ZIBELINE INTERNATIONAL PUBLISHING

Matriks Sains Matematik (MSMK)

DOI : http://doi.org/10.26480/msmk.02.2018.40.49

Print ISSN : 2521-0831
Online ISSN : 2521-084X
CODEN: MSMADH

CrossMark

REVIEW ARTICLE

EXTREMAL IOTA ENERGY OF A SUBCLASS OF TRICYCLIC DIGRAPHS AND SIDIGRAPHS

Fareeha Jamal[1], Mehtab Khan[2*]

[1]School of Natural Sciences, National University of Sciences and Technology, Islamabad, Pakistan.
[2]Department of Mathematics & Statistics, Bacha Khan University Charsadda, Pakistan.
**Corresponding Author Email: mehtabkhan85@gmail.com*

ARTICLE DETAILS

Article History:

Received 26 June 2018
Accepted 2 July 2018
Available online 1 August 2018

ABSTRACT

The iota energy of an n-vertex digraph D is defined by $E_c(D) = \sum_{k=1}^{n} |\mathrm{Im}(z_k)|$, where $z_1, \ldots, z_n$ are eigenvalues of D and $\mathrm{Im}(z_k)$ is the imaginary part of eigenvalue z_k. The iota energy of an n-vertex sidigraph can be defined analogously. In this paper, we define a class F_n of n-vertex tricyclic digraphs containing five linear subdigraphs such that one of the directed cycles does not share any vertex with the other two directed cycles and the remaining two directed cycles are of same length sharing at least one vertex. We find the digraphs in F_n with minimal and maximal iota energy. We also consider a similar class of tricyclic sidigraphs and find extremal values of iota energy among the sidigraphs in this class.

KEYWORDS

iota energy, digraphs, sidigraphs ,tricyclic, n-vertex.

1. INTRODUCTION

A directed graph (henceforth, digraph) D = (V, A) consists of disjoint finite sets V and A, where V is the set of vertices and A is the set of arcs. A signed digraph (henceforth, sidigraph) is an ordered pair S = (D, σ), where D = (V, A) is the underlying digraph of S and $\sigma : A \to \{-1, 1\}$ is called the signing function. The sets of positive and negative arcs of S are respectively denoted by A^+ and A^-. Thus, $A = A^+ \cup A^-$ is the set of signed arcs of S. A directed path P_n of length n − 1 (n ≥ 2) is a digraph whose set of vertices is $\{v_k \mid k = 1, \ldots, n\}$ and set of arcs is $\{v_k v_{k+1} \mid k = 1, \ldots, n-1\}$. A directed cycle C_n of length n (n ≥ 2) is a digraph with the vertex set $\{v_k \mid k = 1, \ldots, n\}$ of n elements and arc set $\{v_k v_{k+1} \mid k = 1, \ldots, n-1\} \cup \{v_n v_1\}$ of n elements. The signed directed path and signed directed cycle are defined in an analogous way. The product of signs of the arcs of a sidigraph is called the sign of a sidigraph. A sidigraph is said to be positive (respectively, negative) if its sign is positive (respectively, negative). A sidigraph is all-positive (respectively, all-negative) if all its arcs are positive (respectively, negative). If each signed directed cycle of a sidigraph has positive sign, then S is said to be cycle-balanced; otherwise non cycle-balanced. A digraph D = (V, A) is unicyclic if $|A| = |V|$ and there is a unique directed cycle. A digraph D = (V, A) is bicyclic if $|A| = |V| + 1$ and there are two directed cycles. A digraph D = (V, A) is tricyclic if $|A| = |V| + 2$ and there are three directed cycles. A sidigraph S = (D, σ) is unicyclic (respectively, bicyclic) if D is unicyclic (respectively, bicyclic). Similarly, a sidigraph S = (D, σ) is tricyclic if D is tricyclic. Two directed cycles that share at least one vertex are said to be joined directed cycles; otherwise disjoint directed cycles. Analogously, we can define joined (respectively, disjoint) signed directed cycles.

The number of arcs entering in a vertex u of a digraph (respectively, sidigraph) is known as in-degree of u and the number of arcs leaving u is called out-degree of that vertex. The in-degree and out-degree of a vertex u in a digraph D (respectively, sidigraph S) are denoted by $d^-(u)$ and $d^+(u)$, respectively. A digraph (respectively, sidigraph) is said to be linear if each of its vertices has both in-degree and out-degree equal to one.

The adjacency matrix of an n-vertex digraph D = (V, A) is an n × n matrix A(D) = $[a_{jk}]$ defined by:

$$a_{jk} = \begin{cases} 1 & \text{if } v_j v_k \in \mathcal{A} \\ 0 & \text{otherwise.} \end{cases}$$

The adjacency matrix of a sidigraph S = (D, σ), where D = (V, A), is the n × n matrix A(S) = $[s_{ij}]$, where

$$s_{ij} = \begin{cases} \sigma(v_i v_j) & \text{if } v_i v_j \in \mathcal{A} \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial det (xI − A(D)) of the adjacency matrix A(D) of digraph D is called the characteristic polynomial of the digraph D and is denoted by $\Phi_D(x)$. Analogously, we can define the characteristic polynomial of a sidigraph S which is denoted by $\varphi_S(x)$. The eigenvalues of A(D) (respectively, A(S)) are called the eigenvalues of D (respectively, S). We observe that the adjacency matrices of both digraphs and sidigraphs are not necessarily symmetric matrices. Thus, the eigenvalues of both D and S may be complex numbers.

The energy of an n-vertex simple graph G was introduced by Gutman [9] which is defined

by $E(G) = \sum_{k=1}^{n} |\lambda_k|$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of G. This concept of graph energy was extended to digraphs by Peña and Rada [14]. The energy of an n-vertex digraph D is defined by:

$$E(D) = \sum_{k=1}^{n} |\mathrm{Re}(z_k)|,$$

where $z_1, \ldots, z_n$ are the eigenvalues of D and $\mathrm{Re}(z_k)$ is the real part of the eigenvalue z_k [14].

Finding smallest and largest energy over a set of digraphs with a fixed order is the fundamental problem in the theory of digraph energy. The extremal values of energy in the class of unicyclic digraphs of fixed order are founded by Peña and Rada [14]. Khan et al. [12] considered a class of those digraphs which contain two vertex-disjoint directed cycles and computed minimal and maximal energy in this class. The extremal energy in the class of digraphs with two linear subdigraphs of equal length is computed by Farooq et al. [7]. Monsalve and Rada [13] found the maximal energy in the general class of bicyclic digraphs. Germina et al. [8] defined energy of a sigraph as the sum of absolute values of its eigenvalues. A generalization of the digraph energy to sidigraphs was proposed by Pirzada and Bhat [15]. Let $z_1, \ldots, z_n$ be the eigenvalues of an n-vertex sidigraph S. Then, the energy of S is defined by:

$$E(S)=\sum_{k=1}^{n}\left|\mathrm{Re}(z_k)\right|,$$

where $\mathrm{Re}(z_k)$ is the real part of the eigenvalue z_k [15]. Bhat and Pirzada [3] gave some interesting results regarding spectra and energy of bipartite sidigraphs. Khan and Farooq [10] found sidigraphs with minimal and maximal energy among all n-vertex sidigraphs which contain two vertex-disjoint signed directed cycles, $n \geq 4$.

Khan et al. [11] introduced the notion of iota energy of digraphs. Let $z_1, \ldots, z_n$ be the eigenvalues of a digraph D of order n. Then, the iota energy of D is defined as follows:

$$E_c(D)=\sum_{k=1}^{n}\left|\mathrm{Im}(z_k)\right|,$$

where $\mathrm{Im}(z_k)$ is the imaginary part of the eigen-value z_k [11]. Khan et al. [11] found the digraphs with extremal iota energy among the class of all unicyclic digraphs of fixed order, $n \geq 2$. Farooq et al. [5] studied the problem of finding the minimal and maximal iota energy among all n-vertex digraphs with two vertex-disjoint directed cycles. Very recently, the concept of iota energy of digraphs was extended to sidigraphs by Farooq et al. [6]. For a sidigraph S of order n, the iota energy of S is defined by:

$$E_c(S)=\sum_{k=1}^{n}\left|\mathrm{Im}(z_k)\right|,$$

where $z_1, \ldots, z_n$ are eigenvalues of S and $\mathrm{Im}(z_k)$ is the imaginary part of the eigenvalue z_k [6].

Motivated by Farooq et al. [7], we take the class of those tricyclic digraphs of fixed order which have five linear subdigraphs such that one of the directed cycles does not share any vertex with the other two directed cycles and the remaining two directed cycles are of same length sharing at least one vertex. We find the digraphs in this class with smallest and largest iota energy. Furthermore, we consider a similar class of n-vertex tricyclic sidigraphs and find sidigraphs in this class with extremal iota energy.

2.KNOWN RESULTS

The following theorem gives the coefficients of the characteristic polynomial of digraphs.

Theorem 2.1 (Cvetković et al. [4]). Let D be a digraph with characteristic polynomial

$$\Phi_D(x)=x^n+\sum_{k=1}^{n}b_k x^{n-k}.$$

Then

$$b_k=\sum_{L\in\mathcal{L}_k}(-1)^{comp(L)},$$

for every k = 1, . . . , n, where $\mathcal{L}_k$ is the set of all linear subdigraphs L of D with exactly k vertices, comp(L) denotes the number of components of L.

The following is the coefficient theorem for sidigraphs.

Theorem 2.2 (Acharya et al. [2]). If S is a sidigraph with characteristic polynomial

$$\phi_S(x)=x^n+\sum_{i=1}^{n}c_i x^{n-i},$$

then

$$c_i=\sum_{L\in\mathcal{L}_i}(-1)^{p(L)}\prod_{Z\in c(L)}s(Z),$$

for all i = 1, . . . , n, where L_i is the set of all linear subdigraphs L of S of order i, p(L) denotes the number of components of L, c(L) denotes the set of all cycles of L and s(Z) the sign of cycle Z.

The spectral criterion for cycle-balanced sidigraphs is given by the following result.

Theorem 2.3 (Acharya [1]). A sidigraph S = (D, σ) is cycle-balanced if and only if S and D are cospectral.

The following lemmas will be useful in proving a few results.

Lemma 2.4. The function f defined by f (x) =

$$\sqrt[x]{2}\cot\frac{\pi}{2x} \text{ is strictly increasing on } [2,\infty).$$

Lemma 2.5. Consider the sequences < a_n > and < b_n >, where a_n and b_n are given by:

$$a_n=\begin{cases}2\cot\frac{\pi}{n} & \text{if } n\equiv 0(\bmod 2)\\ \cot\frac{\pi}{2n} & \text{if } n\equiv 1(\bmod 2).\end{cases}$$

and

$$b_n=\begin{cases}2\csc\frac{\pi}{n} & \text{if } n\equiv 0(\bmod 2)\\ \cot\frac{\pi}{2n} & \text{if } n\equiv 1(\bmod 2).\end{cases}$$

Then $\{a_n\}$ is strictly increasing for $n \geq 4$ and $\{b_n\}$ is strictly increasing for $n \geq 3$.

Lemma 2.6 (Farooq et al. [5]). For $x \in (0, \frac{\pi}{2}]$, the following inequality holds:

$$\frac{1}{x}-0.429x\leq\cot x\leq\frac{1}{x}-\frac{x}{3}.$$

Lemma 2.7 (Khan et al. [12]). Let x, a, b be real numbers such that $x \geq a > 0$ and $b > 0$. Then

$$\frac{\pi x}{bx^2-\pi^2}\leq\frac{\pi a}{ba^2-\pi^2}.$$

It is known that for any real number x with $0 \leq x \leq \frac{\pi}{2}$, the sine function satisfies the following inequality:

$$x-\frac{x^3}{3!}\leq\sin x\leq x. \qquad (2.1)$$

3. IOTA ENERGY OF DIGRAPHS

For $n \geq 4$, we define a set F_n which consists of n-vertex tricyclic digraphs containing five linear subdigraphs such that one of the directed cycles does not share any vertex with the other two directed cycles and the remaining two directed cycles are of same length sharing at least one vertex. Let $D_p^m \in F_n$, where superscript m denotes the length of each joined directed cycle and subscript p denotes the length of disjoint directed cycle such that $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. From Theorem 2.1, the characteristic polynomial of D_p^m is given by:

$$\Phi_{D_p^m}(x)=x^{n-m-p}(x^m-2)(x^p-1).$$

The eigenvalues of D_p^m are 0, $\sqrt[m]{2}\ exp\left(\frac{2k\pi i}{m}\right)$ and $\exp\left(\frac{2j\pi i}{p}\right)$, where k =0,1,.

. . , m − 1 and $j = 0, 1, \ldots, p - 1$ and the multiplicity of eigenvalue 0 is $n - m - p$. Thus, the iota energy of D_p^m is given by:

$$E_c(D_p^m) = \sqrt[m]{2}\sum_{k=0}^{m-1}|\sin\frac{2k\pi}{m}| + \sum_{j=0}^{p-1}|\sin\frac{2j\pi}{p}|. \quad (3.2)$$

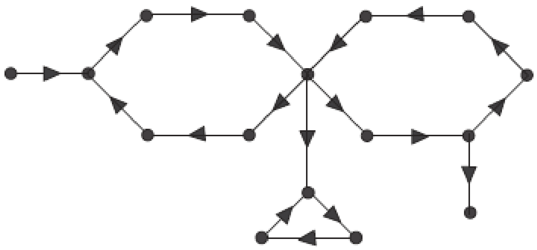


Figure 1: $D_3^6 \in F_{16}$

Example 3.1. Consider $D_3^6 \in F_{16}$ shown in Figure 1. By (3.2), the iota energy of D_3^6 is given by:

$$E_c(D_3^6) = \sqrt[6]{2}\sum_{k=0}^{5}|\sin\frac{2k\pi}{6}| + \sum_{j=0}^{2}|\sin\frac{2j\pi}{3}|.$$

It is shown in Khan et al. [11] that

$$\sum_{j=0}^{m-1}|\sin\frac{2j\pi}{m}| = \begin{cases} 2\cot\frac{\pi}{m} & \text{if } m \equiv 0(\text{mod}\,2) \\ \cot\frac{\pi}{2m} & \text{if } m \equiv 1(\text{mod}\,2). \end{cases} \quad (3.3)$$

Using (3.2) and (3.3), we get the following iota energy formulas for a digraph $D_p^m \in F_n$:

$$E_c(D_p^m) = \begin{cases} 2\left(\sqrt[m]{2}\cot\frac{\pi}{m} + \cot\frac{\pi}{p}\right) & \text{if } m \equiv 0(\text{mod}\,2) \text{ and } p \equiv 0(\text{mod}\,2) \\ 2\sqrt[m]{2}\cot\frac{\pi}{m} + \cot\frac{\pi}{2p} & \text{if } m \equiv 0(\text{mod}\,2) \text{ and } p \equiv 1(\text{mod}\,2) \\ \sqrt[m]{2}\cot\frac{\pi}{2m} + 2\cot\frac{\pi}{p} & \text{if } m \equiv 1(\text{mod}\,2) \text{ and } p \equiv 0(\text{mod}\,2) \\ \sqrt[m]{2}\cot\frac{\pi}{2m} + \cot\frac{\pi}{2p} & \text{if } m \equiv 1(\text{mod}\,2) \text{ and } p \equiv 1(\text{mod}\,2). \end{cases} \quad (3.4)$$

Next, we find the minimal and maximal iota energy among the digraphs in F_n. The following lemma shows that the iota energy of digraphs in F_n increases monotonically with the increase in length of their disjoint directed cycles.

Lemma 3.2. Let $n \geq 7$, $m = 2$ *and* $4 \leq p < n - 2$. Take $D_p^2, D_{p+1}^2 \in F_n$. *Then*

$$E_c(D_{p+1}^2) > E_c(D_p^2).$$

Proof. Let $p \equiv 0(\text{mod}\,2)$, that is, $p + 1 \equiv 1(\text{mod}\,2)$. Then by formula (3.4) and Lemma 2.5, we find

$$E_c(D_{p+1}^2) - E_c(D_p^2) = \cot\frac{\pi}{2(p+1)} - 2\cot\frac{\pi}{p} > 0. \quad (3.5)$$

Next, let $p \equiv 1(\text{mod}\,2)$. In this case, $p \geq 5$ and $p + 1 \equiv 0(\text{mod}\,2)$. Using formula (3.4) and Lemma 2.5, we obtain:

$$E_c(D_{p+1}^2) - E_c(D_p^2) = 2\cot\frac{\pi}{p+1} - \cot\frac{\pi}{2p} > 0. \quad (3.6)$$

From (3.5) and (3.6), we have

$$E_c(D_{p+1}^2) > E_c(D_p^2).$$

This gives the required result.

An easy consequence of Lemma 3.2 is given below:

Corollary 3.3. Let $n \geq 7$, $m = 2$ *and* $4 \leq p_2 \leq p_1 \leq n - 2$. *Take* $D_{p_1}^2, D_{p_2}^2 \in F_n$. *Then*

$$E_c(D_{p_1}^2) \geq E_c(D_{p_2}^2).$$

Next lemma illustrates that the iota energy of digraphs in F_n increases monotonically with the increase in length of their joined directed cycles when $m \equiv 0(\text{mod}\,2)$.

Lemma 3.4. *Let* $n \geq 7$, $p = 2$, $4 \leq m < n - 2$ *and* $m \equiv 0(\text{mod}\,2)$. *Take* $D_2^m, D_2^{m+1} \in F_n$. *Then*

$$E_c(D_2^{m+1}) > E_c(D_2^m).$$

Proof. As $m \equiv 0(\text{mod}\,2)$, we have $m + 1 \equiv 1(\text{mod}\,2)$. Then by formula (3.4), we obtain:

$$\begin{aligned} & E_c(D_2^{m+1}) - E_c(D_2^m) \\ = \; & \sqrt[m+1]{2}\cot\frac{\pi}{2(m+1)} - 2\sqrt[m]{2}\cot\frac{\pi}{m} \\ = \; & \sqrt[m+1]{2}\cot\frac{\pi}{2(m+1)} - 2\sqrt[m]{2}\left(\frac{\cos^2\frac{\pi}{2m} - \sin^2\frac{\pi}{2m}}{2\sin\frac{\pi}{2m}\cos\frac{\pi}{2m}}\right) \\ = \; & \sqrt[m+1]{2}\cot\frac{\pi}{2(m+1)} - \sqrt[m]{2}\left(\cot\frac{\pi}{2m} - \tan\frac{\pi}{2m}\right) \\ = \; & \sqrt[m+1]{2}\cot\frac{\pi}{2(m+1)} - \sqrt[m]{2}\cot\frac{\pi}{2m} + \sqrt[m]{2}\tan\frac{\pi}{2m}. \end{aligned}$$

Using Lemma 2.4, we have:

$$\sqrt[m+1]{2}\cot\frac{\pi}{2(m+1)} - \sqrt[m]{2}\cot\frac{\pi}{2m} > 0.$$

Also, $\sqrt[m]{2}\tan\frac{\pi}{2m} > 0$. Thus

$$E_c(D_2^{m+1}) - E_c(D_2^m) > 0.$$

This proves the assertion.

The iota energy of digraphs in F_n also increases monotonically with the increase in length of their joined directed cycles when $m \equiv 1(\text{mod}\,2)$.

Lemma 3.5. Let $n \geq 7$, $p = 2$, $4 \leq m \leq n - 2$ and $m \equiv 1(\text{mod}\,2)$. *Take* D_2^{m-1}, $D_2^m \in F_n$, *where* $m \geq 5$. *Then:*

$$E_c(D_2^m) > E_c(D_2^{m-1}).$$

Proof. As $m \equiv 1(\text{mod}\,2)$, it holds that $m - 1 \equiv 0(\text{mod}\,2)$. Using formula (3.4), we obtain

$$\begin{aligned} & E_c(D_2^m) - E_c(D_2^{m-1}) \\ = \; & \sqrt[m]{2}\cot\frac{\pi}{2m} - 2\sqrt[m-1]{2}\cot\frac{\pi}{m-1}. \end{aligned}$$

Rest of the proof follows from the proof of Lemma 3.4.

The following is a consequence of Lemmas 3.4 and 3.5.

Corollary 3.6. Let $n \geq 7$, $p = 2$ and $4 \leq m_2 \leq m_1 \leq n-2$. *Take* $D_2^{m_1}, D_2^{m_2} \in F_n$. *Then,*

$$E_c(D_2^{m_1}) \geq E_c(D_2^{m_2}).$$

The following lemma is useful in proving our main result.

Lemma 3.7. *Let* $n \geq 7$ and $D_2^k, D_k^2 \in F_n$, *where* $4 \leq k \leq n-2$. *Then we have the following:*

$$E_c(D_2^k) > E_c(D_k^2).$$

Proof. By formula (3.4), we obtain:

$$E_c(D_2^k) = \begin{cases} 2\sqrt[k]{2}\cot\frac{\pi}{k} & \text{if } k \equiv 0(\bmod 2) \\ \sqrt[k]{2}\cot\frac{\pi}{2k} & \text{if } k \equiv 1(\bmod 2). \end{cases} \quad (3.7)$$

Also, formula (3.4) gives:

$$E_c(D_k^2) = \begin{cases} 2\cot\frac{\pi}{k} & \text{if } k \equiv 0(\bmod 2) \\ \cot\frac{\pi}{2k} & \text{if } k \equiv 1(\bmod 2). \end{cases} \quad (3.8)$$

As $\sqrt[k]{2} > 1$, formulas (3.7) and (3.8) clearly show that the following inequality holds:

$$E_c(D_2^k) > E_c(D_k^2).$$

This gives the required result.

Using Corollary 3.3, one can easily see that the following inequality holds:

$$E_c(D_{n-2}^2) \geq E_c(D_p^2), \quad (3.9)$$

where $4 \leq p \leq n-2$. From Corollary 3.6, the following inequality is easily seen:

$$E_c(D_2^{n-2}) \geq E_c(D_2^m), \quad (3.10)$$

where $4 \leq m \leq n-2$. By (3.9), (3.10) and Lemma 3.7, we find

$$E_c(D_2^{n-2}) > E_c(D_{n-2}^2). \quad (3.11)$$

The following theorem gives the digraphs in F_n with minimal and maximal iota energy.

Theorem 3.8. *Let* $n \geq 4$ *and* $D_p^m \in F_n$, *where* $2 \leq m \leq n-2$ *and* $2 \leq p \leq n-2$. *Then* D_p^m *has minimal iota energy when* $m = p = 2$ *and maximal iota energy when* $m = n-2$ *and* $p = 2$ *among all digraphs of* F_n.

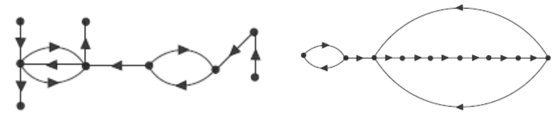
Proof. It can easily be seen from formula (3.4) that:

$$E_c(D_p^m) = \begin{cases} 0 & \text{if } m = 2 \text{ and } p = 2 \\ \sqrt{3} & \text{if } m = 2 \text{ and } p = 3 \\ 2 & \text{if } m = 2 \text{ and } p = 4 \\ \sqrt{3}\sqrt[3]{2} & \text{if } m = 3 \text{ and } p = 2. \end{cases} \quad (3.12)$$

Let [$m = 2$ and $p \geq 5$] or [$p = 2$ and $m \geq 4$]. Then from formula (3.4) and inequalities (3.9) – (3.11), we obtain:

$$\sqrt{3}\sqrt[3]{2} < 2\sqrt[4]{2} \leq E_c(D_p^m) < E_c(D_2^{n-2}). \quad (3.13)$$

It is clear from (3.12) and (3.13) that $E_c(D_p^m)$ is minimal when $m = 2$ and p = 2. Furthermore, from (3.13) it is evident that $E_c(D_p^m)$ is maximal when $m = n-2$ and $p = 2$.



(a) (b)

Figure 2: (a) The digraph with minimal iota energy in F_9, (b) The digraph with maximal iota energy in F_9.

Example 3.9. The digraphs with minimal and maximal iota energy in F_9 are shown in Figure 2.

4.IOTA ENERGY OF SIDIGRAPHS

For $n \geq 4$, we define a set J_n which consists of n-vertex tricyclic sidigraphs containing five linear subdigraphs such that one of the signed directed cycles does not share any vertex with the other two signed directed cycles and the remaining two signed directed cycles are of same length sharing at least one vertex. We denote a sidigraph in J_n by S_p^m, where m is the length of those signed directed cycles which share at least one vertex and p is the length of the signed directed cycle which does not share any vertex with the other two signed directed cycles such that $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$.

Let $J_n^{+++} \subset J_n$ be the subset of all those sidigraphs whose all three signed directed cycles are positive. Also, let $J_n^{++-} \subset J_n$ be the subset of all those sidigraphs whose joined signed directed cycles are positive and disjoint signed directed cycle is negative. Let $J_n^{--+} \subset J_n$ be the subset of all those sidigraphs whose joined signed directed cycles are negative and disjoint signed directed cycle is positive. Similarly, let $J_n^{+-+} \subset J_n$ be the subset of all those sidigraphs whose one of the joined signed directed cycles is positive and the other is negative whereas the disjoint signed directed cycle is positive. Also, let $J_n^{+--} \subset J_n$ be the subset of all those sidigraphs whose one of the joined signed directed cycles is positive and the other is negative whereas the disjoint signed directed cycle is negative. Lastly, let $J_n^{---} \subset J_n$ be the subset of all those sidigraphs whose all three signed directed cycles are negative. There are following six possible cases:

Case 1

If $S_p^m \in J_n^{+++}$ then by Theorem 2.3, we have the following result which is analogue of Theorem 3.8.

Theorem 4.1. Let $n \geq 4$ and $S_p^m \in J_n^{+++}$, where $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. Then S_p^m has minimal iota energy when $m = p = 2$ and maximal iota energy when $m = n-2$ and $p = 2$ among all sidigraphs of J_n^{+++}.

Case 2

If $S_p^m \in J_n^{++-}$ then from Theorem 2.2, the characteristic polynomial of S_p^m is given by

$$\phi_{S_p^m}(x) = x^{n-m-p}(x^m - 2)(x^p + 1).$$

The eigenvalues of S_p^m are 0, $\sqrt[m]{2}\, exp\left(\frac{2k\pi i}{m}\right)$ and $\exp\left(\frac{(2j+1)\pi i}{p}\right)$, where $k = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, p-1$ and the multiplicity of eigenvalue 0 is $n-m-p$. Thus, the iota energy of S_p^m is given by:

$$E_c(S_p^m) = \sqrt[m]{2}\sum_{k=0}^{m-1}\left|\sin\frac{2k\pi}{m}\right| + \sum_{j=0}^{p-1}\left|\sin\frac{(2j+1)\pi}{p}\right|. \quad (4.14)$$

It is shown in Farooq et al. [6] that:

$$\sum_{j=0}^{p-1}\left|\sin\frac{(2j+1)\pi}{p}\right| = \begin{cases} 2\csc\frac{\pi}{p} & \text{if } p \equiv 0(\bmod 2) \\ \cot\frac{\pi}{2p} & \text{if } p \equiv 1(\bmod 2). \end{cases} \quad (4.15)$$

Using (3.3), (4.14) and (4.15), the iota energy formulas for $S_p^m \in J_n^{++-}$ are given by:

$$E_c(S_p^m) = \begin{cases} 2\left(\sqrt[m]{2}\cot\frac{\pi}{m} + \csc\frac{\pi}{p}\right) & \text{if } m \equiv 0(\text{mod}2) \\ & \text{and } p \equiv 0(\text{mod}2) \\ 2\sqrt[m]{2}\cot\frac{\pi}{m} + \cot\frac{\pi}{2p} & \text{if } m \equiv 0(\text{mod}2) \\ & \text{and } p \equiv 1(\text{mod}2) \\ \sqrt[m]{2}\cot\frac{\pi}{2m} + 2\csc\frac{\pi}{p} & \text{if } m \equiv 1(\text{mod}2) \\ & \text{and } p \equiv 0(\text{mod}2) \\ \sqrt[m]{2}\cot\frac{\pi}{2m} + \cot\frac{\pi}{2p} & \text{if } m \equiv 1(\text{mod}2) \\ & \text{and } p \equiv 1(\text{mod}2) \end{cases} \quad (4.16)$$

We find the minimal and maximal iota energy among the sidigraphs in J_n^{++-}. The following lemma shows that the iota energy of sidigraphs in J_n^{++-} increases monotonically with the increase in length of their disjoint signed directed cycles.

Lemma 4.2. *Let $n \geq 7$, $m = 2$ and $4 \leq p < n - 2$. Take $S_p^2, S_{p+1}^2 \in J_n^{++-}$. Then:*

$$E_c(S_{p+1}^2) > E_c(S_p^2).$$

Proof. Let $p \equiv 0(\text{mod}2)$, that is, $p + 1 \equiv 1(\text{mod}2)$. Then by formula (4.16) and Lemma 2.5, we find

$$\begin{aligned} &E_c(S_{p+1}^2) - E_c(S_p^2) \\ &= \cot\frac{\pi}{2(p+1)} - 2\csc\frac{\pi}{p} > 0. \end{aligned} \quad (4.17)$$

Next, let $p \equiv 1(\text{mod}2)$. In this case, $p \geq 5$ and $p + 1 \equiv 0(\text{mod}2)$. Using formula (4.16) and Lemma 2.5, we obtain:

$$\begin{aligned} &E_c(S_{p+1}^2) - E_c(S_p^2) \\ &= 2\csc\frac{\pi}{p+1} - \cot\frac{\pi}{2p} > 0. \end{aligned} \quad (4.18)$$

From (4.17) and (4.18), we have

$$E_c(S_{p+1}^2) > E_c(S_p^2).$$

This gives the required result.

An easy consequence of Lemma 4.2 is the following.

Corollary 4.3. *Let $n \geq 7$, $m = 2$ and $4 \leq p_2 \leq p_1 \leq n - 2$. Take $S_{p_1}^2, S_{p_2}^2 \in J_n^{++-}$. Then*

$$E_c(S_{p_1}^2) \geq E_c(S_{p_2}^2).$$

Next lemma illustrates that the iota energy of sidigraphs in J_n^{++-} increases monotonically with the increase in length of their joined signed directed cycles when $m \equiv 0(\text{mod}2)$.

Lemma 4.4. *Let $n \geq 7$, $p = 2$, $4 \leq m < n - 2$ and $m \equiv 0(\text{mod}2)$.* Take $S_2^m, S_2^{m+1} \in J_n^{++-}$. *Then:*

$$E_c(S_2^{m+1}) > E_c(S_2^m).$$

Proof. As $m \equiv 0(\text{mod}2)$, we have $m + 1 \equiv 1(\text{mod}2)$. Then by formula (4.16), we obtain:

$$\begin{aligned} &E_c(S_2^{m+1}) - E_c(S_2^m) \\ =\ &\sqrt[m+1]{2}\cot\frac{\pi}{2(m+1)} + 2 - 2\sqrt[m]{2}\cot\frac{\pi}{m} - 2. \end{aligned}$$

Rest of the proof follows from the proof of Lemma 3.4.

The iota energy of sidigraphs in J_n^{++-} also increases monotonically with the increase in length of their joined signed directed cycles when $m \equiv 1(\text{mod}2)$.

Lemma 4.5. *Let $n \geq 7$, $p = 2$, $4 \leq m \leq n - 2$ and $m \equiv 1(\text{mod}2)$. Take $S_2^{m-1}, S_2^m \in J_n^{++-}$, where $m \geq 5$. Then:*

$$E_c(S_2^m) > E_c(S_2^{m-1}).$$

Proof. As $m \equiv 1(\text{mod}2)$, it holds that $m - 1 \equiv 0(\text{mod}2)$. Using formula (4.16), we obtain:

$$\begin{aligned} &E_c(S_2^m) - E_c(S_2^{m-1}) \\ =\ &\sqrt[m]{2}\cot\frac{\pi}{2m} + 2 - 2\sqrt[m-1]{2}\cot\frac{\pi}{m-1} - 2. \end{aligned}$$

The rest of the proof follows from the proof of Lemma 3.4.

The following corollary is an easy consequence of Lemmas 4.4 and 4.5.

Corollary 4.6. *Let $n \geq 7$, $p = 2$ and $4 \leq m_2 \leq m_1 \leq n - 2$. Take $S_2^{m_1}, S_2^{m_2} \in J_n^{++-}$. Then:*

$$E_c(S_2^{m_1}) \geq E_c(S_2^{m_2}).$$

The following lemma is useful in proving our main result.

Lemma 4.7. *Let $n \geq 7$ and $S_2^k, S_k^2 \in J_n^{++-}$, where $4 \leq k \leq n - 2$. Then we have the following:*

$$E_c(S_2^k) > E_c(S_k^2).$$

Proof. By formula (4.16), we obtain:

$$E_c(S_2^k) = \begin{cases} 2\sqrt[k]{2}\cot\frac{\pi}{k} + 2 & \text{if } k \equiv 0(\text{mod}2) \\ \sqrt[k]{2}\cot\frac{\pi}{2k} + 2 & \text{if } k \equiv 1(\text{mod}2). \end{cases} \quad (4.19)$$

Also, formula (4.16) gives:

$$E_c(S_k^2) = \begin{cases} 2\csc\frac{\pi}{k} & \text{if } k \equiv 0(\text{mod}2) \\ \cot\frac{\pi}{2k} & \text{if } k \equiv 1(\text{mod}2). \end{cases} \quad (4.20)$$

Let $k \equiv 0(\text{mod}2)$. Note that $\sqrt[k]{2} > 1$. This along with (4.19) and (4.20) imply:

$$\begin{aligned} E_c(S_2^k) - E_c(S_k^2) &= 2\left(\sqrt[k]{2}\cot\frac{\pi}{k} + 1 - \csc\frac{\pi}{k}\right) \\ &> \cot\frac{\pi}{k} + 1 - \csc\frac{\pi}{k}. \end{aligned} \quad (4.21)$$

Let $a_k = \cot\frac{\pi}{k} + 1$ and $b_k = \csc\frac{\pi}{k}$. Then by Lemma 2.6, we find:

$$\begin{aligned} a_k &= \cot\frac{\pi}{k} + 1 \\ &\geq \frac{k}{\pi} - 0.429\frac{\pi}{k} + 1. \end{aligned} \quad (4.22)$$

As $k \geq 4$, we have $-0.429\frac{\pi}{k} \geq -0.429\frac{\pi}{4}$. Thus,

$$\begin{aligned} a_k &\geq \frac{k}{\pi} - 0.429\frac{\pi}{4} + 1 \\ &\geq \frac{k}{\pi} + 0.663. \end{aligned} \quad (4.23)$$

Also, using inequality (2.1) and Lemma 2.7, we obtain:

$$\begin{aligned} b_k &= \csc\frac{\pi}{k} \\ &\le \frac{1}{\frac{\pi}{k}(1-\frac{\pi^2}{6k^2})} \\ &= \frac{k}{\pi}\left(1+\frac{\pi^2}{6k^2-\pi^2}\right) \\ &= \frac{k}{\pi}+\frac{k\pi}{6k^2-\pi^2} \\ &\le \frac{k}{\pi}+\frac{4\pi}{6\times 4^2-\pi^2} \\ &\le \frac{k}{\pi}+0.146. \end{aligned} \quad (4.24)$$

From (4.22) – (4.24), we obtain:

$$\begin{aligned} a_k - b_k &= \cot\frac{\pi}{k}+1-\csc\frac{\pi}{k} \\ &\ge \frac{k}{\pi}+0.663-\frac{k}{\pi}-0.146 \\ &> 0. \end{aligned} \quad (4.25)$$

Using (4.21) – (4.25), we obtain:

$$\begin{aligned} E_c(S_2^k)-E_c(S_k^2) &> \cot\frac{\pi}{k}+1-\csc\frac{\pi}{k} \\ &> 0. \end{aligned} \quad (4.26)$$

Next, let $k \equiv 1(\text{mod}2)$. From (4.19) and (4.20), one can easily see that the following inequality holds:

$$E_c(S_2^k)-E_c(S_k^2) > 0. \quad (4.27)$$

From (4.26) and (4.27), we obtain the required result.

Using Corollary 4.3, one can easily see that the following inequality holds:

$$E_c(S_{n-2}^2) \ge E_c(S_p^2), \quad (4.28)$$

where $4 \le p \le n-2$. From Corollary 4.6, the following inequality is easily seen:

$$E_c(S_2^{n-2}) \ge E_c(S_2^m), \quad (4.29)$$

where $4 \le m \le n-2$. By (4.28), (4.29) and Lemma 4.7, we find:

$$E_c(S_2^{n-2}) > E_c(S_{n-2}^2). \quad (4.30)$$

The following theorem gives the sidigraphs in J_n^{++-} with minimal and maximal iota energy.

Theorem 4.8. *Let $n \ge 4$ and $S_p^m \in J_n^{++-}$, where $2 \le m \le n-2$ and $2 \le p \le n-2$. Then S_p^m has minimal iota energy when $m = 2$ and $p = 3$ and maximal iota energy when $m = n-2$ and $p = 2$ among all sidigraphs of J_n^{++-}.*

Proof. Let $S_p^m \in J_n^{++-}$, where $2 \le m \le n-2$ and $2 \le p \le n-2$. Then, it can easily be seen from formula (4.16) that:

$$E_c(S_p^m) = \begin{cases} 2 & \text{if } m=2 \text{ and } p=2 \\ \sqrt{3} & \text{if } m=2 \text{ and } p=3 \\ 2\sqrt{2} & \text{if } m=2 \text{ and } p=4 \\ \sqrt{5+2\sqrt{5}} & \text{if } m=2 \text{ and } p=5 \\ 4 & \text{if } m=2 \text{ and } p=6 \\ \sqrt{3}\sqrt[3]{2}+2 & \text{if } m=3 \text{ and } p=2. \end{cases} \quad (4.31)$$

Let [$m = 2$ and $p \ge 7$] or [$p = 2$ and $m \ge 4$]. Also, let $S_2^{n-2} \in J_n^{++-}$. Then from formula (4.16) and inequalities (4.28) – (4.30), we obtain:

$$\sqrt{3}\sqrt[3]{2}+2 < 2\sqrt[4]{2}+2 \le E_c(S_p^m) < E_c(S_2^{n-2}). \quad (4.32)$$

It is clear from (4.31) and (4.32) that $E_c(S_p^m)$ is minimal when $m = 2$ and $p = 3$. Furthermore, from (4.32) it is evident that $E_c(S_p^m)$ is maximal when $m = n-2$ and $p = 2$.

Case 3

If $S_p^m \in J_n^{--+}$ then from Theorem 2.2, the characteristic polynomial of S_p^m is given by:

$$\phi_{S_p^m}(x) = x^{n-m-p}(x^m+2)(x^p-1).$$

The eigenvalues of S_p^m are 0, $\sqrt[m]{2}\, exp\left(\frac{(2k+1)\pi i}{m}\right)$ and $\exp\left(\frac{2j\pi i}{p}\right)$, where $k = 0,1 \ldots, m-1$ and $j = 0,1,\ldots, p-1$ and the multiplicity of eigenvalue 0 is $n-m-p$. Thus, the iota energy of S_p^m is given by:

$$E_c(S_p^m) = \sqrt[m]{2}\sum_{k=0}^{m-1}\left|\sin\frac{(2k+1)\pi}{m}\right| + \sum_{j=0}^{p-1}\left|\sin\frac{2j\pi}{p}\right|. \quad (4.33)$$

Using (3.3), (4.15) and (4.33), the iota energy formulas for $S_p^m \in J_n^{--+}$ are given by:

$$E_c(S_p^m) = \begin{cases} 2\left(\sqrt[m]{2}\csc\frac{\pi}{m}+\cot\frac{\pi}{p}\right) & \text{if } m \equiv 0(\text{mod}2) \text{ and } p \equiv 0(\text{mod}2) \\ 2\sqrt[m]{2}\csc\frac{\pi}{m}+\cot\frac{\pi}{2p} & \text{if } m \equiv 0(\text{mod}2) \text{ and } p \equiv 1(\text{mod}2) \\ \sqrt[m]{2}\cot\frac{\pi}{2m}+2\cot\frac{\pi}{p} & \text{if } m \equiv 1(\text{mod}2) \text{ and } p \equiv 0(\text{mod}2) \\ \sqrt[m]{2}\cot\frac{\pi}{2m}+\cot\frac{\pi}{2p} & \text{if } m \equiv 1(\text{mod}2) \text{ and } p \equiv 1(\text{mod}2). \end{cases} \quad (4.34)$$

Next, we find the minimal and maximal iota energy among the sidigraphs in J_n^{--+}. The following lemma shows that the iota energy of sidigraphs in J_n^{--+} increases monotonically with the increase in length of their disjoint signed directed cycles.

Lemma 4.9. *Let $n \ge 7$, $m = 2$ and $4 \le p < n-2$. Take $S_p^2, S_{p+1}^2 \in J_n^{--+}$. Then:*

$$E_c(S_{p+1}^2) > E_c(S_p^2).$$

Proof. Let $p \equiv 0(\text{mod}2)$, that is, $p+1 \equiv 1(\text{mod}2)$. Then by formula (4.34) and Lemma 2.5, we find:

$$\begin{aligned} &E_c(S_{p+1}^2)-E_c(S_p^2) \\ &= 2\sqrt[2]{2}+\cot\frac{\pi}{2(p+1)}-2\sqrt[2]{2}-2\cot\frac{\pi}{p} > 0. \end{aligned} \quad (4.35)$$

Next, let $p \equiv 1(\text{mod}2)$. In this case, $p \ge 5$ and $p+1 \equiv 0(\text{mod}2)$. Using formula (4.34) and Lemma 2.5, we obtain:

$$\begin{aligned} &E_c(S_{p+1}^2)-E_c(S_p^2) \\ &= 2\sqrt[2]{2}+2\cot\frac{\pi}{p+1}-2\sqrt[2]{2}-\cot\frac{\pi}{2p} > 0. \end{aligned} \quad (4.36)$$

From (4.35) and (4.36), we have:

$$E_c(S_{p+1}^2) > E_c(S_p^2).$$

This gives the required result.

The following result is an easy consequence of Lemma 4.9.

Corollary 4.10. *Let $n \ge 7$, $m = 2$ and $4 \le p_2 \le p_1 \le n-2$. Take $S_{p_1}^2, S_{p_2}^2 \in J_n^{--+}$. Then*

$$E_c(S_{p_1}^2) \ge E_c(S_{p_2}^2).$$

Next lemma illustrates that the iota energy of sidigraphs in J_n^{--+} increases monotonically with the increase in length of their joined signed directed cycles when $m \equiv 0(\text{mod}2)$.

Lemma 4.11. *Let $n \geq 7$, $p = 2$, $4 \leq m \leq n-2$ and $m \equiv 0(\text{mod}2)$. Take S_2^{m-1}, $S_2^m \in J_n^{--+}$, where $m \geq 5$. Then:*

$$E_c(S_2^m) > E_c(S_2^{m-1}).$$

Proof. As $m \equiv 0(\text{mod}2)$, we have $m-1 \equiv 1(\text{mod}2)$. Then by formula (4.34), we obtain:

$$\begin{aligned} & E_c(S_2^m) - E_c(S_2^{m-1}) \\ = \; & 2\sqrt[m]{2}\csc\frac{\pi}{m} - \sqrt[m-1]{2}\cot\frac{\pi}{2(m-1)} \\ = \; & 2\sqrt[m]{2}\left(\frac{1}{2\sin\frac{\pi}{2m}\cos\frac{\pi}{2m}}\right) - \sqrt[m-1]{2}\cot\frac{\pi}{2(m-1)} \\ = \; & \sqrt[m]{2}\left(\frac{\sin^2\frac{\pi}{2m}+\cos^2\frac{\pi}{2m}}{\sin\frac{\pi}{2m}\cos\frac{\pi}{2m}}\right) - \sqrt[m-1]{2}\cot\frac{\pi}{2(m-1)} \\ = \; & \sqrt[m]{2}\left(\tan\frac{\pi}{2m}+\cot\frac{\pi}{2m}\right) - \sqrt[m-1]{2}\cot\frac{\pi}{2(m-1)} \\ = \; & \sqrt[m]{2}\tan\frac{\pi}{2m} + \sqrt[m]{2}\cot\frac{\pi}{2m} - \sqrt[m-1]{2}\cot\frac{\pi}{2(m-1)}. \end{aligned}$$

By Lemma 2.4, we have

$$\sqrt[m]{2}\cot\frac{\pi}{2m} - \sqrt[m-1]{2}\cot\frac{\pi}{2(m-1)} > 0,$$

where $4 \leq m \leq n-2$. Also, we know that $\sqrt[m]{2}\tan\frac{\pi}{2m} > 0$ for $4 \leq m \leq n-2$. Thus:

$$E_c(S_2^m) - E_c(S_2^{m-1}) > 0.$$

This proves the assertion.

The iota energy of sidigraphs in J_n^{--+} also increases monotonically with the increase in length of their joined signed directed cycles when $m \equiv 1(\text{mod}2)$.

Lemma 4.12. *Let $n \geq 7$, $p = 2$, $4 \leq m < n-2$ and $m \equiv 1(\text{mod}2)$. Take S_2^m, $S_2^{m+1} \in J_n^{--+}$. Then:*

$$E_c(S_2^{m+1}) > E_c(S_2^m).$$

Proof. As $m \equiv 1(\text{mod}2)$, it holds that $m+1 \equiv 0(\text{mod}2)$. Using formula (4.34), we obtain:

$$E_c(S_2^{m+1}) - E_c(S_2^m) = 2\sqrt[m+1]{2}\csc\frac{\pi}{m+1} - \sqrt[m]{2}\cot\frac{\pi}{2m}.$$

The rest of the proof follows from the proof of Lemma 4.11.

The following corollary is an easy consequence of Lemmas 4.11 and 4.12.

Corollary 4.13. *Let $n \geq 7$, $p = 2$ and $4 \leq m_2 \leq m_1 \leq n-2$. Take $S_2^{m_1}$, $S_2^{m_2} \in J_n^{--+}$. Then*

$$E_c(S_2^{m_1}) \geq E_c(S_2^{m_2}).$$

The following lemma is useful in proving our main result.

Lemma 4.14. *Let $n \geq 7$ and S_2^k, $S_k^2 \in J_n^{--+}$, where $4 \leq k \leq n-2$. Then we have the following:*

$$E_c(S_k^2) > E_c(S_2^k).$$

Proof. By formula (4.34), we obtain:

$$E_c(S_k^2) = \begin{cases} 2\left(\sqrt[2]{2}+\cot\frac{\pi}{k}\right) & \text{if } k \equiv 0(\text{mod}2) \\ 2\sqrt[2]{2}+\cot\frac{\pi}{2k} & \text{if } k \equiv 1(\text{mod}2). \end{cases} \quad (4.37)$$

Also, formula (4.34) gives

$$E_c(S_2^k) = \begin{cases} 2\sqrt[k]{2}\csc\frac{\pi}{k} & \text{if } k \equiv 0(\text{mod}2) \\ \sqrt[k]{2}\cot\frac{\pi}{2k} & \text{if } k \equiv 1(\text{mod}2). \end{cases} \quad (4.38)$$

One can easily see from formulas (4.37) and (4.38) that the following inequality holds:

$$E_c(S_k^2) > E_c(S_2^k).$$

This gives the required result.

Using Corollary 4.10, one can easily see that the following inequality holds:

$$E_c(S_{n-2}^2) \geq E_c(S_p^2), \quad (4.39)$$

where $4 \leq p \leq n-2$. From Corollary 4.13, the following inequality is easily seen:

$$E_c(S_2^{n-2}) \geq E_c(S_2^m), \quad (4.40)$$

where $4 \leq m \leq n-2$. By (4.39), (4.40) and Lemma 4.14, we find:

$$E_c(S_{n-2}^2) > E_c(S_2^{n-2}). \quad (4.41)$$

The following theorem gives the sidigraphs in J_n^{--+} with minimal and maximal iota energy.

Theorem 4.15. *Let $n \geq 4$ and $S_p^m \in J_n^{--+}$, where $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. Then S_p^m has minimal iota energy when $m = 3$ and $p = 2$ and maximal iota energy when $m = 2$ and $p = n-2$ among all sidigraphs of J_n^{--+}*

Proof. It can easily be seen from formula (4.34) that:

$$E_c(S_p^m) = \begin{cases} 2\sqrt[2]{2} & \text{if } p = 2 \text{ and } m = 2 \\ \sqrt{3}\sqrt[3]{2} & \text{if } p = 2 \text{ and } m = 3 \\ 2\sqrt{2}\sqrt[4]{2} & \text{if } p = 2 \text{ and } m = 4 \\ \sqrt[5]{2}\sqrt{5+2\sqrt{5}} & \text{if } p = 2 \text{ and } m = 5 \\ 4\sqrt[6]{2} & \text{if } p = 2 \text{ and } m = 6 \\ 2\sqrt[2]{2}+\sqrt{3} & \text{if } p = 3 \text{ and } m = 2. \end{cases} \quad (4.42)$$

Let [$p = 2$ and $m \geq 7$] or [$m = 2$ and $p \geq 4$]. Also, let $S_{n-2}^2 \in J_n^{--+}$. Then from formula (4.34) and inequalities (4.39) – (4.41), we obtain:

$$2\sqrt[2]{2}+\sqrt{3} < 2\sqrt[2]{2}+2 \leq E_c(S_p^m) < E_c(S_{n-2}^2). \quad (4.43)$$

It is clear from (4.42) and (4.43) that $E_c(S_p^m)$ is minimal when $m = 3$ and $p = 2$. Furthermore, from (4.43) it is evident that $E_c(S_p^m)$ is maximal when $m = 2$ and $p = n-2$.

Case 4

If $S_p^m \in J_n^{+-+}$ then from Theorem 2.2, the characteristic polynomial of S_p^m is given by:

$$\phi_{S_p^m}(x) = x^{n-p}(x^p - 1).$$

The eigenvalues of S_p^m are 0 and $\exp\left(\frac{2j\pi i}{p}\right)$, where $j = 0, 1, \ldots, p-1$ and the multiplicity of eigenvalue 0 is $n - p$. Thus, the iota energy of S_p^m is given by:

$$E_c(S_p^m) = \sum_{j=0}^{p-1} |\sin \frac{2j\pi}{p}|. \qquad (4.44)$$

Using (3.3) and (4.44), the iota energy formulas for $S_p^m \in J_n^{+-+}$ are given by:

$$E_c(S_p^m) = \begin{cases} 2\cot\frac{\pi}{p} & \text{if } p \equiv 0(\text{mod}2) \\ \cot\frac{\pi}{2p} & \text{if } p \equiv 1(\text{mod}2). \end{cases} \qquad (4.45)$$

The following theorem gives the sidigraphs in J_n^{+-+} with minimal and maximal iota energy.

Theorem 4.16. *Let $n \geq 4$ and $S_p^m \in J_n^{+-+}$, where $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. Then S_p^m has minimal iota energy when $p = 2$ and maximal iota energy when $p = n - 2$ among all sidigraphs of J_n^{+-+}.*

Proof. It can easily be seen from formula (4.45) that:

$$E_c(S_p^m) = \begin{cases} 0 & \text{if } p = 2 \\ \sqrt{3} & \text{if } p = 3. \end{cases} \qquad (4.46)$$

Let $p \geq 4$ and $S_r^m \in J_n^{+-+}$ is a sidigraph with length of the disjoint signed directed cycle r, where $r \geq p$. Then from formula (4.45) and Lemma 2.5, we obtain:

$$\sqrt{3} < 2 \leq E_c(S_p^m) \leq E_c(S_r^m). \qquad (4.47)$$

It is clear from (4.46) and (4.47) that $E_c(S_p^m)$ is minimal when $p = 2$. Furthermore, J_n^{+-+} contains a sidigraph with disjoint signed directed cycle of maximum length $n - 2$. Therefore, from (4.47) it is evident that $E_c(S_p^m)$ is maximal when $p = n - 2$.

Case 5

If $S_p^m \in J_n^{+--}$ then from Theorem 2.2, the characteristic polynomial of S_p^m is given by:

$$\phi_{S_p^m}(x) = x^{n-p}(x^p + 1).$$

The eigenvalues of S_p^m are 0 and $\exp\left(\frac{(2j+1)\pi i}{p}\right)$, where $j = 0, 1, \ldots, p-1$ and the multiplicity of eigenvalue 0 is $n - p$. Thus, the iota energy of S_p^m is given by:

$$E_c(S_p^m) = \sum_{j=0}^{p-1} |\sin \frac{(2j+1)\pi}{p}|. \qquad (4.48)$$

Using (4.15) and (4.48), the iota energy formulas for $S_p^m \in J_n^{+--}$ are given by:

$$E_c(S_p^m) = \begin{cases} 2\csc\frac{\pi}{p} & \text{if } p \equiv 0(\text{mod}2) \\ \cot\frac{\pi}{2p} & \text{if } p \equiv 1(\text{mod}2). \end{cases} \qquad (4.49)$$

Next theorem gives the sidigraphs in J_n^{+--} with minimal and maximal iota energy.

Theorem 4.17. *Let $n \geq 4$ and $S_p^m \in J_n^{+--}$, where $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. Then S_p^m has minimal iota energy when $p = 3$ and maximal iota energy when $p = n - 2$ among all sidigraphs of J_n^{+--}*

Proof. Let $S_p^m \in J_n^{+--}$, where $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. Then, it can easily be seen from formula (4.49) that:

$$E_c(S_p^m) = \begin{cases} 2 & \text{if } p = 2 \\ \sqrt{3} & \text{if } p = 3. \end{cases} \qquad (4.50)$$

Let $p \geq 4$ and $S_r^m \in J_n^{+--}$, where $r \geq p$. Then from formula (4.49) and Lemma 2.5, we obtain:

$$2 < 2\sqrt{2} \leq E_c(S_p^m) \leq E_c(S_r^m). \qquad (4.51)$$

It is clear from (4.50) and (4.51) that $E_c(S_p^m)$ is minimal when $p = 3$. Furthermore, J_n^{+--} contains a sidigraph with disjoint signed directed cycle of maximum length $n - 2$. Therefore, from (4.51) it is evident that $E_c(S_p^m)$ is maximal when $p = n - 2$.

Case 6

If $S_p^m \in J_n^{---}$ then from Theorem 2.2, the characteristic polynomial of S_p^m is given by:

$$\phi_{S_p^m}(x) = x^{n-m-p}(x^m + 2)(x^p + 1).$$

The eigenvalues of S_p^m are 0, $\sqrt[m]{2}\exp\left(\frac{(2k+1)\pi i}{m}\right)$ and $\exp\left(\frac{(2j+1)\pi i}{p}\right)$, where $k = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, p-1$ and the multiplicity of eigenvalue 0 is $n - m - p$. Thus, the iota energy of S_p^m is given by:

$$E_c(S_p^m) = \sqrt[m]{2}\sum_{k=0}^{m-1} |\sin \frac{(2k+1)\pi}{m}| + \sum_{j=0}^{p-1} |\sin \frac{(2j+1)\pi}{p}|. \qquad (4.52)$$

Using (4.15) and (4.52), the iota energy formulas for $S_p^m \in J_n^{---}$ are given by:

$$E_c(S_p^m) = \begin{cases} 2\left(\sqrt[m]{2}\csc\frac{\pi}{m} + \csc\frac{\pi}{p}\right) & \text{if } m \equiv 0(\text{mod}2) \text{ and } p \equiv 0(\text{mod}2) \\ 2\sqrt[m]{2}\csc\frac{\pi}{m} + \cot\frac{\pi}{2p} & \text{if } m \equiv 0(\text{mod}2) \text{ and } p \equiv 1(\text{mod}2) \\ \sqrt[m]{2}\cot\frac{\pi}{2m} + 2\csc\frac{\pi}{p} & \text{if } m \equiv 1(\text{mod}2) \text{ and } p \equiv 0(\text{mod}2) \\ \sqrt[m]{2}\cot\frac{\pi}{2m} + \cot\frac{\pi}{2p} & \text{if } m \equiv 1(\text{mod}2) \text{ and } p \equiv 1(\text{mod}2). \end{cases} \qquad (4.53)$$

We find the minimal and maximal iota energy among the sidigraphs in J_n^{---}. The following lemma shows that the iota energy of sidigraphs in J_n^{---} increases monotonically with the increase in length of their disjoint signed directed cycles.

Lemma 4.18. *Let $n \geq 7$, $m = 2$ and $4 \leq p < n - 2$. Take $S_p^2, S_{p+1}^2 \in J_n^{---}$. Then:*

$$E_c(S_{p+1}^2) > E_c(S_p^2).$$

Proof. Let $p \equiv 0(\text{mod}2)$, that is, $p + 1 \equiv 1(\text{mod}2)$. Then by formula (4.53) and Lemma 2.5, we find:

$$E_c(S_{p+1}^2) - E_c(S_p^2) = 2\sqrt[2]{2} + \cot\frac{\pi}{2(p+1)} - 2\sqrt[2]{2} - 2\csc\frac{\pi}{p} > 0. \qquad (4.54)$$

Next, let $p \equiv 1(\text{mod}2)$. In this case, $p \geq 5$ and $p + 1 \equiv 0(\text{mod}2)$. Using formula (4.53) and Lemma 2.5, we obtain:

$$E_c(S_{p+1}^2) - E_c(S_p^2) = 2\sqrt[2]{2} + 2\csc\frac{\pi}{p+1} - 2\sqrt[2]{2} - \cot\frac{\pi}{2p} > 0. \qquad (4.55)$$

From (4.54) and (4.55), we have

$$E_c(S^2_{p+1}) > E_c(S^2_p).$$

This gives the required result.

The following result is an easy consequence of Lemma 4.18.

Corollary 4.19. *Let $n \geq 7$, $m = 2$ and $4 \leq p_2 \leq p_1 \leq n - 2$. Take $S^2_{p_1}$, $S^2_{p_2} \in J_n^{---}$. Then:*

$$E_c(S^2_{p_1}) \geq E_c(S^2_{p_2}).$$

Next lemma illustrates that the iota energy of sidigraphs in J_n^{---} increases monotonically with the increase in length of their joined signed directed cycles when $m \equiv 0(\text{mod}2)$.

Lemma 4.20. *Let $n \geq 7$, $p = 2$, $4 \leq m \leq n - 2$ and $m \equiv 0(\text{mod}2)$. Take S^{m-1}_2, $S^m_2 \in J_n^{---}$, where $m \geq 5$. Then:*

$$E_c(S^m_2) > E_c(S^{m-1}_2).$$

Proof. As $m \equiv 0(\text{mod}2)$, we have $m - 1 \equiv 1(\text{mod}2)$. Then by formula (4.53), we obtain:

$$E_c(S^m_2) - E_c(S^{m-1}_2) = 2\sqrt[m]{2}\csc\frac{\pi}{m} + 2 - \sqrt[m-1]{2}\cot\frac{\pi}{2(m-1)} - 2.$$

The rest of the proof follows from the proof of Lemma 4.11.

The iota energy of sidigraphs in J_n^{---} also increases monotonically with the increase in length of their joined signed directed cycles when $m \equiv 1(\text{mod}2)$.

Lemma 4.21. *Let $n \geq 7$, $p = 2$, $4 \leq m < n - 2$ and $m \equiv 1(\text{mod}2)$. Take S^m_2, $S^{m+1}_2 \in J_n^{---}$. Then:*

$$E_c(S^{m+1}_2) > E_c(S^m_2).$$

Proof. As $m \equiv 1(\text{mod}2)$, it holds that $m + 1 \equiv 0(\text{mod}2)$. Using formula (4.53), we obtain:

$$E_c(S^{m+1}_2) - E_c(S^m_2) = 2\sqrt[m+1]{2}\csc\frac{\pi}{m+1} + 2 - \sqrt[m]{2}\cot\frac{\pi}{2m} - 2.$$

Rest of the proof follows from the proof of Lemma 4.11.

The following corollary is an easy consequence of Lemmas 4.20 and 4.21.

Corollary 4.22. *Let $n \geq 7$, $p = 2$ and $4 \leq m_2 \leq m_1 \leq n - 2$. Take $S^{m_1}_2$, $S^{m_2}_2 \in J_n^{---}$. Then*

$$E_c(S^{m_1}_2) \geq E_c(S^{m_2}_2).$$

The following lemma is useful in proving our main result.

Lemma 4.23. *Let $n \geq 7$ and S^k_2, $S^2_k \in J_n^{---}$, where $4 \leq k \leq n - 2$. Then we have the following:*

$$E_c(S^2_k) > E_c(S^k_2).$$

Proof. By formula (4.53), we obtain:

$$E_c(S^2_k) = \begin{cases} 2\left(\sqrt[2]{2} + \csc\frac{\pi}{k}\right) & \text{if } k \equiv 0(\text{mod}2) \\ 2\sqrt[2]{2} + \cot\frac{\pi}{2k} & \text{if } k \equiv 1(\text{mod}2). \end{cases} \quad (4.56)$$

Also, formula (4.53) gives

$$E_c(S^k_2) = \begin{cases} 2\sqrt[k]{2}\csc\frac{\pi}{k} + 2 & \text{if } k \equiv 0(\text{mod}2) \\ \sqrt[k]{2}\cot\frac{\pi}{2k} + 2 & \text{if } k \equiv 1(\text{mod}2). \end{cases} \quad (4.57)$$

One can easily see from formulas (4.56) and (4.57) that the following inequality holds:

$$E_c(S^2_k) > E_c(S^k_2).$$

This gives the required result.

Using Corollary 4.19, one can easily see that the following inequality holds:

$$E_c(S^2_{n-2}) \geq E_c(S^2_p), \quad (4.58)$$

where $4 \leq p \leq n - 2$. From Corollary 4.22, the following inequality is easily seen:

$$E_c(S^{n-2}_2) \geq E_c(S^m_2), \quad (4.59)$$

where $4 \leq m \leq n - 2$. By (4.58), (4.59) and Lemma 4.23, we find

$$E_c(S^2_{n-2}) > E_c(S^{n-2}_2). \quad (4.60)$$

The following theorem gives the sidigraphs in J_n^{---} with minimal and maximal iota energy.

Theorem 4.24. *Let $n \geq 4$ and $S^m_p \in J_n^{---}$, where $2 \leq m \leq n - 2$ and $2 \leq p \leq n - 2$. Then S^m_p has minimal iota energy when $m = 3$ and $p = 2$ and maximal iota energy when $m = 2$ and $p = n - 2$ among all sidigraphs of J_n^{---}.*

Proof. It can easily be seen from formula (4.53) that:

$$E_c(S^m_p) = \begin{cases} 2\sqrt[2]{2} + 2 & \text{if } m = 2 \text{ and } p = 2 \\ 2\sqrt[2]{2} + \sqrt{3} & \text{if } m = 2 \text{ and } p = 3 \\ \sqrt{3}\sqrt[3]{2} + 2 & \text{if } m = 3 \text{ and } p = 2. \end{cases} \quad (4.61)$$

Let [$m = 2$ and $p \geq 4$] or [$p = 2$ and $m \geq 4$]. Also, let $S^2_{n-2} \in J_n^{---}$. Then from formula (4.53) and inequalities (4.58) – (4.60), we obtain:

$$2\sqrt[2]{2} + 2 < 2\sqrt{2}\sqrt[4]{2} + 2 \leq E_c(S^m_p) < E_c(S^2_{n-2}). \quad (4.62)$$

It is clear from (4.61) and (4.62) that $E_c(S^m_p)$ is minimal when $m = 3$ and $p = 2$. Furthermore, from (4.62) it is evident that $E_c(S^m_p)$ is maximal when $m = 2$ and $p = n - 2$.

We know that $\sin x < x < \tan x$ for each $x \in \left(0, \frac{\pi}{2}\right)$. This gives:

$$\csc x > \cot x, \quad (4.63)$$

for each $x \in \left(0, \frac{\pi}{2}\right)$. The next theorem follows from formulas (4.16), (4.34), (4.45), (4.49),(4.53), inequality (4.63), Theorems 4.1, 4.8, 4.15 – 4.17 and 4.24.

Theorem 4.25. *Let $n \geq 4$ and $S^m_p \in J_n$, where $2 \leq m \leq n - 2$ and $2 \leq p \leq n - 2$. Then:*

(1) S^m_p has minimal iota energy when:
(i) $S^m_p \in J_n^{+++}$ and $m = p = 2$, or
(ii) $S^m_p \in J_n^{+-+}$ and $p = 2$.

(2) S_p^m has maximal iota energy when m = 2 and p = n – 2 such that:
(i) $S_p^m \in J_n^{---}$ and p ≡ 0(mod2), or
(ii) [$S_p^m \in J_n^{--+}$ or $S_p^m \in J_n^{---}$] and p ≡ 1(mod2).

Example 4.26. The sidigraphs with minimal and maximal iota energy in J_9 are shown in Figure 3.

5.CONCLUSION

In this paper, we introduced a new class F_n of n-vertex tricyclic digraphs containing five linear subdigraphs such that one of the directed cycles does not share any vertex with the other two directed cycles and the remaining two directed cycles are of same length sharing at least one vertex. We found digraphs in F_n with smallest and largest iota energy. We have also introduced a similar class J_n of n-vertex tricyclic sidigraphs. We found sidigraphs in J_n with smallest and largest iota energy. It will be interesting to find digraphs (respectively, sidigraphs) with extremal iota energy among the general class of tricyclic digraphs (respectively, sidigraphs) of fixed order.

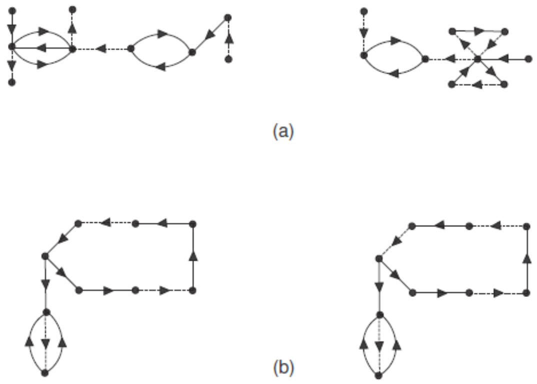


Figure 3: (a) The sidigraphs with minimal iota energy in J_9. (b) The sidigraphs with maximal iota energy in J_9.

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