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RESEARCH ARTICLE

BERNOULLI COLLOCATION FOR SOLVING TWO-POINT BVP IN MODELLING VISCOELASTIC FLOWS

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ABSTRACT

Bernoulli bases are developed to approximate the solutions of two-point BVP in modelling viscoelastic flows in which the shifted Chebyshev collocation points are used as collocation nodes. Properties of Bernoulli bases are then used to reduce the two-point BVP in modelling viscoelastic flows to systems of nonlinear algebraic equations. The results show the agreement between the exact solutions and the approximate solutions. Form the numerical results we see that the proposed method gives accurate results.

KEYWORDS

Bernoulli, Collocation, Viscoelastic flows, Fifth-order, Nonlinear.

1. INTRODUCTION

Differential equations of elliptical-hyperbolic operator forms occur in the simulation of viscoelastic flows. The essential attributes of these elliptical-hyperbolic operators may be described in a nonlinear fifth order two-point boundary value problem in one dimension. Numerical evaluation of viscoelastic streams seems to be the issue of concern of many scientists. There's often a difficulty in the Numerical treatment of certain problems by traditional approaches due to the existence of high gradients in velocity, pressure and stress. Several methods for solving viscoelastic flows model have been developed such as, the Galerkin method, collocation algorithm, the Runge kutta, the homotopy scheme, finite elements, Chebyshev differentiation matrix method and the shooting method in (Davies et al., 1988; Davies et al., 1988; Attili, 2000; Syam and Attili, 2005; Marchal and Crochet, 1986; El-Gamel et al., 2020; Attili, 1993; Elgindi and Langer, 1994). Some authors have discussed other methods of solution of this model (O'Malley, 1974; Kevorkian and Cole, 1981; Philips, 1989). The purpose of this paper is to implement a new method, based on Bernoulli polynomials for solving in the form

$$\left[1+\varepsilon\frac{dy}{d\eta}\frac{d}{d\eta}\right]\frac{d^4y}{d\eta^4}=f(\eta),\ -\tfrac{1}{2}\le\eta\le\tfrac{1}{2} \tag{1}$$

subject to the boundary conditions

$$y\left(\pm\tfrac{1}{2}\right)=\frac{dy}{d\eta}\left(\pm\tfrac{1}{2}\right)=0,\qquad \frac{d^2y}{d\eta^2}\left(-\tfrac{1}{2}\right)=\gamma\ \ at\ \ \varepsilon>0 \tag{2}$$

such that ε and γ are positive constants which denote as elasticity parameter and a boundary stress, respectively. In addition, γ will be unity in this paper.

In modern years, a lot of attention has been devoted to the study of Bernoulli methods to investigate various scientific models. Using these methods made it possible to solve the Bratu equation, linear Fredholm integro-differential-difference equations, telegraph equation, volterra integro-differential equations of high order, linear second order initial value problems, generalized pantograph equation, nonhomogeneous time-dependent problems, nonlinear BVPs which arise from the problems in calculus of variation, nonlinear Fredholm integro-differential equations with piecewise intervals, multidimensional diffusion and wave equations with Dirichlet boundary conditions, high-order linear complex differential equations in a rectangular domain, second order linear partial differential equations, and systems of high-order linear Volterra integral equations (El-Gamel et al., 2018; Erdem et al., 2013; Bicer et al., 2018; Matinfar et al., 2017; Napoli, 2016; Tohidi et al., 2013; El-Gamel, 2006; Tohidi and Kiliman, 2013; Bhrawy et al., 2012; Zogheib et al., 2017; Toutounian et al., 2013; Toutounian and Tohidi, 2013; Mirzaee and Bimesl, 2014). Recently, has been made numerical comparison of collocation method based on Bernoulli and Galerkin method based on wavelet for solving nonhomogeneous heat and wave equations (El-Gamel et al., 2018). Up to date, there is no study on Bernoulli-collocation scheme to two-point BVP in modelling viscoelastic flows.

The rest of the paper is structured as follows. Section 2, In Section 2, below briefly references, in which the reader can find an excellent summary of Bernoulli methods, along with their proofs (Nafyeh, 1981; Lehmer, 1988; Natalini and Bernaridini, 2003; Costabile and Accio, 2001). Section 3 Bernoulli method is developed for solving two-point BVP in modelling viscoelastic flows. In Section 4, we present a Convergence and error estimation for the full discrete problem. Numerical comparison and discussion are provided in Section 5. Lastly, Section 6 concludes the paper.

2. PRELIMINARIES

As was already mentioned in the above introduction, we have excluded the presentation of Bernoulli methods, in order to save space, deferring instead to the excellent references in which Bernoulli methods along with their proofs are given (Nafyeh, 1981; Lehmer, 1988; Natalini and Bernaridini, 2003; Costabile and Accio, 2001).

3. DIRECT SHIFTED BERNOULLI COLLOCATION METHOD

In this part, we will approximate the solution of the equation (1) as follows

$$y(\eta)=\sum_{i=0}^{N}a_i\beta_i(\eta),\quad -\tfrac{1}{2}\le\eta\le\tfrac{1}{2} \tag{3}$$

$$y^{(k)}(\eta)=\sum_{i=0}^{N}a_i{\beta_i}^{(k)}(\eta),\ \ k=0,1,2,\dots,5. \tag{4}$$

Where N is chosen as any positive integer such that $0\le i\le N$.

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Specifically, the compositions of the approximate solution and its derivatives in the matrix form are:

$$[y(\eta)] = \boldsymbol{\beta}(\eta)\mathbf{A}\,, \quad [y^{(k)}(\eta)] = \boldsymbol{\beta}(\eta)(\mathbf{M_\eta})^{\mathbf{k}}\mathbf{A}, \tag{5}$$

Where

$$\boldsymbol{\beta}(\eta) = [\beta_0(\eta) \;\; \beta_1(\eta) \;\; \beta_2(\eta) \ldots \;\; \beta_N(\eta)],$$

and

$$\mathbf{M_\eta} = \frac{1}{b-a} M, \qquad \mathbf{A} = [a_0, a_1, \ldots, a_N]^\tau$$

and $\mathbf{M}$ is the $(\mathbf{N+1}) \times (\mathbf{N+1})$ operational matrix of derivative given by

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

We need the following lemma

Lemma 3.1. ℓ and k are both positive integers, then following relation holds (El-Gamel, 2015; Akyuz-Dascioglu and Cerdik-Yaslan, 2011).

$$\begin{bmatrix} y^{(k)}(\eta_0)y^{(\ell)}(\eta_0) \\ y^{(k)}(\eta_1)y^{(\ell)}(\eta_1) \\ \vdots \\ y^{(k)}(\eta_N)y^{(\ell)}(\eta_N) \end{bmatrix} = \begin{bmatrix} y^{(k)}(\eta_0) & 0 & \cdots & 0 \\ 0 & y^{(k)}(\eta_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y^{(k)}(\eta_N) \end{bmatrix} \begin{bmatrix} y^{(\ell)}(\eta_0) \\ y^{(\ell)}(\eta_1) \\ \vdots \\ y^{(\ell)}(\eta_N) \end{bmatrix}$$

$$= \left(\overline{\boldsymbol{\beta}}\; \overline{\mathbf{M_\eta}}^{k}\; \overline{\boldsymbol{A}}\right) \boldsymbol{\beta}\, \mathbf{M_\eta}^{\ell} \boldsymbol{A}$$

Where

$$\overline{\boldsymbol{\beta}} = \begin{bmatrix} \boldsymbol{\beta}(\eta_0) & 0 & \cdots & 0 \\ 0 & \boldsymbol{\beta}(\eta_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{\beta}(\eta_N) \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \boldsymbol{\beta}(\eta_0) \\ \boldsymbol{\beta}(\eta_1) \\ \vdots \\ \boldsymbol{\beta}(\eta_N) \end{bmatrix},$$

And

$$\overline{\mathbf{M_\eta}} = \begin{bmatrix} \mathbf{M_\eta} & 0 & \cdots & 0 \\ 0 & \mathbf{M_\eta} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{M_\eta} \end{bmatrix}, \quad \overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & 0 & \cdots & 0 \\ 0 & \mathbf{A} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A} \end{bmatrix}$$

In our study, we use the shifted Chebyshev collocation points as collocation nodes

$$\eta_i = -\frac{1}{2}cos\left(\frac{i\pi}{N}\right) \quad i = 0,1, \ldots, N$$

in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ as well as subrogating $\eta = \eta_i$ to the approximate solution and its derivatives (5). We ultimately realize the BVP in equation (1) at this point and acquire the following theorem

$$\mathbf{WA = F}, \tag{6}$$

where

$$\boldsymbol{W} = \boldsymbol{\beta}\, \mathbf{M_\eta}^{4} + \varepsilon\left(\overline{\boldsymbol{\beta}}\; \overline{\mathbf{M_\eta}}\; \overline{\boldsymbol{A}}\right) \boldsymbol{\beta}\, \mathbf{M_\eta}^{5}$$

The matrix forms to the boundary conditions subjected to equation (2) are

$$\boldsymbol{\beta}\left(\tfrac{1}{2}\right)\boldsymbol{A} = \boldsymbol{\beta}\left(\tfrac{1}{2}\right) \mathbf{M_\eta}\boldsymbol{A}=0,$$

$$\boldsymbol{\beta}\left(-\tfrac{1}{2}\right)\boldsymbol{A} = \boldsymbol{\beta}\left(-\tfrac{1}{2}\right) \mathbf{M_\eta}\boldsymbol{A}=0 \tag{7}$$

$$\boldsymbol{\beta}\left(-\tfrac{1}{2}\right) \mathbf{M_\eta}^{2}\boldsymbol{A} = c$$

so, in the matrix [W; F] we will replace the last 5_{th} rows by the Eq. (7), we have the augmented matrix$[\widetilde{W}; \tilde{F}]$

$$\widetilde{W}A = \tilde{F}, \tag{8}$$

This equation generates $\mathbf{N+1}$ sets of nonlinear equations respectively. These nonlinear equationscan be solved which can be solved using Newton's iterative method for unknown coefficients of the vector $\mathbf{A}$ and solution $\mathbf{y(\eta)}$ can be calculated easily.

4. CONVERGENCE AND ERROR ESTIMATION

4.1 Convergence of Bernoulli Polynomial

Now, we will introduce the convergence and error bound to Bernoulli polynomials. Theorem 4.1. Suppose that the function $y: [0,1] \to R$ is k+1 time continuously differentiable, $y \in C^{k+1}[0,1]$, and $Y = Span\{\beta_0, \beta_1, \beta_2, \ldots, \beta_N\}$ is vector space. If $\boldsymbol{\beta}(\eta)\boldsymbol{A}$ is the best approximation of y out of Y such that:

$$y(\eta) \simeq y_N(\eta) = \sum_{i=0}^{N} a_i\beta_i(\eta)$$

Then

$$\|y(\eta) - y_N(\eta)\|_2 \le \frac{\xi}{(k+1)!\sqrt{2k+3}}$$

where $\|.\|_2$ refers to the $L^2[0,1]$ norm defined as

$$\|y(\eta)\|_2^2 = \int_0^1 |y(t)|^2\, dt$$

And

$$\xi = \max_{\eta\in[0,1]} \left|y^{(k+1)}(t)\right|$$

Proof. Consider the polynomial g(η), that can be expanded in truncated Taylor expansion as:

$$g(\eta) = y(0) + y'(0)\eta + \frac{y''(0)}{2!}\eta^2 + \cdots + \frac{y^{(k)}(0)}{k!}\eta^k$$

from the previous Taylor expansion, there exist $\zeta \in (0,1)$ such that:

$$\|y(\eta) - y_N\|_2^2 \le \|y(\eta) - g(\eta)\|_2^2$$

$$= \int_0^1 \left(\frac{y^{(k+1)}(\eta)\,\eta^{k+1}}{(k+1)!}\right)^2 d\eta$$

$$\le \frac{\xi^2}{[(k+1)!]^2(2k+3)}$$

Then, the required results will be found by taking the square root to the above in equality

4.2 Error Estimation of Bernoulli-Collocation Method

In this section, the error estimation for the Bernoulli-collocation method has been employed with the residual error function (Celik, 2005; Shahmorad, 2005). One can obtain the residual function first, we can display the residual function $R_N(\eta)$ as

$$R_N(\eta) = \left[1 + \varepsilon\frac{dy}{d\eta}\frac{d}{d\eta}\right]\frac{d^4y}{d\eta^4} - f(\eta) \tag{9}$$

Here, $\hat{y}(\eta)$ is the Bernoulli polynomial solution given by (6) of the equation (1) with the condition (2). Thus, $\hat{y}(\eta)$ fulfill the equation:

$$\left[1 + \varepsilon\frac{dy}{d\eta}\frac{d}{d\eta}\right]\frac{d^4y}{d\eta^4} = f(\eta) + R_N(\eta),$$

$$y\left(\pm\frac{1}{2}\right) = \frac{dy}{d\eta}\left(\pm\frac{1}{2}\right) = 0, \qquad \frac{d^2y}{d\eta^2}\left(-\frac{1}{2}\right) = \gamma$$

so, we can obtain the error function

$$\zeta(\eta) = y(\eta) - \hat{y}(\eta) \tag{10}$$

such that $y(\eta)$ is the exact solution of equation (1). Accordingly, the error differential equation is:

$$\left[1 + \varepsilon\frac{d\zeta}{d\eta}\frac{d}{d\eta}\right]\frac{d^4\zeta}{d\eta^4} = f(\eta) + R_N(\eta),$$

$$y\left(\pm\tfrac{1}{2}\right) = \tfrac{dy}{d\eta}\left(\pm\tfrac{1}{2}\right) = 0, \qquad \tfrac{d^2\zeta}{d\zeta^2}\left(-\tfrac{1}{2}\right) = 0 \tag{11}$$

The solution of equation (11) is:

$$\zeta(\eta) = \sum_{i=0}^{N} \hat{a}_i\beta_i(\eta)$$

In the same manner as Section 3, we obtain unknown coefficients $\hat{a}_i$, i = 0, 1, 2, ...N . so, the maximum absolute error can be determined by

$$E_{max} = max|\zeta(\eta)| \qquad -\frac{1}{2} \le \eta \le \frac{1}{2}$$

Using maximum error estimation, we can test the reliability of the results especially if the exact solution is unknown.

5. Examples and Comparison

In this section, we present several examples to show the feasibility and robustness of the proposed technique. The formula of l_2 norm error $\|E_{Bernoulli}\|$ is

$$\|E_{Bernoulli}\| = \sqrt{\sum_{i=0}^{N} |y_{exact}(\eta_i) - y_{Bernoulli}(\eta_i)|^2}$$

Example 1: Consider

$$\left[1 + \varepsilon \frac{dy}{d\eta}\frac{d}{d\eta}\right]\frac{d^4y}{d\eta^4} = 12, \ -\frac{1}{2} \le \eta \le \frac{1}{2} \qquad (12)$$

whose exact solution is

$$y(\eta) = \frac{1}{2}\left(\eta^2 - \frac{1}{4}\right)^2$$

We seek the approximate solutions $y(\eta)$ by the truncated Bernoulli series for N = 4 is

$$y(\eta) = a_0\beta_0(\eta) + a_1\beta_1(\eta) + a_2\beta_2(\eta) + a_3\beta_3(\eta) + a_4\beta_4(\eta)$$

by applying our scheme we obtain

$$y(\eta) = \frac{1}{60}\beta_0(\eta) + \frac{1}{2}\beta_4(\eta) = \frac{1}{2}\left(\eta^2 - \frac{1}{4}\right)^2$$

the approximate solution which is the exact solution.

Table 1 exhibits a comparison between l_2 norm error obtained by using the present scheme, Chebyshev Collocation, Chebyshev function, Beam-Galerkin, Beam-Collocation, Runge-Kutta methods and Chebyfun for Example 1. This comparison shows the efficiency of present scheme.

Table 1: Comparison of the l_2 norm error at $\varepsilon = 10^{-1}$ for Example 1.

Method	l_2 norm error
Present method, N=4	0
Chebyshev-collocation, N = 5	1.74E-18
Chebyfun	8.664E-18
Runge-Kutta method, N =100	6.899E-03
Beam-Galerkin, N = 4	0.975E-04
Beam-Collocation, N = 4	0.855E-03

Example 2 : Consider

$$\left[1 + \varepsilon \frac{dy}{d\eta}\frac{d}{d\eta}\right]\frac{d^4y}{d\eta^4} = -120\eta + 600\varepsilon\left(\eta^2 - \frac{1}{4}\right)\left(\eta^2 - \frac{1}{20}\right),$$
$$-\frac{1}{2} \le \eta \le \frac{1}{2} \qquad (13)$$

whose exact solution is

$$y(\eta) = -\eta\left(\eta^2 - \frac{1}{4}\right)^2$$

Table 2 represents a comparison between l_2 norm errors with the methods in [1, 2, 3, 6] for Example 2.

Table 2: Comparison of l_2 norm errors for Example 2, for the present method, Chebyshev, Galerkin, collocation methods and Runge-Kutta method. the approximate solution which is the exact solution.

ε	$\|E_{\text{Bernoulli}}\|$, $N = 5$	$\|E_{\text{Chebyshev}}\|$, $N = 5$ [6]	$\|E_{Runge-Kutta}\|$, $N = 100$ [3]	$\|E_{Galerkin}\|$, $N = 5$ [1]	$\|E_{Collocation}\|$, $N = 5$ [2]
10^{-1}	5.0545E-17	4.7300E-13	3.98537E-03	0.25E-03	0.24E-02
10^{-2}	5.0545E-17	1.7820E-13	3.98585E-03	0.16E-03	0.22E-02
10^{-3}	5.0545E-17	1.2704E-16	3.98599E-03	0.16E-03	0.22E-02
10^{-4}	5.0545E-17	2.0230E-16	3.98586E-03	×	×

Example 3:[1, 2, 3] Consider

$$\left[1 + \varepsilon \frac{dy}{d\eta}\frac{d}{d\eta}\right]\frac{d^4y}{d\eta^4} = f(\eta), \qquad -\frac{1}{2} \le \eta \le \frac{1}{2}$$

whose exact solution is

$$y(\eta) = \frac{1}{2\pi}\left(\eta^2 - \frac{1}{4}\right)\sin\pi\left(\eta - \frac{1}{2}\right)$$

And

$$f(\eta) = \frac{\pi}{2}\left[\pi^2\left(\eta^2 - \frac{1}{4}\right) - 12\right]\sin\pi\left(\eta - \frac{1}{2}\right) - 4\pi^2\eta\cos\pi\left(\eta - \frac{1}{2}\right)$$

$$+\frac{1}{8}\varepsilon\pi^2\left[\pi^2\left(\eta^2 - \frac{1}{4}\right)^2 - 40\left(\eta^2 - \frac{1}{8}\right)\right]\cos 2\pi\left(\eta - \frac{1}{2}\right)$$

$$+\frac{1}{8}\varepsilon\pi\eta\left[12\pi^2\left(\eta^2 - \frac{1}{4}\right) - 40\right]\sin 2\pi\left(\eta - \frac{1}{2}\right)$$

$$+\frac{1}{8}\varepsilon\pi^2\left[\pi^2\left(\eta^2 - \frac{1}{4}\right)^2 + 5\right]$$

Table 3 illustrate the comparison of l_2 norm errors between result of Bernoulli polynomial method and result of Runge-Kutta method in [1] and [2] at $\varepsilon = 10^{-1}$, 10^{-2} and 10^{-3} with N = 16 for Example 3 (Davies et al., 1988; Davies et al., 1988).

Table 3: Comparison of l_2 norm errors for Example 3, for the present method and Runge-Kutta method.

ε	$\|E_{\text{Bernoulli}}\|, N = 16$	$\|E_{Runge-Kutta}\|, = 100$ (Attili, 2000)
10^{-1}	3.5150E-12	2.52017E-02
10^{-2}	1.8158-12	9.238658E-03
10^{-3}	1.7832E-12	8.836376E-03

Example 4: Now we turn to a singular problem

$$\left[1 + \varepsilon \frac{dy}{d\eta}\frac{d}{d\eta}\right]\frac{d^4y}{d\eta^4} + \frac{1}{\eta}y' + \frac{1}{\eta^2}y = f(\eta), \qquad -\frac{1}{2} \le \eta \le \frac{1}{2}$$

where

$$f(\eta) = 1440\varepsilon\eta\left[4\eta\left(\eta^2 - \frac{1}{4}\right)^2 + 8\eta^3\left(\eta - \frac{1}{2}\right)\right] +$$

$$6\left(\eta^2 - \frac{1}{4}\right)^2 + 8\eta^2\left(\eta^2 - \frac{1}{2}\right) + 720\eta^2 - 24$$

whose exact solution is

$$y(\eta) = 2\eta^2\left(\eta^2 - \frac{1}{4}\right)^2$$

Applying L'Hospital rule to BVP in order to remove singularity. The latter form is

$$\left[1 + \varepsilon \frac{dy}{d\eta}\frac{d}{d\eta}\right]\frac{d^4y}{d\eta^4} + a(\eta)y'' + b(\eta)y' + c(\eta)y = f(\eta)$$

$$a(\eta) = \begin{cases} 0, & \eta \ne 0; \\ 1.5, & \eta = 0; \end{cases} \quad b(\eta) = \begin{cases} 1/\eta, & \eta \ne 0; \\ 0, & \eta = 0; \end{cases} \quad c(\eta) = \begin{cases} 1/\eta^2, & \eta \ne 0; \\ 0, & \eta = 0; \end{cases}$$

Table 4 exhibits l_2 norm errors for the present method and other method in [6] at N = 6 for Example 4 (El-Gamel et al., 2020)

Table 4: Comparison of l_2 norm errors of the present method and Chebyshev collocation method for Example 4

ε	$\|E_{\text{Bernoulli}}\|, N = 6$	$\|E_{\text{Chebyshev}}\|, N = 6$ (El – Gamel et al., 2020)
10^{-1}	4.0707E-12	8.7184E-12
10^{-2}	3.7836E-16	2.9893E-14
10^{-3}	3.7836E-16	9.3948E-15

6. Conclusion

This paper discusses how Bernoulli matrix collocation method can be applied for obtaining solutions of viscoelastic flows. We present four examples, the first three examples are nonlinear fifth-order differential equation and fourth example is singular nonlinear. The concerned results have been testified by an examples. Hence Bernoulli approach is a powerful technique to investigate various nonlinear problems which have many applications in modelling viscoelastic flows. It is advisable to use it for other nonlinear differential equations.

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References

Akyuz-Dascioglu, A., Cerdik-Yaslan, H., 2011. The solution of high-order nonlinear ordinary differential equations by Chebyshev Series, Appl. Math. Comput., 217, 5658-5666.

Attili, A., 2000. Initial value methods for the primary two-point boundary value problems in modelling viscoelastic flows, Int. J. Computer Math., 74, Pp. 379-390.

Attili, S., 1993. On the numerical implementation of the shooting methods to the one-dimensional singular boundary value problems. Int. J. Computer Math., 47, Pp. 65-75.

Bhrawy, A.E., Tohidi, E., Soleymani, F., 2012. A new Bernoulli matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals, Appl. Math. Comput., 219, Pp. 482-497.

BiC¸ er K., YalC¸inbaS¸ S., 2018. Numerical solution of telegraph equation using Bernoulli collocation method, Proc. Natl. Acad. Sci. India. Sect. A: Phys. Sci., Pp. 1-7.

Celik, I., 2005. Approximate calculation of eigenvalues with the method of weighted residuals collocation method. Appl. Math. Comput., 160, Pp. 401-410.

Costabile, F., Accio, F., 2001. Expansions over a rectangle of real functions in Bernoulli polynomials and applications. BIT Numer. Math., 41, Pp. 451-464.

Davies, A., Karageorghis, A., Phillips, T., 1988. Spectral collocation methods for the primary two-point boundary value problem in modelling viscoelastic flows. Int. J. Numer. Methods, 26, Pp. 805-813.

Davies, A., Karageorghis, A.T., Phillips, T., 1988. Spectral Galerkin methods for the primary two-point boundary value problem in modelling viscoelastic flows. Int. J. Numer. methods, 26, Pp. 647-662.

El-Gamel, M., 2006. A numerical scheme for solving nonhomogeneous time-dependent problems. Z. angew. Math. Phys., 57, Pp. 369-383.

El-Gamel, M., 2015. Chebychev polynomial solutions of twelfth-order boundary-value problems, British J. Math. Computer Sci., 6, Pp. 13-23.

El-Gamel, M., Adel, W., El-Azab, M., 2018. Two very accurate and efficient methods for solving time-dependent problems. Applied Mathematics, 9, Pp. 1270-1280.

El-Gamel, M., Adel, W., El-Azab, S., 2018. Collocation method based on Bernoulli polynomial and shifted Chebychev for solving the Bratu equation. J. Appl. Computat. Math., 7, Pp. 1-7.

El-Gamel, M., Mohamed, O., El-Shamy, N., 2020. A robust and effective method for solving two-point BVP in modelling viscoelastic flows. Applied Mathematics, 11, Pp. 23-34.

Elgindi, B., Langer, W., 1994. On the shooting method for a class of two-point singular nonlinear boundary value problems. Int. J. Computer Math., 51, Pp. 107-118.

Erdem, K., Yal, C., inbaS, S., Sezer, M., 2013. A Bernoulli polynomial approach with residual correction for solving mixed linear Fredholm integro-differential-difference equations. J. Difference Equ. Appl., 19, Pp. 1619-1631.

Kevorkian, J., and Cole, J., 1981. Perturbation Methods in Applied Mathematics, Springer, Berlin. https://doi.org/10.1007/978-1-4757-4213-8.

Lehmer, D., 1988. A new approach to Bernoulli polynomials. Amer. Math. Month., 95, Pp. 905- 911.

Marchal, J., Crochet, M., 1986. Hermitian finite elements for calculating viscoelastic flow. J. Non- Newtonian Fluid Mech., 20, Pp. 187-208.

Matinfar, M., Lashaki, H., Akbari, M., 2017. Numerical approximation based on the Bernoulli polynomials for solving volterra integro-differential equations of high order. Math. Rese., 3, Pp. 19-32.

Mirzaee, F., Bimesl, S., 2014. An efficient numerical approach for solving systems of high-order linear Volterra integral equations. Scientia Iranica, 21, Pp. 2250-2263.

Napoli, A., 2016. Solutions of linear second order initial value problems by using Bernoulli polynomials. Appl. Numer. Math., 99, Pp. 109-120.

Natalini, P., Bernaridini, A., 2003. A generalization of the Bernoulli polynomials. J. Appl. Math., 3, Pp. 155-163.

Nayfeh, A., 1981. Introduction to Perturbation Techniques, Wiley, New York.

O' Mally, R., 1974. Introduction to Singular Perturbation, Academic Press, New York.

Oliveira, F., 1980. Collocation and residual correction. Numer. Math., 36, Pp. 27-31.

Phillips, T., 1989. On the potential of spectral methods to solve problems in Non- Newtonian fluid mechanics Numer Methods Partial Diff. Equ., 5, Pp. 35-43. https://doi.org/10.1002/num.1690050104

Shahmorad, S., 2005. Numerical solution of the general form linear Fredholm-Volterra integro-differential equations by the Tau method with an error estimation. Appl. Math. Comput., 167, Pp. 1418-1429.

Syam, M., Attili, B., 2005. Numerical solution of singularly perturbed fifth order two-point boundary value problem, Appl. Math. Comput., 170, Pp. 1085-1094.

Tohidi, E., Bhrawy, E.A., Erfani, E., 2013. A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation. Appl. Math. Model., 37, Pp. 4283-4294.

Tohidi, E., Kiliman, A., 2013. A Collocation method Based on the Bernoulli operational matrix for solving nonlinear BVPs which arise from the problems in calculus of variation, Math. Probl. Eng., Article ID 757206 — https://doi.org/10.1155/2013/757206.

Toutounian, F., Tohidi, E., 2013. A new Bernoulli matrix method for solving second order linear partial differential equations with the convergence analysis. Appl. Math. Comput., 223, Pp. 298-310.

Toutounian, F., Tohidi, E., Shateyi, S., 2013. A collocation method based on Bernoulli operational matrix for solving high-order linear complex differential equations in a rectangular domain. Abstract Appl. Anal., (1) DOI: 10.1155/2013/823098

Zogheib, B., Tohidi, E., and Shateyi, S., 2017. Bernoulli collocation method for solving linear multidimensional diffusion and wave equations with Dirichlet boundary conditions,BIT Numer. Math., Pp. 451-464.