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RESEARCH ARTICLE

MODIFIED COMPUTATIONAL METHOD BASED ON INTEGRAL TRANSFORM FOR SOLVING FRACTIONAL ZAKHAROV-KUZNETSOV EQUATIONS

Mudhafar Hamed Hamadamen[a], Sadeq Taha Abdulazeez[b,c], Muhammad Amin Sadiq Murad[d*]

[a] Department of Mathematics, College of Education, Salahaddin University, Erbil, Iraq
[b] Department of Mathematics, College of Basic Education, University of Duhok, Duhok, Iraq
[c] Department of Computer Science, College of Science, Nawroz University, Duhok, Iraq
[d] Department of Mathematics, College of Science, University of Duhok, Duhok, Iraq
[*]Corresponding Author Email: muhammad.math@uod.ac

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ABSTRACT

This work proposes a new computational method, namely the He-Elzaki transform method (HETM) formulated by He's variation iteration method and modified Laplace transform called Elzaki integral transform to solve nonlinear fractional Zakharov-Kuznetsov equations. The fractional derivatives are described by Caputo sense. The beauty of this technique is that one has no need to evaluate the Lagrange multiplier by integration or taking the convolution theorem. The suggested method is implemented on two examples and the results obtained are compared with those of the Variation iteration method (VIM), homotopy perturbation transform method (HPTM), and new iteration Sumudu transform method (NISTM), and optimum homotopy analysis method (OHAM). The innovative computational technique is an efficient high accurate method and facilitates solving fractional differential equations.

KEYWORDS

Fractional Zakharov–Kuznetsov, Variation Iteration Method, Elzaki Integral Transform

1. INTRODUCTION

Fractional calculus has been studied by numerous researchers and extensively implement on many real problems which are modeled in different fields of science. Besides fractional calculus is the focus of many studies due to their frequent appearance in various applications such as engineering, biology, thermodynamics, and fluid mechanics. Many phenomena in different fields of science can only be introduced by differential equations of fractional orders; therefore the fractional differential equations (FDEs) have been applied in various problems, especially in physics and engineering (Oldham and Spanier 1974; Luchko and Gorenflo 1998). The exact and analytical solutions of (FDEs) are quite difficult to be found; therefor the numerical methods are used to find the solutions of (FDEs) (Murad 2018; Murad and Hamasalh 2022).

In the last decade, various techniques are combined to solve linear and nonlinear differential equations of fractional and integer orders such as Laplace Adomian decomposition (LADM), Laplace homotopy perturbation method (LHPM), homotopy analysis Sumudu transform method (HASTM), Elzaki homotopy perturbation method (EHPM), He-Elzaki method (HEM), Laplace variation iteration method (LVIM), and there are other combined techniques which give significant results in solving (FDEs) (Shah et al., 2019; Khan et al., 2019; Ul Rahman et al., 2019; Elzaki and Biazar 2013; Nadeem et al., 2019; Pandey and Mishra, 2015; Aminikhah 2012). Recently, Elzaki homotopy transformation perturbation method has been used to solve a class of problems such as a family of Fisher's equation, spatial diffusion of biological population, nonlinear oscillators (Loyinmi and Akinfe 2020; Ul Rahman et al. 2019; Anjum et al. 2019). In this paper, the Elzaki transform combined with the modified (VIM) is used to solve the well-known fractional Zakharov–Kuznetsov equation (FZK). Consider the following FZK (k_1, k_2, k_3) equation:

$$D_t^{\beta} z + a(z^{k_1})_{\theta} + b(z^{k_2})_{\theta\theta\theta} + c(z^{k_3})_{\theta\vartheta\vartheta} = 0, \tag{1}$$

where $= z(\theta, \vartheta, t)$, $0 < \beta \leq 1$, and a, b, and c are constants. k_1, k_2 and k_3 are integers numbers and conductors of weak nonlinear acoustic ion vibrators in a plasma comprising cool ions and hot exothermic electronic at electric field system (Kumar, Singh, and Kumar 2014). Due to its widely used and applications, the (FZK) has been solved by many researchers using different techniques like (HPM), VIM, LADM, and perturbation-iteration algorithm and residual power series method (Shah et al., 2019; Molliq et al. 2009; Biazar and Azimi, 2009; Şenol and Kasmaei, 2018).

The VIM first was suggested by mathematician Inokuti, while the Lagrange multiplier was complex to be introduced. In 1998 the concept VIM was developed by Chinese mathematician He, and has been applied via numbers of researchers in different types of nonlinear problems (Rau et al., 1978; He, 1999; Inokuti et al., 1980; Biazar and Ghazvini 2007; Akbarzade and Langari, 2011; Geng and Lin 2009; Saadatmandi and Dehghan, 2009). Elzaki integral transform is a modification of the Laplace and Sumudu transforms which proposed by Tariq (Elzaki, 2011). Elzaki transformation is used to find the exact solutions to a class of differential equations which cannot be solved by Sumudu transform (Hilal, 2012). Elzaki integral equation is a powerful and efficient technique that has been used to solve linear and nonlinear differential equations of integer and fractional orders see (Elzaki, 2012; Ziane and Cherif, 2015; Ige et al., 2019; Jena and Chakraverty, 2019; Murad 2022; Malo et al., 2021).

2. PRELIMINARIES

In this section, we introduce some definitions and properties of fractional calculus and Elzaki transform which are used in this article.

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Definition 2.1. A real valued function $h(z) \in C_\sigma,\ z > 0$, is said to be in the space $C_\sigma, \sigma \in R$ if there exists at least a real number $d > \sigma$, s.t $h(z) = z^d h_1(z)$ where $h_1(z) \in C(0,\infty)$, and it is said to be in the space C_σ^m if $h^m \in C_\sigma$, $m \in N$ (Prakash and Verma, 2019).

Definition 2.2. The function $f(x)$ is called Riemann-Liouvill fractional integral of order $\alpha > 0$ if it defines as (Prakash and Verma, 2019):

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt \qquad ,t > 0.$$

In particular $D^0 f(x) = f(x)$.

For $\mu \geq 0$ and $\tau \geq -1$, some properties of the operator D^α

1. $D^\alpha D^\mu f(x) = D^{\alpha+\mu} f(x)$,
2. $D^\alpha D^\mu f(x) = D^\mu D^\alpha f(x)$,
3. $D^\alpha y^\tau = \frac{\Gamma(\tau+1)}{\Gamma(\alpha+\tau+1)} y^{\alpha+\tau}$.

Definition 2.3. The function $f \in C_{-1}^n$, $n \in N$ is called Caputo fractional derivative if it defines as (Prakash and Verma, 2019)

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1} f^n(t)dt, \qquad n-1 < \alpha \leq n.$$

Definition 2.1. The Elzaki-transform of the function $f(y)$ is defined as (Elzaki, 2011):

$$E[f(y)] = T(s) = s\int_0^\infty f(y)e^{\frac{-y}{s}}dy \qquad y > 0.$$

3. Elzaki Transform of The Caputo Derivative

Suppose that f is piecewise continuous, then we can calculate $E\left[\frac{\partial f}{\partial x}\right]$ as follows:

$$E\left[\frac{\partial f(x,y)}{\partial x}\right] = \int_0^\infty s e^{\frac{-y}{s}} \frac{\partial f(x,y)}{\partial x} dy = \frac{\partial}{\partial x}\int_0^\infty s e^{\frac{-y}{s}} f(x,y)dy = \frac{d}{dx}T(x,s),$$

Similarly, we can have:

$$E\left[\frac{\partial^2 f(x,s)}{\partial x^2}\right] = \frac{d^2 T(x,s)}{dx^2}.$$

Assume that $\frac{\partial f}{\partial y} = g$, then we have:

$$E\left[\frac{\partial^2 f(x,y)}{\partial y^2}\right] = E\left[\frac{\partial g(x,y)}{\partial y}\right] = \frac{1}{s}E[g(x,y)] - sg(x,0)$$

$$E\left[\frac{\partial^2 f(x,y)}{\partial y^2}\right] = \frac{T(x,s)}{s^2} - f(x,0) - s\frac{\partial f}{\partial y}(x,0)$$

By mathematical induction one can extend this result to the m^{th} partial derivative as follows:

$$E\left[\frac{\partial^m f(x,y)}{\partial y^m}\right] = \frac{T(x,s)}{s^n} - \sum_{i=0}^{m-1} s^{2-m+i}\frac{\partial^i f(x,0)}{\partial y^i}. \tag{2}$$

4. Homotopy Perturbation Method

In this section, we study the concept of homotropy perturbation method. Consider the following differential equation

$$Rz - Nz = p, \tag{3}$$

where p be a source term, R and N represent linear and non-linear operators, respectively, and z is the solution of the function.

According to the homtopy theory $H(w,p)$, $H(w,p): R \times [0,1] \to R$ that satisfies the equation

$$H(u,p) = (1-p)[R(u) - R(u_0)] + p[R(u) - N(u) - p] = 0,$$

Here, we obtain

$$R(u) - pN(u) = p, \tag{4}$$

where the embedding parameter $p \in [0,1]$, and z_0 represents the initial approximation of the solution of equation (3), further u is the homotopy function with $R(z_0) = k$.

Since, u can be written as:

$$u = \lim_{p\to 1}(u_0 + pu_1 + p^2u_2 + \cdots) \tag{5}$$

Using (4) and (5), we have

$$u_0 + pu_1 + p^2u_2 + p^3u_3 \ldots = k + pN(u)$$

Equating the powers of p can be written as follows:

p^0: $u_0 = k$,

p^1: $u_1 = N(u_0)$,

p^2: $u_2 = u_1 N'(u_0)$,

p^3: $u_3 = N'(u_0) + \frac{{u_1}^2 N''(u_0)}{2}$,

As p approach to 1 the approximate solution of (3) is

$$u = u_0 + u_1 + u_2 + u_3 \ldots. \tag{6}$$

Finally, the convergence and uniqueness of the series (6) were investigated in (Biazar and Aminikhah, 2009; Turkyilmazoglu 2011).

5. He-Elzaki Transform Method (HETM)

In this section, we construct the scheme of the innovative technique that employed to study the solution of the proposed problem. The Lagrange multiplier is defined in a new pattern and compute without using integration or convolution theorem. First, we take the Elzaki transform of the suggested fractional differential equation and multiply it by the Lagrange multiplier to get the recurrence relation. The series solution of the proposed problem is found by the well-known semi-analytic technique (HPM). The nonlinear terms are computed by the Adomin polynomial. Consider the following nonlinear FZK (k_1,k_2,k_3) equation:

$$D_t^\beta z + a(z^{k_1})_\theta + b(z^{k_2})_{\theta\theta\theta} + c(z^{k_3})_{\theta\vartheta\vartheta} = 0,\ 0 < \beta \leq 1, \tag{7}$$

where $D_t^\beta = \frac{\partial^\beta}{\partial t}$ is the Caputo fractional derivative. Assume that $z(\theta,\vartheta,0) = l(\theta,\vartheta)$ is the initial condition.

Applying the Elzaki transform on both sides, we obtain:

$$E\left[D_t^\beta z + a(z^{k_1})_\theta + b(z^{k_2})_{\theta\theta\theta} + c(z^{k_3})_{\theta\vartheta\vartheta}\right] = 0$$

Taking the Lagrange multiplier $\mu(v)$

$$\mu(v)\left\{E\left[D_t^\beta z + a(z^{k_1})_\theta + b(z^{k_2})_{\theta\theta\theta} + c(z^{k_3})_{\theta\vartheta\vartheta}\right]\right\} = 0.$$

Here, we have

$$Z(v) = Z(v) + \mu(v)\left\{E\left[D_t^\beta z + a(z^{k_1})_\theta + b(z^{k_2})_{\theta\theta\theta} + c(z^{k_3})_{\theta\vartheta\vartheta}\right]\right\}$$

Here, we can have the recurrence relation as given below

$$Z_{j+1}(v) = Z_j(v) + \mu(v)\left\{E\left[D_t^\beta z_j + a({z_j}^{k_1})_\theta + b({z_j}^{k_2})_{\theta\theta\theta} + c({z_j}^{k_3})_{\theta\vartheta\vartheta}\right]\right\}. \tag{8}$$

The variation of the above equation gives

$$\rho Z_{j+1}(\theta,\vartheta,v) = \rho Z_j(\theta,\vartheta,v) + \mu(v)\rho\left\{\frac{Z_j(\theta,\vartheta,v)}{v^\beta} - v^{2-\beta}\hat{Z}_j(\theta,\vartheta,0) + E\left[b({\hat{z}_j}^2(\theta,\vartheta,0))_\theta + b({\hat{z}_j}^2(\theta,\vartheta,0))_{\theta\theta\theta} + c({\hat{z}_j}^2(\theta,\vartheta,0))_{\theta\vartheta\vartheta}\right]\right\} \tag{9}$$

The above recurrence represents the modified He-Elzaki transform relation. We apply the optimal condition to introduce the Lagrange multiplier $\mu(v)$, Note that, $\hat{z}_j = \hat{z}_j(\theta,\vartheta,0) = \hat{Z}_j(\theta,\vartheta,0)$ are restricted variables, these leads $\rho\hat{z}_j(\theta,\vartheta,0) = \rho\hat{Z}_j(\theta,\vartheta,0) = 0$ and

$$\frac{\rho Z_{j+1}(v)}{\rho Z_j(v)} = 0.$$

Now, we have

$$\rho Z_{j+1}(\theta,\vartheta,v) = \rho Z_j(\theta,\vartheta,v) + \mu(v)\rho\left\{\frac{Z_j(\theta,\vartheta,v)}{v^\beta}\right\}.$$

Here, we obtain

$$\mu(v) = -v^\beta.$$

Now, taking the inverse Elzaki transform of equation (8) to achieve the estimate solution as:

$$z_{j+1}(v) = z_j(v) - E^{-1}\left[v^\beta\left\{E\left[D_t^\beta z + a(z^{k_1})_\theta + b(z^{k_2})_{\theta\theta\theta} + c(z^{k_3})_{\theta\vartheta\vartheta}\right]\right\}\right] \tag{10}$$

The nonlinear terms are calculated by the Adomian polynomial as follows:

$$z^k = \sum_{j=0}^\infty A_j,$$

$$A_j = \frac{1}{j!}\frac{d^j}{d\tau^j}(N(\sum_{j=0}^\infty z_j\tau^j)), \qquad j = 0,1,2,3,\ldots.$$

At the end, one can apply (HPM) to find the series approximate solution of equation (10).

6. APPLICATIONS

The numerical models are used to confirm the efficiency and accuracy of the suggested technique for solving the FZK. For this purpose, we illustrate two test examples to convey our modification method.

Example 6.1 Consider the following nonlinear FZK (2, 2, 2) of fractional order $0 < \beta \le 1$.

$$D_t^{\beta} z + (z^2)_\theta + \frac{1}{8}(z^2)_{\theta\theta\theta} + \frac{1}{8}(z^2)_{\theta\vartheta\vartheta} = 0, \quad (11)$$

with the initial condition $z(\theta, \vartheta, 0) = \frac{4}{3}\delta \sinh^2(\theta + \vartheta)$, where δ is an arbitrary constant.

Taking the Elzaki transform of equation (11), we have:

$$E\left[\frac{\partial^\beta z}{\partial t^\beta} + (z^2)_\theta + \frac{1}{8}(z^2)_{\theta\theta\theta} + \frac{1}{8}(z^2)_{\theta\vartheta\vartheta}\right] = 0.$$

Here, multiplying both sides of the above equation by Lagrange multiplier $\mu(v)$

$$\mu(v)E\left[\frac{\partial^\beta z}{\partial t^\beta} + (z^2)_\theta + \frac{1}{8}(z^2)_{\theta\theta\theta} + \frac{1}{8}(z^2)_{\theta\vartheta\vartheta}\right] = 0.$$

The recurrence relation has the following form

$$Z_{j+1}(x, v) = Z_j(x, v) + \mu(v)E\left[\frac{\partial^\beta z_j}{\partial t^\beta} + ({z_j}^2)_\theta + \frac{1}{8}({z_j}^2)_{\theta\theta\theta} + \frac{1}{8}({z_j}^2)_{\theta\vartheta\vartheta}\right]. \quad (12)$$

Using the Elzaki differentiation property (2) and taking the variation of the above equation, we obtain

$$\rho Z_{j+1}(\theta, \vartheta, v) = \rho Z_j(\theta, \vartheta, v) + \mu(v)\rho\left\{\frac{Z_j(\theta,\vartheta,v)}{v^{2\alpha}} - v^{2-\alpha}\hat{Z}_j(\theta, \vartheta, 0) + E\left[({\hat{z}_j}^2(\theta, \vartheta, v))_\theta + \frac{1}{8}({\hat{z}_j}^2(\theta, \vartheta, v))_{\theta\theta\theta} + \frac{1}{8}({\hat{z}_j}^2(\theta, \vartheta, v))_{\theta\vartheta\vartheta}\right]\right\}. \quad (13)$$

Note that, $\hat{z}_j = \hat{z}_j(\theta, \vartheta, 0) = \hat{Z}_j(\theta, \vartheta, 0)$ are restricted variables and these leads $\rho\hat{z}_j(\theta, \vartheta, 0) = \rho\hat{Z}_j(\theta, \vartheta, 0) = 0$ and since $\frac{\rho Z_{j+1}(\theta,\vartheta,v)}{\rho Z_j(\theta,\vartheta,v)} = 0$.

Substituting restricted variables in equation (13), gives

$$\rho Z_{j+1}(\theta, \vartheta, v) = \rho Z_j(\theta, \vartheta, v) + \frac{1}{v^\beta}\mu(v)\rho Z_j(\theta, \vartheta, v).$$

Thus, the Lagrange multiplier $\mu(v) = -v^\beta$.

Substituting the Lagrange multiplier in equation (10), we obtain

$$Z_{j+1}(\theta, \vartheta, v) = Z_j(\theta, \vartheta, v) - v^\beta E\left[\frac{\partial^\beta z_j}{\partial t^\beta} + ({z_j}^2)_\theta + \frac{1}{8}({z_j}^2)_{\theta\theta\theta} + \frac{1}{8}({z_j}^2)_{\theta\vartheta\vartheta}\right].$$

Applying Elzaki inverse, we obtain

$$z_{j+1}(\theta, \vartheta, v) = z_j(\theta, \vartheta, v) - E^{-1}\left[v^\beta E\left[\frac{\partial^\beta z_j(\theta,\vartheta,v)}{\partial t^\beta} + ({z_j}^2(\theta, \vartheta, v))_\theta + \frac{1}{8}({z_j}^2(\theta, \vartheta, v))_{\theta\theta\theta} + \frac{1}{8}({z_j}^2(\theta, \vartheta, v))_{\theta\vartheta\vartheta}\right]\right]$$

Since $\frac{\partial^\beta z_j}{\partial t^\beta} = 0, j = 0,1,2, \ldots$, and to get He's polynomial, we use HPM

$$z_0 + pz_1 + p^2 z_2 + p^3 z_3 \ldots = z_j(\theta, \vartheta, t) - pE^{-1}\left[v^\beta E\left[(A_j)_\theta + \frac{1}{8}(A_j)_{\theta\theta\theta} + \frac{1}{8}(A_j)_{\theta\vartheta\vartheta}\right]\right],$$

where $A_j,\ j = 0,1,2, \ldots,$ is an Adomian polynomial of $(z_0,\ z_1,\ z_2, z_3 \ldots)$, thus we have

$$A_0 = z_0^2,$$

$$A_1 = 2z_0 z_1,$$

$$A_2 = 2z_0 z_2 + z_1^2,$$

$$z_0 + pz_1 + p^2 z_2 + p^3 z_3 \ldots = z_j(\theta, \vartheta, t) - pE^{-1}\left[v^\beta E\left[\left(({z_0}^2)_\theta + \frac{1}{8}({z_0}^2)_{\theta\theta\theta} + \frac{1}{8}({z_0}^2)_{\theta\vartheta\vartheta}\right) + p\left((2z_0z_1)_\theta + \frac{1}{8}(2z_0z_1)_{\theta\theta\theta} + \frac{1}{8}(2z_0z_1)_{\theta\vartheta\vartheta}\right) + p^2\left((2z_0z_2 + z_1^2)_\theta + \frac{1}{8}(2z_0z_2 + z_1^2)_{\theta\theta\theta} + \frac{1}{8}(2z_0z_2 + z_1^2)_{\theta\vartheta\vartheta}\right)\right]\right].$$

Equating the highest powers of p

$$p^0\ :\ z_0 = z_0(x, t) + tz_{0_t}(x, t)$$

$$p^1\ :\ z_1 = -E^{-1}\left[v^{2\alpha}E\left[\left(({z_0}^2)_\theta + \frac{1}{8}({z_0}^2)_{\theta\theta\theta} + \frac{1}{8}({z_0}^2)_{\theta\vartheta\vartheta}\right)\right]\right]$$

$$p^2\ :\ z_2 = -E^{-1}\left[v^{2\alpha}E\left[\left((2z_0z_1)_\theta + \frac{1}{8}(2z_0z_1)_{\theta\theta\theta} + \frac{1}{8}(2z_0z_1)_{\theta\vartheta\vartheta}\right)\right]\right]$$

$$p^3\ :\ z_3 = -E^{-1}\left[v^{2\alpha}E\left[\left((2z_0z_2 + z_1^2)_\theta + \frac{1}{8}(2z_0z_2 + z_1^2)_{\theta\theta\theta} + \frac{1}{8}(2z_0z_2 + z_1^2)_{\theta\vartheta\vartheta}\right)\right]\right]$$

Therefore, we obtain

$$z_0(\theta, \vartheta, t) = \frac{4}{3}\delta\, sinh^2(\theta + \vartheta).$$

$$z_1(\theta, \vartheta, t) = -\delta^2\left(\frac{32}{3}sinh(\theta + \vartheta)\, cosh^3(\theta + \vartheta) + \frac{224}{9}cosh(\theta + \vartheta)\, sinh^3(\theta + \vartheta)\right)\frac{t^\beta}{\Gamma(\beta+1)},$$

$$z_2(\theta, \vartheta, t) = -\delta^3(79 - 1200cosh^6(\theta + \vartheta) + 2080\, cosh^4(\theta + \vartheta) - 968cosh^2(\theta + \vartheta))\frac{t^{2\beta}}{\Gamma(2\beta+1)},$$

One can expressed these results in a series such as:

$$z(\theta, \vartheta, t) = z_0(\theta, \vartheta, t) + z_1(\theta, \vartheta, t) + z_2(\theta, \vartheta, t) + z_3(\theta, \vartheta, t) + \cdots.$$

$$z(\theta, \vartheta, t) = \frac{4}{3}\delta\, sinh^2(\theta + \vartheta) - \delta^2\left(\frac{32}{3}sinh(\theta + \vartheta)\, cosh^3(\theta + \vartheta) + \frac{224}{9}cosh(\theta + \vartheta)\, sinh^3(\theta + \vartheta)\right)\frac{t^\beta}{\Gamma(\beta + 1)} - \delta^3(79 - 1200cosh^6(\theta + \vartheta) + 2080\, cosh^4(\theta + \vartheta) - 968cosh^2(\theta + \vartheta))\frac{t^{2\beta}}{\Gamma(2\beta + 1)} \ldots,$$

when $\beta = 1$, the HETM solution for equation (9) is

$$z(\theta, \vartheta, t) = \frac{4}{3}\delta\, sinh^2(\theta + \vartheta - \delta t).$$

Table 1: The Absolute Error Comparison at $\beta = 1$ and $\delta = 0.001$

	$\theta, \vartheta \backslash t$	0.00	0.04	0.08	0.1
HPTM	0.00	2.00×10^{-8}	3.01×10^{-7}	6.35×10^{-7}	8.11×10^{-7}
NISTM		2.00×10^{-8}	3.01×10^{-7}	6.35×10^{-7}	8.12×10^{-7}
VIM		2.00×10^{-8}	7.90×10^{-6}	1.59×10^{-5}	2.00×10^{-5}
HETM		0.00	3.20×10^{-11}	1.27×10^{-10}	1.99×10^{-10}
HPTM	0.04	3.01×10^{-7}	6.35×10^{-7}	9.98×10^{-7}	1.2×10^{-6}
NISTM		3.01×10^{-7}	6.35×10^{-7}	9.98×10^{-7}	1.2×10^{-6}
VIM		7.90×10^{-6}	1.59×10^{-5}	2.41×10^{-5}	2.84×10^{-5}
HETM		0.00	2.64×10^{-8}	5.27×10^{-8}	6.58×10^{-8}
HPTM	0.08	6.35×10^{-7}	9.98×10^{-7}	1.41×10^{-6}	1.63×10^{-6}
NISTM		6.35×10^{-7}	9.98×10^{-7}	1.41×10^{-6}	1.63×10^{-6}
VIM		1.59×10^{-5}	2.41×10^{-5}	3.27×10^{-5}	3.72×10^{-5}
HETM		0.00	5.79×10^{-8}	1.15×10^{-7}	1.44×10^{-7}
HPTM	0.1	8.12×10^{-7}	1.2×10^{-6}	1.63×10^{-6}	1.88×10^{-6}
NISTM		8.12×10^{-7}	1.2×10^{-6}	1.63×10^{-6}	4.18×10^{-5}
VIM		2.00×10^{-5}	2.84×10^{-5}	3.72×10^{-5}	1.88×10^{-6}
HETM		0.00	7.74×10^{-8}	1.54×10^{-7}	1.93×10^{-7}

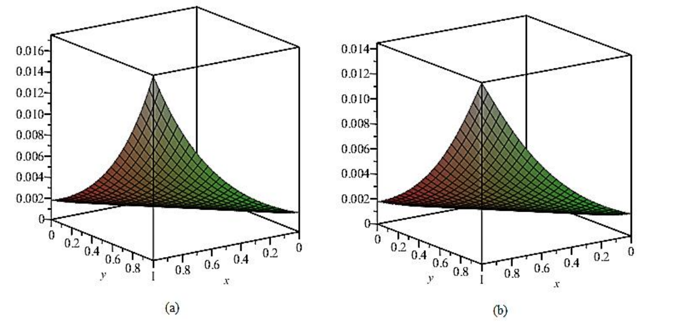

Figure 1: (a) The exact solution and (b) the approximate solution for FZK (2, 2, 2) at $\beta = 1$

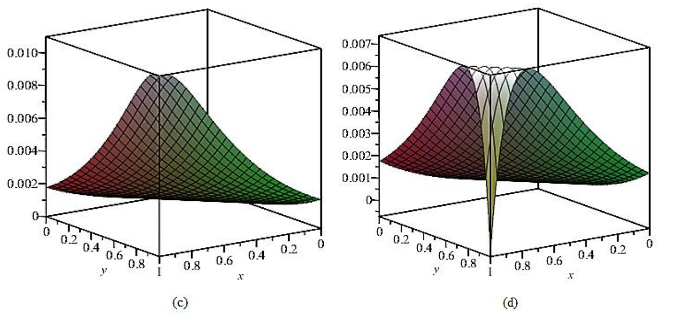

Figure 2: The approximate solution for FZK (2, 2, 2) at (c) $\beta = 0.75$ and (d) $\beta = 0.5$

Example 6.2 Consider the following nonlinear FZK (3,3,3) of fractional order $0 < \beta \le 1$.

$$D_t^\beta z + (z^3)_\theta + (z^3)_{\theta\theta\theta} + 2(z^3)_{\theta\vartheta\vartheta} = 0, \tag{12}$$

with the initial condition $z(\theta,\vartheta,0) = \frac{3}{2}\delta \sinh^2\left(\frac{\theta}{6}+\frac{\vartheta}{6}\right)$.

Taking the Elzaki transform of equation (9), we have

$$E\left[\frac{\partial^\beta z}{\partial t^\beta} + (z^3)_\theta + 2(z^3)_{\theta\theta\theta} + 2(z^3)_{\theta\vartheta\vartheta}\right] = 0.$$

Here, multiplying both sides of the above equation by Lagrange multiplier $\mu(v)$

$$\mu(v)E\left[\frac{\partial^\beta z}{\partial t^\beta} + (z^3)_\theta + 2(z^3)_{\theta\theta\theta} + 2(z^3)_{\theta\vartheta\vartheta}\right] = 0.$$

The recurrence relation has the following form

$$Z_{j+1}(x,v) = Z_j(x,v) + \mu(v)E\left[\frac{\partial^\beta z_j}{\partial t^\beta} + ({z_j}^3)_\theta + 2({z_j}^3)_{\theta\theta\theta} + 2({z_j}^3)_{\theta\vartheta\vartheta}\right] \tag{13}$$

Using the Elzaki differentiation property (2) and taking the variation of the above equation, we obtain:

$$\rho Z_{j+1}(\theta,\vartheta,v) = \rho Z_j(\theta,\vartheta,v) + \mu(v)\rho\left\{\frac{Z_j(\theta,\vartheta,v)}{v^\beta} - v^{2-\beta}\hat{Z}_j(\theta,\vartheta,0) + E\left[({\hat{z}_j}^3(\theta,\vartheta,v))_\theta + 2({\hat{z}_j}^3(\theta,\vartheta,v))_{\theta\theta\theta} + 2({\hat{z}_j}^3(\theta,\vartheta,v))_{\theta\vartheta\vartheta}\right]\right\} \tag{14}$$

Note that, $\hat{z}_j = \hat{z}_j(\theta,\vartheta,0) = \hat{Z}_j(\theta,\vartheta,0)$ are restricted variables, these leads $\rho\hat{z}_j(\theta,\vartheta,0) = \rho\hat{Z}_j(\theta,\vartheta,0) = 0$ and since $\frac{\rho Z_{j+1}(\theta,\vartheta,v)}{\rho Z_j(\theta,\vartheta,v)} = 0$

Substituting restricted variables in equation (14), gives

$$\rho Z_{j+1}(\theta,\vartheta,v) = \rho Z_j(\theta,\vartheta,v) + \frac{1}{v^\beta}\mu(v)\rho Z_j(\theta,\vartheta,v).$$

Thus, the Lagrange multiplier $\mu(v) = -v^\beta$.

Substituting the Lagrange multiplier in equation (10), we obtain

$$Z_{j+1}(\theta,\vartheta,v) = Z_j(\theta,\vartheta,v) - v^\beta E\left[\frac{\partial^\beta z_j}{\partial t^\beta} + ({z_j}^3)_\theta + 2({z_j}^3)_{\theta\theta\theta} + 2({z_j}^3)_{\theta\vartheta\vartheta}\right].$$

Applying Elzaki inverse, we have

$$z_{j+1}(\theta,\vartheta,v) = z_j(\theta,\vartheta,v) - E^{-1}\left[v^\beta E\left[\frac{\partial^\beta z_j(\theta,\vartheta,v)}{\partial t^\beta} + ({z_j}^2(\theta,\vartheta,v))_\theta + 2({z_j}^2(\theta,\vartheta,v))_{\theta\theta\theta} + 2({z_j}^2(\theta,\vartheta,v))_{\theta\vartheta\vartheta}\right]\right]$$

Since $\frac{\partial^\beta z_j}{\partial t^\beta} = 0$, $j = 0,1,2,\ldots$, and to get He's polynomial, we use HPM

$$z_0 + pz_1 + p^2z_2 + p^3z_3 \ldots = z_j(\theta,\vartheta,t) - pE^{-1}\left[v^\beta E\left[(A_j)_\theta + \frac{1}{8}(A_j)_{\theta\theta\theta} + \frac{1}{8}(A_j)_{\theta\vartheta\vartheta}\right]\right],$$

where A_j, $j = 0,1,2,\ldots$, is an Adomian polynomial of $(z_0, z_1, z_2, z_3 \ldots)$, thus we have

$$A_0 = z_0^3,$$
$$A_1 = 3z_0^2z_1,$$
$$A_2 = 3z_0^2z_2 + 3z_0^2z_1^2,$$

$$z_0 + pz_1 + p^2z_2 + p^3z_3 \ldots = z_j(\theta,\vartheta,t) - pE^{-1}\Big[v^\beta E\Big[\Big((z_0^3)_\theta + \frac{1}{8}(z_0^3)_{\theta\theta\theta} + \frac{1}{8}(z_0^3)_{\theta\vartheta\vartheta}\Big) + p((3z_0^2z_1)_\theta + 2(3z_0^2z_1)_{\theta\theta\theta} + 2(3z_0^2z_1)_{\theta\vartheta\vartheta}) + p^2((3z_0^2z_2 + 3z_0^2z_1^2)_\theta + 2(3z_0^2z_2 + 3z_0^2z_1^2)_{\theta\theta\theta} + 2(3z_0^2z_2 + 3z_0^2z_1^2)_{\theta\vartheta\vartheta})\Big]\Big].$$

Equating the highest powers of p

$$p^0 \ : \ z_0 = z_0(x,t) + tz_{0_t}(x,t)$$

$$p^1 \ : \ z_1 = -E^{-1}\left[v^{2\alpha}E\left[\left((z_0^3)_\theta + \frac{1}{8}(z_0^3)_{\theta\theta\theta} + \frac{1}{8}(z_0^3)_{\theta\vartheta\vartheta}\right)\right]\right]$$

$$p^2 \ : \ z_2 = -E^{-1}\left[v^{2\alpha}E[((3z_0^2z_1)_\theta + 2(3z_0^2z_1)_{\theta\theta\theta} + 2(3z_0^2z_1)_{\theta\vartheta\vartheta})]\right]$$

$$p^3 \ : \ z_3 = -E^{-1}\left[v^{2\alpha}E[((3z_0^2z_2 + 3z_0^2z_1^2)_\theta + 2(3z_0^2z_2 + 3z_0^2z_1^2)_{\theta\theta\theta} + 2(3z_0^2z_2 + 3z_0^2z_1^2)_{\theta\vartheta\vartheta})]\right]$$

Therefore, we obtain

$$z_0(\theta,\vartheta,t) = \frac{3}{2}\delta\sinh^2\left(\frac{\theta}{6}+\frac{\vartheta}{6}\right).$$

$$z_1(\theta,\vartheta,t) = -3\delta^3\left(\cosh\left(\frac{\theta}{6}+\frac{\vartheta}{6}\right)sinh^2\left(\frac{\theta}{6}+\frac{\vartheta}{6}\right) + \frac{1}{8}\cos h^3\left(\frac{\theta}{6}+\frac{\vartheta}{6}\right)\right)\frac{t^\beta}{\Gamma(\beta+1)},$$

$$z_2(\theta,\vartheta,t) = \frac{3}{32}\delta^5 sinh\left(\frac{\theta}{6}+\frac{\vartheta}{6}\right)\left(91 - 729cosh^2\left(\frac{\theta}{6}+\frac{\vartheta}{6}\right) + 765\,cosh^4\left(\frac{\theta}{6}+\frac{\vartheta}{6}\right)\right)\frac{t^{2\beta}}{\Gamma(2\beta+1)},$$

One can expressed these results in a series such as:

$$z(\theta,\vartheta,t) = z_0(\theta,\vartheta,t) + z_1(\theta,\vartheta,t) + z_2(\theta,\vartheta,t) + z_3(\theta,\vartheta,t) + \cdots.$$

$$z(\theta,\vartheta,t) = \frac{3}{2}\delta\sinh^2\left(\frac{\theta}{6}+\frac{\vartheta}{6}\right) - 3\delta^3\left(cosh\left(\frac{\theta}{6}+\frac{\vartheta}{6}\right)sinh^2\left(\frac{\theta}{6}+\frac{\vartheta}{6}\right) + \frac{1}{8}cosh^3\left(\frac{\theta}{6}+\frac{\vartheta}{6}\right)\right)\frac{t^\beta}{\Gamma(\beta+1)} + \frac{3}{32}\delta^5 sinh\left(\frac{\theta}{6}+\frac{\vartheta}{6}\right)\left(91 - 729cosh^2\left(\frac{\theta}{6}+\frac{\vartheta}{6}\right) + 765\,cosh^4\left(\frac{\theta}{6}+\frac{\vartheta}{6}\right)\right)\frac{t^{2\beta}}{\Gamma(2\beta+1)}.$$

Table 2: 1st term approximate solution obtains by the HETM in comparison with 1st order and 3 terms approximate solutions obtained by OHAM and VIM respectively, and exact solution for FZK (3, 3, 3) at $\beta = 1$ and $\delta = 0.001$.

θ,ϑ	t	VIM	OHAM	HETM	Exact
0.1	0.2	5.00091×10^{-5}	5.00092×10^{-5}	5.00091×10^{-5}	4.99592×10^{-5}
	0.3	5.00091×10^{-5}	5.00091×10^{-5}	5.00091×10^{-5}	4.99342×10^{-5}
	0.4	5.00091×10^{-5}	5.00091×10^{-5}	5.00091×10^{-5}	4.99092×10^{-5}
0.6	0.2	3.02003×10^{-4}	3.02004×10^{-4}	3.02003×10^{-4}	3.01953×10^{-4}
	0.3	3.02003×10^{-4}	3.02004×10^{-4}	3.02003×10^{-4}	3.01927×10^{-4}
	0.4	3.02003×10^{-4}	3.02004×10^{-4}	3.02003×10^{-4}	3.01901×10^{-4}
0.9	0.2	4.56780×10^{-4}	4.56780×10^{-4}	4.56780×10^{-4}	4.56728×10^{-4}
	0.3	4.56780×10^{-4}	4.56780×10^{-4}	4.56780×10^{-4}	4.56702×10^{-4}
	0.4	4.56780×10^{-4}	4.56780×10^{-4}	4.56780×10^{-4}	4.56675×10^{-4}

Table 3: 1st term approximate solution obtains by the HETM in comparison with 1st order and 3 terms approximate solutions obtained by OHAM and VIM respectively, for FZK (3, 3, 3) at $\delta = 0.001$

	$\beta = 0.67$			$\beta = 0.75$		
θ, ϑ	0.1	0.6	0.9	0.1	0.6	0.9
t	VIM	OHAM	HETM	VIM	OHAM	HETM
0.2	5.00091×10^{-5}	5.00092×10^{-5}	5.00091×10^{-5}	5.00091×10^{-5}	5.00091×10^{-5}	5.00091×10^{-5}
0.3	5.00090×10^{-5}	5.00091×10^{-5}	5.00090×10^{-5}	5.00090×10^{-5}	5.00091×10^{-5}	5.00090×10^{-5}
0.4	5.00090×10^{-5}	5.00091×10^{-5}	5.00090×10^{-5}	5.00090×10^{-5}	5.00091×10^{-5}	5.00090×10^{-5}
0.2	3.02003×10^{-4}	3.02004×10^{-4}	3.02003×10^{-4}	3.02003×10^{-4}	3.02004×10^{-4}	3.02003×10^{-4}
0.3	3.02003×10^{-4}	3.02004×10^{-4}	3.02003×10^{-4}	3.02003×10^{-4}	3.02004×10^{-4}	3.02003×10^{-4}
0.4	3.02003×10^{-4}	3.02004×10^{-4}	3.02003×10^{-4}	3.02003×10^{-4}	3.02004×10^{-4}	3.02003×10^{-4}
0.2	4.56780×10^{-4}	4.56780×10^{-4}	4.56780×10^{-4}	4.56780×10^{-4}	4.56780×10^{-4}	4.56780×10^{-4}
0.3	4.56780×10^{-4}	4.56780×10^{-4}	4.56780×10^{-4}	4.56780×10^{-4}	4.56780×10^{-4}	4.56780×10^{-4}
0.4	4.56780×10^{-4}	4.56780×10^{-4}	4.56780×10^{-4}	4.56780×10^{-4}	4.56780×10^{-4}	4.56780×10^{-4}

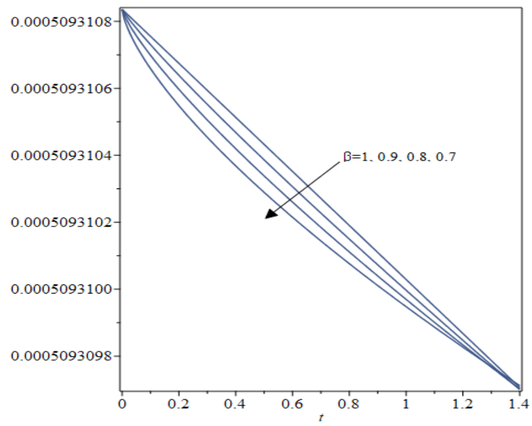

Figure 3: Convergence at different values of β using HETM for FZK (3, 3, 3) at $\theta = \vartheta = 1$

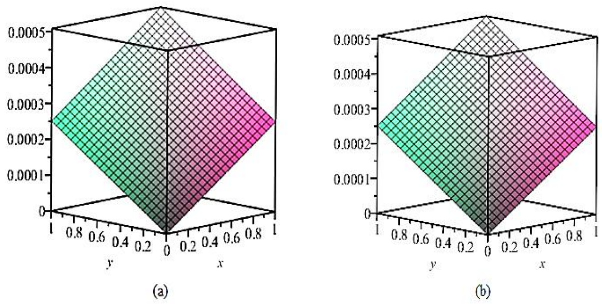

Figure 4: (a) The exact solution and (b) the approximate solution for FZK (3, 3, 3) at $\beta = 1$

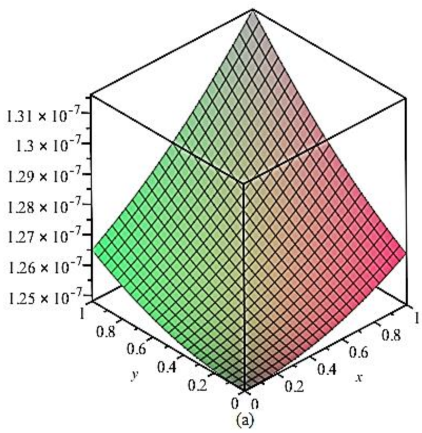

Figure 5: The absolute error for FZK (3, 3, 3) at $\beta = 1$.

7. Result Analysis

Figure 1presents 3D plots of (a) the exact solution and (b) the numerical solution using HETM for FZK (2, 2, 2) when $\beta = 1$, $t = 0.5$ and $\delta = 0.001$. Figure 2 illustrates 3D plots of the numerical solutions using HETM for FZK (2, 2, 2) at (c) $\beta = 0.75$ and (d) $\beta = 0.5$. It can be observed from Figure 1 that the numerical solution obtained by HETM for FZK (2, 2, 2) when $\beta = 1$ is quite agree with the exact solution. Table 1 the comparison of absolute error among different methods for various values of θ, ϑ and t is illustrated, it can be observed from Table 1 that the HETM technique converges faster than other methods. Figure 3 shows 2D plot of numerical solutions obtained by the proposed technique for different values of β and θ when $\vartheta = 1$. Figure 4 shows 3D plots of (a) the exact solution (b) numerical solution using HETM for FZK (3, 3, 3) at $\beta = 1, \delta = 0.001$, and $t = 0.5$. The equivalency between (a) and (b) can be observed in Figure 4. Finally, Figure 5 presents 3Dplot of the absolute error which illustrates the accuracy of the proposed technique in solving nonlinear fractional Zakharov–Kuznetsov equations. Table 2-3 demonstrate the comparison among exact solution, first term HETM numerical solution, first order OHAM solution, and third terms of VIM solution for FZK (3, 3, 3) at different values of β. It is observed from Tables1-3 that the new computational method is superior and more efficient than other methods and almost agree with the exact solution when $\beta = 1$.

8. Conclusion

In this work, a new technique of He's variation iteration method combined with the Elzaki integral transform is proposed to investigate the solution of nonlinear fractional Zakharov–Kuznetsov equations. The beauty of the present innovative method is that one has no need to depend on the integration or taking the convolution theorem to compute the Lagrange multiplier. This technique is used to avoid integral computation and convolution terms. The suggested computational technique is applied for two models of nonlinear fractional Zakharov–Kuznetsov equations. The analytical and approximate solutions are obtained for the proposed models and compared with those obtained by VIM, HPTM, NISTM, and OPAM. Finally, according to the obtained results, He- Elzaki transform method HETM an accurate and powerful tool for solving linear and nonlinear fractional differential equations.

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