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RESEARCH ARTICLE

NUMERICAL COMPUTATIONS OF GENERAL NON-LINEAR THIRD ORDER BOUNDARY VALUE PROBLEMS BY GALERKIN WEIGHTED RESIDUAL TECHNIQUE WITH MODIFIED LEGENDRE AND BEZIER POLYNOMIALS

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ABSTRACT

Several different approaches are implemented and used to solve higher order non-linear boundary value problems (BVPs). Galerkin weighted residual technique (GWRT) are commonly used to solve linear and non-linear BVPs. In this paper, we have proposed GWRT for the numerical computations of general third order three-point non-linear BVPs. Modified Legendre and Bezier Polynomials, over the interval [0, 1], are chosen separately as a basis functions. The main advantage of this method is its efficiency and simple applicability. Numerical result is presented to illustrate the performance of the proposed method. The results clearly show that the proposed method is suitable for solving third order nonlinear BVPs

KEYWORDS

Third order non-linear BVPs, Galerkin weighted residual technique, Numerical solutions, Legendre and Bezier polynomials.

1. INTRODUCTION

Non-linear boundary value problems control a variety of phenomena in engineering and applied science fields. Because of their mathematical significance and applications, higher order non-linear boundary value problems have been studied. Therefore, BVPs have received a huge attention from mathematicians, physicists, and engineers for the sake of finding and analyzing their solutions. Seeking analytical solutions for non-linear BVPs is far from easy and often it's impossible. To solve such problems, a large number of works have been identified. But researchers have faced troubles to solve problems when the problems are non-linear. Galerkin weighted residual method for the solution of BVPs are very important in recent literature and third order boundary value problems which is one in the family of ODEs is also a well searched area for the application of different methods.

Third order BVPs are used in the study of electromagnetic waves, three layer beams, beam deflection theory, incompressible flows and other natural and technical science topics. To solve such problems, a variety of numerical techniques have been devised. These techniques include Adomian's decomposition method, homotopy perturbation method, variational iteration method, optimal homotopy asymptotic method, operational matrices techniques based on various orthogonal polynomials and wavelets, finite difference method, Galerkin method and spectral methods (Wazwaz, 2017; Al-Mdallal et al, 2017; Al-Mdallal, 2015; Nazrul and Islam, 2018; Nazrul, 2021; Zaimi et al, 2014; Wazwaz, 1999). Manisha Patel, Jayshri Patel and Timol G. M. (Patel et al, 2015) have tried to establish numerical solution of third order ordinary differential equations using the method of Finite Difference. On Sisko Fluid's Passing Surface in MHD flow Nakone Bello, Abubakar Roko and Aminu Mustafa (Nakone et al, 2018) proposed numerical solution of a linear third order multi-point boundary value problem using the fixed point iterative approach. Olasunmbo Agboola, Abiodun A. Opanuga, Jacob Agboola Gbadeyan (Agboola et al, 2015) tried to developed the third order ordinary differential equations using the process of differential transformation. The writers, Ghazala Akram, Muhammad Tehseen, Shahid S. Siddiqi and Hamood ur Rehman (Akram et al, 2013) addressed computations of a linear third order multi-point boundary value problems using the reproducing kernel methods. A subdivision of the third order boundary value was formed by Manan, A. Ghaffarm, M. Rizwan, G. Rahaman and G. Kanwal (Manan et al, 2018). Abd El-Salam, A.A. El-Sabbagh and Z.A. Zaki (Salam et al, 2010) presented second and fourth order convergent methods based on quartic non-polynomial spline function for the numerical solution of a third order two-point boundary value problems. The numerical solutions of higher order nonlinear BVPs were introduced by Nazrul Islam (Nazrul, 2021) using a new iterative method. In (Tirmizi et al, 2005) the authors developed a second-order method for solving third-order three-point boundary value problems based on Pade approximant in a recurrence relation. In (Tatari and Dehghan, 2006), the authors introduced Adomian decomposition method for multipoint boundary value problems. With advent of computers it gained important to develop more accurate numerical methods to solve higher order boundary value problems.

The objective of this research is to construct a rigorous matrix formulation to calculate approximate solutions of third order three-point non-linear BVPs by Galerkin weighted residual technique. Some numerical examples have been considered to investigate the feasibility and efficiency of the

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proposed method. Due to calculation, the trial functions namely, Modified Legendre and Bezier Polynomials are assumed which is used to satisfy the given essential boundary conditions. Obtained results are represented in a tabular form and also graphically for better understanding. The numerical solutions are in good agreement with the exact solutions, and the solutions have a higher level of accuracy. So it has been observed that GWRM converges to the exact solution faster than all other existing methods and needs less cognitive complexity.

The rest of the paper will be described as follows: In section two, we provide a short discussion on Legendre and Bezier polynomials which are relevant for the analysis of the problems under investigation. In third section, we drive rigorous matrix formulation for solving non-linear third orders BVPs using GWRM and particular attention is given to how the polynomials fulfill the essential boundary conditions. In section four, three numerical examples be examined and verified the result with actual result. In the lattermost part, the conclusion is given.

2. Piecewise Polynomials

Bezier and Legendre polynomials are widely used because their mathematical descriptions are succinct, intuitive, and elegant. For solving higher order non-linear boundary value problems, they are simple to compute and use.

2.1 Bezier Polynomials

The Bezier polynomials $B_{m,n}(x)$ of order n are defined as follows

$$B_{m,n}(x)=\sum_{m=0}^{n}\binom{n}{m}x^m(1-x)^{n-m}P_m\ ,0\le x\le 1$$

Some Bezier polynomials over the interval [0, 1] are shown as follows:

$B_0(x)=(1-x)^{17}$	$B_6(x)=12376(1-x)^{11}x^6$	$B_{12}(x)=6188(1-x)^5x^{12}$
$B_1(x)=17(1-x)^{16}x$	$B_7(x)=19448(1-x)^{10}x^7$	$B_{13}(x)=2380(1-x)^4x^{13}$
$B_2(x)=136(1-x)^{15}x^2$	$B_8(x)=24310(1-x)^9x^8$	$B_{14}(x)=680(1-x)^3x^{14}$
$B_3(x)=680(1-x)^{14}x^3$	$B_9(x)=24310(1-x)^8x^9$	$B_{15}(x)=136(1-x)^2x^{15}$
$B_4(x)=2380(1-x)^{13}x^4$	$B_{10}(x)=19448(1-x)^7x^{10}$	$B_{16}(x)=17(1-x)x^{16}$
$B_5(x)=6188(1-x)^{12}x^5$	$B_{11}(x)=12376(1-x)^6x^{11}$	$B_{17}(x)=x^{17}$

Each of the Bezier polynomials having order n satisfies the following special properties:

(i) $B_{m,n}(x)=0$ when $m<0$ or $m>n$

(ii) $\sum_{m=0}^{n}B_{m,n}(x)=1$

(iii) $B_{m,n}(\xi)=B_{m,n}(\eta)=0,\ m=1,2,\dots,n-1$

For these special properties, Bezier polynomials are easy to use and easy to compute for solving higher order non-linear BVPs.

2.2 Legendre Polynomials

The Legendre polynomials $L_{m,n}(x)$ of order n are defined as follows

$$L_m(x)=\frac{(-1)^m}{2^m(m!)}\frac{d^m}{dx^m}[(1-x^2)]^m,\ \ m\ge 1$$

Now we modify above Legendre polynomials as

$$L_m(x)=\left[\frac{1}{(m!)}\frac{d^m}{dx^m}(x^2-x)^m-(-1)^m\right](x-1),\ \ m\ge 1$$

Some modified Legendre polynomials over the interval [0, 1] are given below:

$L_1(x)=-2x+2x^2,\qquad L_2(x)=6x-12x^2+6x^3,\qquad L_3(x)=-12x+42x^2-50x^3+20x^4$

$L_4(x)=20x-110x^2+230x^3-210x^4+70x^5$

$L_5(x)=-30x+240x^2-770x^3+1190x^4-882x^5+252x^6$

$L_6(x)=42x-462x^2+2100x^3-48300x^4+5922x^5-3696x^6+924x^7$

$L_7(x)=-56x+812x^2-4956x^3+1570x^4-28182x^5+28644x^6-15444x^7+3432x^8$

$L_8(x)=72x-1332x^2+10500x^3-43890x^4+106722x^5-156156x^6+135564x^7-64350x^8+12870x^9$

$L_9(x)=-90x+2070x^2-20460x^3+108570x^4-342342x^5+672672x^6-832260x^7+630630x^8-267410x^9+48620x^{10}$

$L_{10}(x)=110x-3080x^2+37290x^3-244530x^4+966966x^5-2438436x^6+4015440x^7-4302870x^8+2892890x^9-1108536x^{10}+184756x^{11}$

Since modified Legendre polynomials have special characteristics at $x=0$ and $x=1$ such that $L_m(0)=0$ and $L_m(1)=0$ respectively, so these polynomials can be used as set of basis functions to fulfill the corresponding homogeneous form of the essential boundary conditions in the GWRT to solve a non-linear BVP over the interval $[0,1]$. In this research paper we used Legendre and Bezier polynomials as basis functions to solve higher order non-linear BVPs.

3. Research Method

In this section we will discuss general Galerkin weighted residual technique for the numerical computations of general third order three-point non-linear boundary value problems. For do this we consider a general third order three-point boundary value problem is defined by

$$c_3\frac{d^3y}{dx^3}+c_2\frac{d^2y}{dx^2}+c_1\frac{dy}{dx}+c_0\,y\ =t,\ \ c\le x\le d \tag{1}$$

Subject to the following boundary conditions

$$y(c)=C_0\ ,\ y(d)=D_0,\ \ y'(c)=C_1 \tag{2}$$

where C_0, C_1, D_0 are real constants and $c_i, i=0,1,2,3$ and t all are differentiable functions of x defined on the closed interval $[c,d]$.

We approximate the solution of differential equation (1) as given as follows:

$$\tilde{y}(x)=\theta_0(x)+\sum_{m=1}^{n-1}\zeta_m B_m(x),\ n\ge 2 \tag{3}$$

Here $\theta_0(x)$ is specified by the essential boundary conditions, $B_m(x)$ are the Bezier polynomials must fulfill the corresponding homogeneous boundary condition in such a way that $B_m(c)=B_m(d)=0$ for each $m=1,2,3,\dots\dots,n-1$.

Applying equation (3) into equation (1) the Galerkin weighted residual equations are:

$$\int_c^d\left[c_3\frac{d^3\tilde{y}}{dx^3}+c_2\frac{d^2\tilde{y}}{dx^2}+c_1\frac{d\tilde{y}}{dx}+c_0\tilde{y}-t\right]B_j(x)dx=0,\ j=1,2,\dots\dots n-1 \tag{4}$$

Integrating the second and third derivative terms on the L.H.S of equation (4), we have

$$\int_c^d\left[c_3\frac{d^3\tilde{y}}{dx^3}B_j(x)\right]=\left[c_3\frac{d^2\tilde{y}}{dx^2}B_j(x)\right]_c^d-\int_c^d\frac{d}{dx}(c_3\,B_j(x))\frac{d^2\tilde{y}}{dx^2}dx=$$
$$-\int_c^d\frac{d}{dx}(c_3\,B_j(x))\frac{d^2\tilde{y}}{dx^2}dx\quad[\text{since } B_j(c)=B_j(d)=0]$$
$$=-\left[\frac{d}{dx}\left(c_3\,B_j(x)\right)\frac{d\tilde{y}}{dx}\right]_c^d+\int_c^d\frac{d^2}{dx^2}(c_3\,B_j(x))\frac{d\tilde{y}}{dx}dx \tag{5}$$

$$\int_c^d\left[c_2\frac{d^2\tilde{y}}{dx^2}B_j(x)\right]=\left[c_2\frac{d\tilde{y}}{dx}B_j(x)\right]_c^d-\int_c^d\frac{d}{dx}\left(c_2\,B_j(x)\right)\frac{d\tilde{y}}{dx}dx\quad[\text{since } B_j(c)=B_j(d)=0] \tag{6}$$

Substituting equations (5) and (6) into equation (4) and using approximations for $\tilde{y}(x)$ given in equation (3) and imposing the boundary conditions given in equation (2) and rearranging the terms we obtain the system of equations in matrix form as follows:

$$\sum_{m=1}^{n-1}\mathcal{F}_{m,j}\,\zeta_m=\mathcal{M}_j\ ,\ j=1,2,3\dots\dots n-1 \tag{7}$$

Where

$$\mathcal{F}_{m,j}=\int_c^d\left[\left\{\frac{d^2}{dx^2}\left(c_3B_j(x)\right)-\frac{d}{dx}\left(c_2B_j(x)\right)+c_1B_j(x)\right\}\frac{d}{dx}(B_m(x))+c_0B_m(x)B_j(x)\right]dx-\left[\frac{d}{dx}\left(c_3B_j(x)\right)\frac{d}{dx}(B_m(x))\right]_{x=d}$$

$$\mathcal{M}_j=\int_c^d\left\{t(x)B_j(x)+\left[-\frac{d^2}{dx^2}\left(c_3B_j(x)\right)+\frac{d}{dx}\left(c_2B_j(x)\right)-c_1B_j(x)\right]\frac{d\theta_0(x)}{dx}-c_0\theta_0(x)B_j(x)\right\}dx-\left[\frac{d}{dx}\left(c_3B_j(x)\right)\right]_{x=c}\times C_1\ ,\ j=1,2,\dots\dots,n-1$$

We solve the systems (7) by neglecting non-linear terms. Then by applying the most common Newton's iterative technique we get the value of parameters ζ_m. These values are then substituted in the system of equations (7) in order to calculate solutions for the desired non-linear BVP. This formulation is described through the numerical examples in the next section.

4. Results and Discussion

Three non-linear BVPs are considered to assess the applicability of the proposed procedure. The proposed method's computations are compared to analytic solutions in all of the cases. The programming language MATLAB R2019a is used to solve all of the computations.

4.1 Investigation by First Example

We consider the following three-point non-linear differential equation for equating our algorithm with the current algorithm (Xueqin and Gao, 2017):

$$\begin{cases} y'''(x) - e^{y(x)}\left(0.5 - e^{y(x)}\right) = t(x),\ x\epsilon\,[0,1] \\ y(0) = \ln(10000),\ y(1) = \ln(10001),\ y'(0) = \frac{1}{10000} \end{cases} \tag{8}$$

where $t(x) = \left(-2 - \frac{1}{2}\sqrt{x}\,(9998 + x)(10000 + x)\right) / (1000 + x)^3$; the exact solution is given by $y(x) = \ln(x + 10000)$; using the method discussed in section three we get the approximate solution of the form $\tilde{y}(x) = \theta_0(x) + \sum_{m=1}^{n-1} \zeta_m B_m(x)$, $n \geq 2$; the results are summarized in Table 1.

Table 1: For example 1, representing the approximate solutions and the absolute errors obtained by the GWRT in y_i using 7 iterations

x	Exact values	11, Bezier polynomials		11, Legendre polynomials		Reference (Xueqin and Gao, 2017)
		Approximate	Relative Error	Approximate	Relative Error	Maximum relative error by Xueqin and Gao
1/10	9.210350372	9.210350372	6.98×10^{-12}	9.210350372	9.90×10^{-12}	2.44×10^{-6}
2/10	9.210360372	9.210360372	7.55×10^{-12}	9.210360372	1.12×10^{-11}	
3/10	9.210370372	9.210370372	1.17×10^{-11}	9.210370372	1.60×10^{-11}	
4/10	9.210380371	9.210380371	6.28×10^{-11}	9.210380371	1.25×10^{-11}	
5/10	9.210390371	9.210390371	7.96×10^{-11}	9.210390371	1.06×10^{-11}	
6/10	9.210400370	9.210400370	2.32×10^{-11}	9.210400370	1.41×10^{-11}	
7/10	9.210410370	9.210410370	5.45×10^{-11}	9.210410370	3.94×10^{-12}	
8/10	9.210420369	9.210420369	8.21×10^{-11}	9.210420369	9.76×10^{-12}	
9/10	9.210430368	9.210430368	3.42×10^{-12}	9.210430368	3.61×10^{-12}	

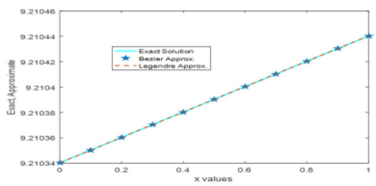

Figure 1: Approximate vs analytical solutions for example 1

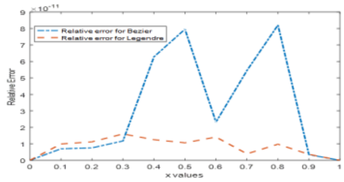

Figure 2: Comparing the numerical errors obtained by the Bezier and Legendre polynomials for example 1

The analytic solutions and the computed solutions obtained are depicted in Fig.1. There is a very good agreement and relationship between the approximate solutions obtained by 7th iterations using GWRT and the exact solutions which are shown in Table 1.

4.2 Investigation by Second Example

We consider the following three-point non-linear differential equation for equating our algorithm with the current algorithm (Lin et al, 2021):

$$\begin{cases} y'''(x) - xy(x) + y'(x)^2 + y^3(x) = t(x),\ x\epsilon\,[0,1] \\ y(0) = 0,\ y(1) = 0,\ y'(0) = \pi \end{cases} \tag{9}$$

whose exact solution is given by $y(x) = e^x(x - x^2)$ such that $t(x)$ can be computed by in setting $y(x) = e^x(x - x^2)$ into equation (9); using the method discussed in section three we get the approximate solution of the form $\tilde{y}(x) = \theta_0(x) + \sum_{m=1}^{n-1} \zeta_m B_m(x)$, $n \geq 2$; the results are summarized in Table 2.

Table 2: For example 2, representing the approximate solutions and the absolute errors obtained by the GWRT in y_i using 10 iterations

x	Exact values	12, Bezier polynomials		13, Legendre polynomials		Reference (Lin et al, 2021)
		Approximate	Relative Error	Approximate	Relative Error	Maximum relative error by Ji Lin, Zhang and Liu
1/10	0.099465383	0.099465383	1.27×10^{-12}	0.099465383	7.09×10^{-13}	5.45×10^{-8}
2/10	0.195424441	0.195424441	9.21×10^{-13}	0.195424441	1.57×10^{-12}	
3/10	0.283470350	0.283470350	6.38×10^{-13}	0.283470350	2.60×10^{-12}	
4/10	0.358037927	0.358037927	4.54×10^{-13}	0.358037927	3.83×10^{-12}	
5/10	0.412180318	0.412180318	4.09×10^{-13}	0.412180318	5.29×10^{-12}	
6/10	0.437308512	0.437308512	5.54×10^{-13}	0.437308512	9.01×10^{-12}	
7/10	0.422888069	0.422888069	9.52×10^{-13}	0.422888069	7.04×10^{-12}	
8/10	0.356086549	0.356086549	1.68×10^{-12}	0.356086549	1.14×10^{-11}	
9/10	0.221364280	0.221364280	2.82×10^{-12}	0.221364280	1.42×10^{-11}	

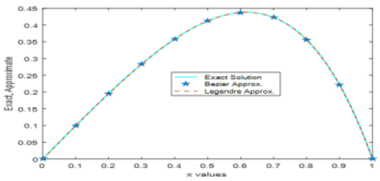

Figure 3: Approximate vs analytical solutions for example 2

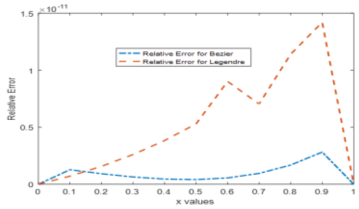

Figure 4: Comparing the numerical errors obtained by the Bezier and Legendre polynomials for example 2

The analytic solutions and the computed solutions obtained are depicted in Fig.3. There is a very good agreement and relationship between the approximate solutions obtained by 10th iterations using GWRT and the exact solutions which are shown in Table 2.

4.3 Investigation by Third Example

We consider the following three-point non-linear differential equation for equating our algorithm with the current algorithm (Lin et al, 2021):

$$\begin{cases} y'''(x) + y''(x) + y^2(x) = t(x), \ x\epsilon\,[0,1] \\ y(0) = 0, \ y(1) = 0\,, \ y'(0) = \pi \end{cases} \tag{10}$$

whose exact solution is given by $y(x) = e^x \sin(\pi x)$ such that $t(x)$ can be computed by in setting by $y(x) = e^x \sin(\pi x)$ into equation (10); using the method discussed in section three we get the approximate solution of the form $\tilde{y}(x) = \theta_0(x) + \sum_{m=1}^{n-1} \zeta_m B_m(x)$, $n \geq 2$; the results are summarized in Table 3.

Table 3: For example 3, representing the approximate solutions and the absolute errors obtained by the GWRT in y_i using 8 iterations

x	Exact values	13, Bezier polynomials		13, Legendre polynomials		Reference (Lin et al, 2021)
		Approximate	Relative Error	Approximate	Relative Error	Maximum relative error by Ji Lin, Zhang and Liu
1/10	0.341516595	0.341516595	1.24×10^{-14}	0.341516595	1.81×10^{-13}	
2/10	0.717922528	0.717922528	1.61×10^{-14}	0.717922528	1.10×10^{-12}	
3/10	1.092058715	1.092058715	2.77×10^{-13}	1.092058715	4.42×10^{-12}	
4/10	1.418809599	1.418809599	3.28×10^{-14}	1.418809599	1.67×10^{-12}	1.40×10^{-8}
5/10	1.648721271	1.648721271	8.98×10^{-13}	1.648721271	6.02×10^{-11}	
6/10	1.732937959	1.732937959	2.54×10^{-14}	1.732937959	4.70×10^{-11}	
7/10	1.629160163	1.629160163	2.01×10^{-13}	1.629160163	1.58×10^{-13}	
8/10	1.308140136	1.308140136	3.85×10^{-13}	1.308140136	2.60×10^{-12}	
9/10	0.760059161	0.760059161	1.02×10^{-13}	0.760059161	1.32×10^{-11}	

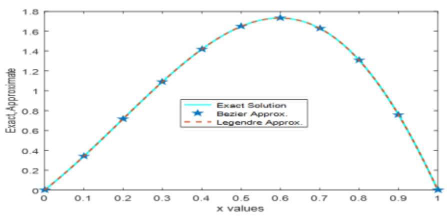

Figure 5: Approximate vs analytical solutions for example 3

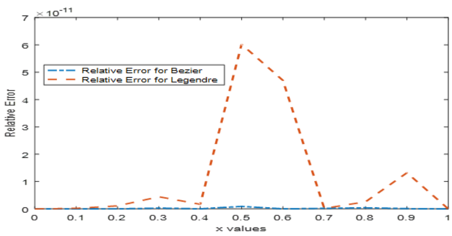

Figure 6: Comparing the numerical errors obtained by the Bezier and Legendre polynomials for example 3

5. Conclusions

The main objective of our work is to determine better numerical computations for higher order three-point non-linear boundary value problems. We applied our mathematical model by investigating some numerical examples and we have found a feasible result. The findings are displayed graphically and in a data structured table. Observing all the figures and tables, it is obvious that the presented result shows the method's higher estimated order of convergence. The proposed method may play a vital role in the field of applied science and engineering where similar category higher order boundary value problems are arising. All computations are performed by the MATLAB R2017a software package.

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