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RESEARCH ARTICLE

THE LUCAS POLYNOMIAL SOLUTION OF LINEAR VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

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ABSTRACT

In this study, linear Volterra-Fredholm integral equations are approximatively solved in terms of Lucas polynomials about any point in this study using a practical matrix approach. This technique uses collocation points and Lucas polynomials to transform the aforementioned linear Volterra-Fredholm integral problem into a matrix equation. Lucas coefficients are unknown in the system of linear algebraic equations. With the use of an error estimation, some illustrated examples are also provided. The outcomes demonstrate how effective and practical the suggested methodology is. Code was created in MATLAB to acquire the matrix equations and answers for the chosen issues.

KEYWORDS

Lucas Polynomials, Volterra and Fredholm Integral Equations, Collocation Method.

1. INTRODUCTION

The integral equations serve as the foundation for several mathematical models in numerous scientific areas, including chemistry, mathematics, and engineering. Integral equations are related to many engineering and applied mathematics topics. Equations with the unknown function under the integral sign are referred to as integral equations (Delves and Mohamed, 1985; Tricomi, 1985; Rashed, 2004; Yalcinbas and Erdem, 2010; Wazwaz, 2010; Brunner, 2004). The Volterra-Fredholm integral equations have a wide range of applications in the sciences and have played a significant role in the development of several techniques in applied mathematics, engineering, and physics. These integral equations are a synthesis of the integral equations of Volterra and Fredholm. There are several techniques that can solve these problems effectively and accurately (Maleknejad et al., 2005; Hwang and Yen-Ping Shih, 1982; Muhammad, 2005; Yalcinbas and Erdem, 2010; Yalcinbas and Erdem, 2014; Yuzbasi, 2009; Yuzbasi and Nurbol, 2017).

Recently, Ahmad proposed a modification of block pulse functions for numerical solution of linear Volterra- Fredholm integral equations (Ahmad, 2021), Esmaeili et. al. introduced a hyperbolic basis functions method for solving Volterra-Fredholm integral equations (Esmaeili et al., 2021). Gecmen and Celik use the Hasoya polynomials for solving Volterra-Fredholm integral equations (Gecmen and Celik, 2021). And also, Lucas polynomials different types were used to solve differential, fractional differential and partial differential equations (Yuzbasi and Yildirim, 2022; Adel, 2022; Yuzbasi and Yildirim, 2022). In this research, Lucas polynomial-based method to resolve Volterra - Fredholm integral equations (VFIE) is presented.

$$y(t)=g(t)+\int_a^t K_v(t,s)y(s)ds+\int_a^b K_f(t,s)y(s)ds \qquad a\le t,s\le b \qquad (1)$$

Where $g(t)$, $K_V(t,s)$, and $K_f(t,s)$ are functions defined on the interval $a \le t$, $s \le b$; $y(t)$ is an unknown solution function to be determined.

For our purpose, we assume the approximate solution of the problem Eq. (1) in the truncated Lucas polynomials form

$$y(t)\cong y_N(t)=\sum_{n=0}^{N} a_n L_n(t), \qquad a\le t\le b \qquad (2)$$

where an, $n = 0, 1, 2, \ldots, N$ are unknown coefficients to be determined and $L_n(t)$ indicates the Lucas polynomials which are originally studied in 1970 by Bicknell. Lucas polynomials are defined recursively as follows (Gumgum et al., 2018; Savasaneril and Sezer, 2008; Gumgum et al., 2019)

$$L_{n+1}(t)=tL_n(t)+L_{n-1}(t), \qquad n\ge 1, \quad L_0(t)=2, \quad L_1(t)=t. \qquad (3)$$

Their explicit form for $n \ge 1$ is

$$L_n(t)=\sum_{k=0}^{\frac{n}{2}} \frac{n}{n-k}\binom{n-k}{k} t^{n-2k} \qquad (4)$$

where x is the largest integer smaller than or equal to x.

By using Eq. (3) and Eq. (4) the first Lucas polynomials respectively are given by

$$L_0(t)=2, \quad L_1(t)=t, \quad L_2(t)=t^2+2, \quad L_3(t)=t^3+3t,$$
$$L_4(t)=t^4+4t^2+2, \quad L_5(t)=t^5+5t^3+5t, \quad L_6(t)=t^6+6t^4+9t^2+2$$

2. MATERIALS AND METHODS

2.1 Matrix Relations

The following process is used in this section to convert the expressions defined in Eq. (2) into matrix form:

$$y(t) \cong y_N(t) = \mathbf{L}(t)\mathbf{A}, \qquad \mathbf{L}(t) = \mathbf{T}(t)\mathbf{D}^T \tag{5}$$

where

$$L(t) = [L_0(t) \quad L_1(t) \quad \cdots \quad L_N(t)], \quad A = [a_0 \quad a_1 \quad \cdots \quad a_N]^T$$
$$T(t) = [1 \quad t \quad \cdots \quad t^N]$$

If N is odd,

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0} & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1} & 0 & \frac{3}{3}\binom{3}{0} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{N-1}{\left(\frac{N-1}{2}\right)}\binom{\frac{N-1}{2}}{\frac{N-1}{2}} & 0 & \frac{N-1}{\left(\frac{N+1}{2}\right)}\binom{\frac{N+1}{2}}{\frac{N-3}{2}} & 0 & \cdots & \cdots & \frac{N-1}{\left(\frac{2N-2}{2}\right)}\binom{\frac{2N-1}{2}}{0} & 0 \\ 0 & \frac{N}{\left(\frac{N+1}{2}\right)}\binom{\frac{N+1}{2}}{\frac{N-1}{2}} & 0 & \frac{N}{\left(\frac{N+3}{2}\right)}\binom{\frac{N+3}{2}}{\frac{N-3}{2}} & \cdots & \cdots & 0 & \frac{N}{N}\binom{N}{0} \end{bmatrix}$$

and if N is even,

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0} & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1} & 0 & \frac{3}{3}\binom{3}{0} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{N-1}{\left(\frac{N}{2}\right)}\binom{\frac{N}{2}}{\frac{N-2}{2}} & 0 & \frac{N-1}{\left(\frac{N+2}{2}\right)}\binom{\frac{N+2}{2}}{\frac{N-4}{2}} & \cdots & \cdots & 0 & \frac{N-1}{N-1}\binom{N-1}{0} \\ \frac{N}{\left(\frac{N}{2}\right)}\binom{\frac{N}{2}}{\frac{N}{2}} & 0 & \frac{N}{\left(\frac{N+2}{2}\right)}\binom{\frac{N+2}{2}}{\frac{N-2}{2}} & 0 & \cdots & \cdots & \frac{N}{\left(\frac{2N}{2}\right)}\binom{\frac{2N}{2}}{0} & 0 \end{bmatrix}$$

From the matrix relations Eq. (5), it follows that

$$y_N(t) = \mathbf{T}(t)\mathbf{D}^T\mathbf{A}, \tag{6}$$

Additionally, the kernel functions $K_v(t,s)$, $K_f(t,s)$, in Eq. (1) is constructed in matrix form as follows:

$$\begin{aligned} K_v(t,s) &= \mathbf{T}(t)\mathbf{K}_v\mathbf{T}(s)^T \\ K_f(t,s) &= \mathbf{T}(t)\mathbf{K}_f\mathbf{T}(s)^T \end{aligned} \tag{7}$$

where

$$\mathbf{K}_f = \mathbf{K}_v = \mathbf{K} = \left[k_{mn}\right], \quad m,n = 0,1,\ldots,N \qquad k_{mn} = \frac{1}{m!n!}\cdot\frac{\partial^{m+n}\mathbf{K}(0,0)}{\partial t^m \partial s^n}$$

$$\begin{aligned} \int_a^b \boldsymbol{K_f(t,s)y(s)ds} &= \boldsymbol{T(t)K_fQ_f(t)}\mathrm{SA} \\ \int_a^t \boldsymbol{K_v(t,s)y(s)ds} &= \boldsymbol{T(t)K_vQ_v(t)}\mathrm{SA} \end{aligned} \tag{8}$$

where

$$\mathbf{Q}_f(t) = \left[q^f{}_{mn}(t)\right] = \int_a^b \mathbf{T}^T(s)\mathbf{T}(s)ds, \quad \mathbf{Q}_v(t) = \left[q^v{}_{mn}(t)\right] = \int_a^t \mathbf{T}^T(s)\mathbf{T}(s)ds,$$

$$\left.\begin{aligned} q^f{}_{mn}(t) &= \frac{b^{m+n+1} - a^{m+n+1}}{m+n+1} \\ q^v{}_{mn}(t) &= \frac{t^{m+n+1} - a^{m+n+1}}{m+n+1} \end{aligned}\right\} \quad m,n = 0,1,\ldots,N$$

To obtain the Lucas polynomial solution of Eq.(1) in the form Eq.(2) we firstly compute the Lucas coefficients by means of the collocation points defined by

$$t_i = a + \frac{b-a}{N}i, \quad i = 0,1,\ldots,N.$$

The following steps are taken to obtain the matrix equation system:

$$y(t_i) = g(t_i) + \int_a^t K_v(t_i,s_i)y(s_i)ds + \int_a^b K_f(t_i,s_i)y(s_i)ds \tag{9}$$

It is constructed the fundamental matrix equation corresponding to the FIDEs, by substituting the matrix relations Eq.(6)-(8) into Eq.(1):

$$\boldsymbol{T(t_i)D^TA = g(t_i) + T(t_i)K_vQ_v(t_i)D^TA + T(t_i)K_fQ_f(t_i)D^TA} \tag{10}$$

or briefly,

$$\mathrm{TD}^T A - \overline{TK_vQ_v}D^T A - \mathrm{TK}_f Q_f D^T A = G \tag{11}$$

where

$$T = \begin{bmatrix} T(t_0) \\ T(t_1) \\ \cdots \\ T(t_N) \end{bmatrix}, \quad \overline{T} = \begin{bmatrix} T(t_0) & 0 & \cdots & 0 \\ 0 & T(t_1) & 0 & \cdots \\ \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & T(t_N) \end{bmatrix}, \quad G = \begin{bmatrix} g(t_0) \\ g(t_1) \\ \cdots \\ g(t_N) \end{bmatrix}, \quad A = \begin{bmatrix} a_0 \\ a_1 \\ \cdots \\ a_N \end{bmatrix},$$

$$\overline{K_v} = \begin{bmatrix} K & 0 & \cdots & 0 \\ 0 & K & 0 & \cdots \\ \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & K \end{bmatrix}, \overline{Q}_v(t) = \begin{bmatrix} Q_v(t_0) \\ Q_v(t_1) \\ \cdots \\ Q_v(t_N) \end{bmatrix},$$

Besides, the fundamental matrix equation Eq.(11) can be expressed in the form

$$\boldsymbol{WA = G} \quad \Leftrightarrow \quad [\boldsymbol{W}:\boldsymbol{G}] \tag{12}$$

Where

$$W = \mathrm{TD}^T - \overline{TK_vQ_v}D^T - \mathrm{TK}_f Q_f D^T = [w_{mn}]; \quad m,n = 0,1,\ldots,N. \tag{13}$$

Then the coefficient matrix A is uniquelly determined and the solution of the problem Eq.(1) is obtained as:

$$y_N(t) = L(t)A = T(t)D^T A$$

3. Accuracy of Solution

The accuracy of the method can be easily checked. Since the truncated Lucas series Eq.(2) is an approximate solution of Eq.(1), when the function $y_N(t)$ and the kernel functions $K_v(t,s)$, $K_f(t,s)$, are substituted in Eq.(1), the resulting equation must be satisfied approximately as follows:

$$E(t_q)=\left\| y(t_q)-\int_a^{t_q} K_v(t_q,s_q)y(s_q)ds-\int_a^b K_f(t_q,s_q)y(s_q)ds-g(t_q)\right\| \cong 0 \quad (13)$$

for $t = t_q \in [a,b]$, $q = 0,1,2,\dots$ and $E(t_q) \leq 10^{-k_q}$, k_q positive integer. If $\max 10^{-k_q} = 10^{-k}$ (k positive integer) is prescribed, then the truncation limit N is increased until the difference $E(t_q)$ at each of the points becomes smaller than the prescribed 10^{-k} (Maleknejad and Mahmoudi, 2003; Kurt and Sezer, 2008; Bulbul et al., 2010; Kurkcu et al., 2017).

4. Numerical Illustrations

Table 1 show the comparison of the error functions of Example 4.1. by the presented method, scaling functions interpolation method (SFI) (Al-Jarrah and En-Bing, 2013) and tan-sec hyperbolic basis functions method (TSHF) (Esmaeili et. al., 2021). It can be said that the proposed method is very useful since the exact solution can be obtained at N=1 and the exact solution can be estimated from other N values

Table 1: Comparison of The Errors In Example 4.1.

t	Scaling Functions Interpolations	Tan-Sec Hyperbolic Basis Functions	Present Method (N=1)
0.0625	1.7249e-01	2.9413e-03	0
0.1875	1.6717e-02	3.0684e-03	0
0.3125	1.5314e-02	3.1962e-03	0
0.4375	1.4215e-02	3.3264e-03	0
0.5625	1.2116e-02	3.4639e-03	0
0.6875	1.1245e-02	3.6236e-03	0
0.8125	1.8516e-03	3.8583e-03	0
0.9375	1.9744e-03	3.9417e-03	0

In order to demonstrate the correctness and efficiency of the procedure, some numerical examples of the problem Eq. (1) are provided in this section.

Example 4.1. Let us first consider the linear VFIE (Esmaeili et. al., 2021; Al-Jarrah and En-Bing, 2013)

$$y(t)=\frac{2}{3}t-\frac{1}{3}t^4+\int_0^t tsy(s)ds+\int_0^1 tsy(s)ds,\quad 0\leq t,s\leq 1$$

We approximate the solution $y(t)$ by the polinomial

$$y(t)\cong y_N(t)=\sum_{n=0}^{1} a_n L_n(t),\qquad 0\leq t\leq 1$$

$g(t)=\frac{2}{3}t-\frac{1}{3}t^4$, $K_v(t,s)=K_f(t,s)=ts$ and the collocation points for $a = 0$, $b = 1$ and $N = 1$ are computed as

$\{t_0=0,\quad t_1=1\}$

Following the procedure in Section2, the fundamental matrix equation of the given equation becomes

$$\mathrm{T}D^TA-\overline{TK_vQ_v}D^TA-\mathrm{T}K_fQ_fD^TA=G$$

Where

$$T=\begin{bmatrix}1&0\\1&1\end{bmatrix},\ D^T=\begin{bmatrix}2&0\\0&1\end{bmatrix},\ \overline{T}=\begin{bmatrix}1&0&0&0\\0&0&1&1\end{bmatrix},\ K_v=K_f=\begin{bmatrix}0&0\\0&1\end{bmatrix},\ \overline{K_v}=\begin{bmatrix}K_v&0\\0&K_v\end{bmatrix},$$

$$Q(0)=\begin{bmatrix}0&0\\0&0\end{bmatrix},\ Q(1)=\begin{bmatrix}1&\frac{1}{2}\\\frac{1}{2}&\frac{1}{3}\end{bmatrix},\ Q_v=\begin{bmatrix}0&0\\0&0\\1&\frac{1}{2}\\\frac{1}{2}&\frac{1}{3}\end{bmatrix},\ Q_f=\begin{bmatrix}1&\frac{1}{2}\\\frac{1}{2}&\frac{1}{3}\end{bmatrix}$$

The augmented matrix for this fundamental matrix equation is

$$[\widetilde{W}\ \ ;\ \ \widetilde{G}]=\begin{bmatrix}2&0&;&0\\0&\frac{1}{3}&;&\frac{1}{3}\end{bmatrix}$$

Solving this system, $A$is obtained as $A=[0\quad 1]$. Thus, the approximate solution of the problem becomes $y_2(t)=t$ which is the exact solution.

Example 4.2. Let us consider another linear VFIE (Esmaeili et. al.,, 2021).

$$y(t)=-\frac{2}{5}t^7-\frac{5}{4}t^4+t^3-\frac{59}{20}t+1+\int_0^t(2t^2s+1)y(s)ds+\int_0^1 t(s+1)y(s)ds,\quad 0\leq t,s\leq 1$$

Following the procedure, for N = 3 the polynomial solution is obtained as follows:

$$y_3(t)=t^3-1.1585\cdot 10^{-14}t^2+1$$

which is the approximate solution of $y(t)=t^3+1$.

Table 2: Comparison of the errors in Example 4.2.

t	Repeated trapezoidal method	Tan-sec hyperbolic basis functions	Present method (N=3)
0.0625	7.8321e-01	2.8883e-03	2.3315e-14
0.1875	7.7222e-02	2.8887e-03	2.3537e-14
0.3125	7.6115e-02	2.8898e-03	2.4425e-14
0.4375	7.5049e-02	2.8917e-03	2.5313e-14
0.5625	7.4231e-02	2.8953e-03	2.6867e-14
0.6875	7.2119e-02	2.9053e-03	2.8644e-14
0.8125	7.1656e-03	2.9556e-03	3.0864e-14
0.9375	7.0119e-03	2.9530e-03	3.3307e-14

The comparison of the error functions of Example 4.2. by the presented method, repeated trapezoidal method (RT) and tan-sec hyperbolic basis functions method (TSHF) is given in Table 2 (Esmaeili et. al., 2021). Errors show that the method is quite useful.

Example 4.3. Consider the problem (Gecmen and Celik, 2021)

$$y(t)=2\cos(t)-t\cos(2)-2t\sin(2)+t-1+\int_0^t(t-s)y(s)ds+\int_0^2 tsy(s)ds,\quad 0\leq t,s\leq 2$$

According to the proposed method. The numerical results obtained from this example for diverse Lucas polynomial solution for N = 5, 8, 10, 13 and exact solution are tabulated in Table 3. It can be seen that, the proposed method gives better results than Hasoya polynomial method (Gecmen and Celik, 2021).

The numerical solution of the absolute errors in Example 4.3 are depicted in Figure 1. As the integer N is increased, the error goes down.

Example 4.4. Consider Volterra–Fredholm integral equations (Gecmen and Celik, 2021)

$$y(t)=2\cos(t)-t\cos(2)-2t\sin(2)+t-1+\int_0^t(t-s)y(s)ds+\int_0^2 tsy(s)ds,\quad 0\leq t,s\leq 2$$

which has the exact solution $y(t)=e^t$. Applying the proposed method to solve the problem for N = 5, 8, 12 becomes as follows:

$$y_5(t)=0.013849t^5+0.03488t^4+0.17039t^3+0.49907t^2+1.00009t+1$$

$y_8(t)=0.00004t^8+0.00016t^7+0.00143t^6+0.00831t^5+0.04168t^4+0.16666t^3+0.5t^2+t+1y_{12}(t)=4.30\cdot 10^{-9}t^{12}+1.56\cdot 10^{-8}t^{11}+2.97\cdot 10^{-7}t^{10}+2.73\cdot 10^{-6}t^9+0.00002t^8+0.00020t^7+0.00139t^6+0.00833t^5+0.04167t^4+0.16667t^3+0.5t^2+t+1$

Table 3: Comparison of The Results for Various N Values in Example 4.3.					
t	**Exact Solution**	**Lucas (N=5)**	**Lucas (N=8)**	**Lucas (N=10)**	**Lucas (N=13)**
0.2	0.98006657784124	0.98001760563704	0.98006658875074	0.98006657784074	0.98006657784125
0.4	0.92106099400289	0.92106558686879	0.92106098908030	0.92106099400067	0.92106099400288
0.6	0.82533561490968	0.82535803538746	0.82533561870648	0.82533561490583	0.82533561490968
0.8	0.69670670934717	0.69671509805512	0.69670670999477	0.69670670934148	0.69670670934716
1.0	0.54030230586814	0.54030257772998	0.54030230821837	0.54030230586041	0.54030230586814
1.2	0.36235775447667	0.36237153409265	0.36235775865209	0.36235775446659	0.36235775447667
1.4	0.16996714290024	0.16999788447242	0.16996714404558	0.16996714288738	0.16996714290024
1.6	-0.029199522301289	-0.029177995326438	-0.029199511359451	-0.029199522317404	-0.029199522301289
1.8	-0.22720209469309	-0.22721167019840	-0.22720210048450	-0.22720209471327	-0.22720209469310
2.0	-0.41614683654714	-0.41611504491043	-0.41614682933905	-0.41614683657090	-0.41614683654715

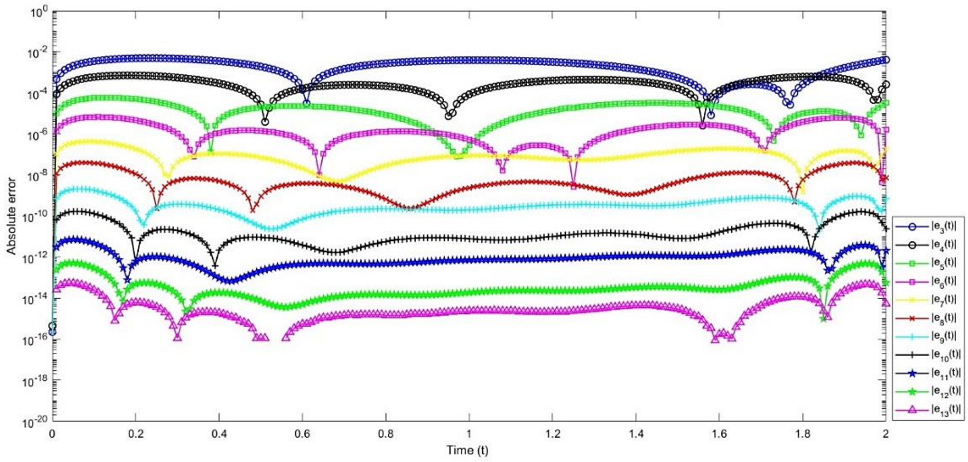

Figure 1: The absolute errors of Example 4.3 for $3 \leq N \leq 13$.

Figure 2 depicts the absolute errors to solution of Example 4.4. As the number N is increased, the error decreases.

In Table 4, we compare absolute errors our obtained results for various N values. From these comparisons, it is seen that the proposed method gives better results than Hosoya polynomial method.

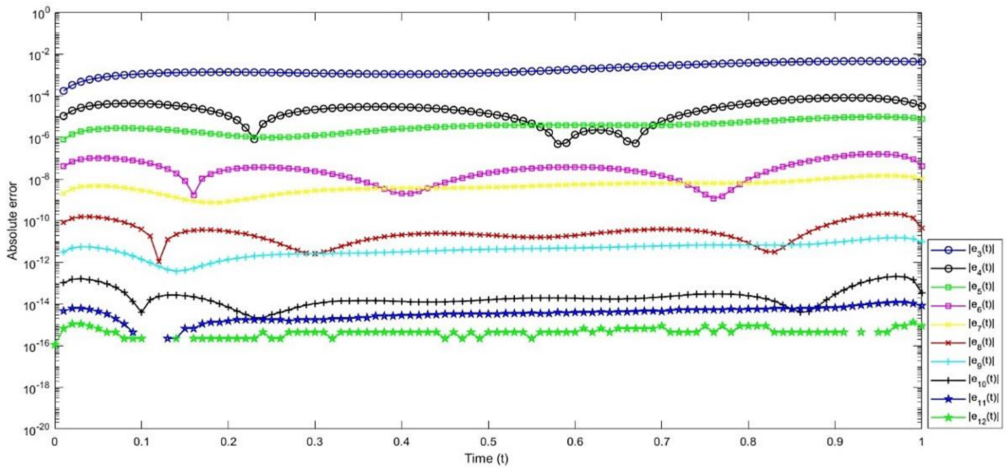

Figure 2: The absolute errors of Example 4.4 for $3 \leq N \leq 12$.

Table 4: Comparison of The Absolute Errors of The Present Method in Example 4.4.			
t	**Present Method** e_5	**Present Method** e_8	**Present Method** e_{12}
0.1	2.67112619334320e-06	3.77680109409084e-11	2.22044604925031e-16
0.2	1.26320138993385e-06	3.08288949923963e-11	2.22044604925031e-16
0.3	1.20827320349015e-06	2.58748578119139e-12	4.44089209850063e-16
0.4	2.55457771114465e-06	2.09001704831735e-11	4.44089209850063e-16
0.5	3.71772876062160e-06	2.02360350698427e-11	4.44089209850063e-16
0.6	3.87866988482699e-06	1.97146743374788e-11	4.44089209850063e-16
0.7	3.88606759793575e-06	3.86179976885614e-11	8.88178419700125e-16
0.8	5.50585681269311e-06	8.50119974415975e-12	8.88178419700125e-16
0.9	8.84410683266168e-06	8.41460234823899e-11	4.44089209850063e-16
1.0	7.75109438899690e-06	4.34599023435567e-11	8.88178419700125e-16

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5. Conclusion

This article investigates the Lucas collocation method's performance in solving Volterra-Fredholm integral equations. The technique was used to solve four test problems using MATLAB. Tables and figures are used to display the numerical solution, estimated solution, and graphs for each test issue. The Lucas collocation method works well to solve the Volterra-Fredholm integral problem, as can be observed when the results are analyzed.

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