. 1994. Derivations in prime near-rings*, Proc. Amer. Math. Soc.,* 121, 361-366.