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REVIEW ARTICLE

FINITE ELEMENT METHOD FOR SOLVING NONLINEAR RANDOM ORDINARY DIFFERENTIAL EQUATIONS

Ibrahim Elkott, Ibrahim.L. El-Kalla, Ahmed Elsaid*, Reda Abdo

Mathematics and Engineering Physics Department, Faculty of Engineering, Mansoura 1University, PO 35516, Mansoura, Egypt.
****Corresponding Author Email: a_elsaid@mans.edu.eg***

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ABSTRACT

In this paper we utilize the finite element method for solving random nonlinear differential equations. In the proposed technique, the nodal coefficients are formulated as functions of the random variable. At certain values of random variable, curve fitting is used to construct the approximate nodal solution. Several numerical examples are presented, and the approximate mean solutions are compared with the exact mean solution to illustrate the ability and effectiveness of this method.

KEYWORDS

Finite element method, Random ordinary differential equations, Nonlinear equations.

1. INTRODUCTION

A random differential equation is a differential equation which contains random variables in the equation or in the prescribed conditions. Many scientific problems and physical phenomena take the form of random differential equations [1-3]. Several numerical and semi-analytic techniques have been considered to approximate the solutions of random differential equations. Some of these methods include Adomian decomposition method, homotopy perturbation method, differential transformation method and variational iteration method [4-7]. In this paper, Galerkin finite element method is used to solve random nonlinear second-order ordinary differential equation (ODE) [8-11].

First, we use the finite element method to solve the considered problem at some selected values of the random variables. This yields a system of nonlinear algebraic equations that is solved using simple iteration method [12-16]. Then by using curve fitting over the selected values of the random variables, and utilizing the basis of the finite element method, we obtain the approximate solution as a function in both space and random variables. The rest of this article is organized as follows. In Section 2, the technique we propose to discretize the considered problems and the technique adopted to deal with nonlinear terms are illustrated. Section 3 is devoted to the numerical examples. The conclusion of this work is presented in Section 4.

2. FINITE ELEMENT APPROACH

The basic idea of using finite element method to solve a given problem is to subdivide the domain of the problem into smaller subdomains that are called finite elements. The simple equations that describe the considered problem over these finite elements are then assembled into one large system of equations that models the entire problem. Then variational methods are utilized to approximate a solution by minimizing an associated error function.

Consider the following second order nonlinear random ordinary differential equation

$$\frac{}{dx^2} + g\left(\beta, u, \frac{du}{dx}\right) = f(\beta, x), \quad a < x < b,\ \beta \in \left(\beta_i, \beta_f\right), \tag{1}$$

with mixed conditions given by

$$u(a) + b_1 \frac{du(a)}{dx} = h_1(\beta), \tag{2a}$$

$$u(b) + b_2 \frac{du(b)}{dx} = h_2(\beta), \tag{2b}$$

where β is a second order random variable, h_1 and h_2 are two function of the random variable β, $f(\beta, x)$ is a given function of β and x, and $g(\beta, u, \frac{du}{dx})$ is a nonlinear function that takes the polynomial form as

$$g(\beta, u, \frac{du}{dx}) = \sum_{n=0}^{N} \sum_{m=0}^{M} c_{n,m}(\beta) u^n (\frac{du}{dx})^m. \tag{2c}$$

The Galerkin finite element method is used to solve Eq (1). This differential equation is written in an integral form as follows

$$\int_\Omega w(x) [\frac{d^2u}{dx^2} + g(\beta, u, \frac{du}{dx}) - f(\beta, x)] dx = 0, \tag{3}$$

where w is the weight function and Ω is the problem domain. By using integration by parts for Eq (3) we have

$$\begin{aligned} -\int_\Omega \frac{dw}{dx}\frac{du}{dx} dx + \int_\Omega w(x) g\left(\beta, u, \frac{du}{dx}\right) dx \\ = \int_\Omega w(x) f(\beta, x) dx \\ - \int_\Gamma w(x) \frac{du}{dx} \eta_x d\Gamma, \end{aligned} \tag{4}$$

where η_x is the component of the unit outward normal of the boundary and Γ is the domain boundary. Eq (4) is called the weak form of Eq (1). By dividing the domain Ω into N^e subdomains (or elements), Eq (4) can be written as

$$-\sum_{e=1}^{N^e}\int_{\Omega^e}\frac{dw}{dx}\frac{du}{dx}dx+\sum_{e=1}^{N^e}\int_{\Omega^e}w\,(x)g\left(\beta,u,\frac{du}{dx}\right)dx = \sum_{e=1}^{N^e}\int_{\Omega^e}w\,(x)f(\beta,x)dx \quad -\sum_{e=1}^{N^e}\int_{\Gamma^e}w\,(x)\frac{du}{dx}\eta_x d\Gamma. \tag{5}$$

We propose that the function u at any point within the element be approximated by

$$u(x;\beta)=\sum_{j=1}^{N_h}u_j\,(\beta)s_j(x), \tag{6}$$

where N_h is the number of nodes in the finite element mesh, $u_j(\beta)$ are the nodal unknown value of u and $s_j(x)$ are the prescribed finite element shape functions

$$s_j(x_i)=\begin{cases}1 & i=j,\\ 0 & i\neq j.\end{cases} \tag{7}$$

According to Galerkin method, we set

$$w(x)=s_j(x). \tag{8}$$

Eq (5) results in the following equation for each element

$$-\sum_{e=1}^{N^e}\int_{\Omega^e}\frac{ds_i(x)}{dx}\frac{d}{dx}\left(\sum_{j=1}^{N_h}u_j\,(\beta)s_j(x)\right)dx + \sum_{e=1}^{N^e}\int_{\Omega^e}s_i\,(x)g\left(\beta,u,\frac{du}{dx}\right)dx = \sum_{e=1}^{N^e}\int_{\Omega^e}s_i\,(x)f(\beta,x)dx-\sum_{e=1}^{N^e}\int_{\Gamma^e}s_i\,(x)\frac{du}{dx}\eta_x d\Gamma. \tag{9}$$

For the random nonlinear term in Eq (9), we approximate it by

$$g(\beta,u,\frac{du}{dx})=c_{0,0}(\beta)+\sum_{n=0}^{N}c_{n,0}\,(\beta)(\overline{u})^{n-1}u + \sum_{m=0}^{M}c_{0,m}\,(\beta)(\frac{d\overline{u}}{dx})^{m-1}\frac{du}{dx} + \sum_{n=1}^{N}\sum_{m=1}^{M}c_{n,m}\,(\beta)(\overline{u})^{n}(\frac{d\overline{u}}{dx})^{m-1}\frac{du}{dx}, \tag{10a}$$

where $\overline{u_j}(\beta)$ is the initial guess of the nodal unknown value approximated by

$$\overline{u}=\sum_{j=1}^{N_h}\overline{u_j}\,(\beta)s_j(x). \tag{10b}$$

In this paper we propose discretizing the random variable. Then at every value β_o of the random variable, Eq (9) is reduced to system of nonlinear algebraic equation of the unknowns $u_j(\beta_o)$. By solving this system and applying curve fitting, we obtain the unknown nodal values as functions of the random variable β.

3. NUMERICAL EXAMPLES

In this section, we apply the proposed technique to several random nonlinear ODEs with different types of nonlinearities.

Example 1

$$\frac{d^2u}{dx^2}+\beta u^2\frac{du}{dx}-\beta\frac{du}{dx}+u=f(x,\beta),\; 0<x<0.1, \tag{11a}$$

where $f(x,\beta)$ is chosen such that the exact solution is given by $u=\beta(x-x^2)$ with boundary condition

$$u(0)=0, \qquad \frac{du}{dx}(0.1)=0.8\beta, \tag{11b}$$

where β is a second order uniform random variable with a range and expectation $E(\beta)$=1.5 [1,2]. Figure 1 illustrates finite element and exact solutions of nonlinear differential equation (11a) and (11b) at selected values of random variable. Figure 2 shows the maximum difference between the finite element and exact solutions at certain values of the random variable β. Table 1 presents the finite element solution at every node as a function of the random variable. Table 2 illustrates the finite element solution at every element as a function of the random variable β and independent variable x. Figure 3 shows the expectation of finite element and exact solutions.

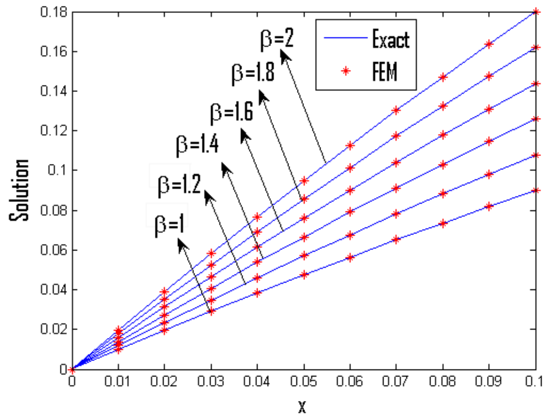

Figure 1: Finite element and exact solution of Ex (1).

Table 1: The nodal value of u at some values of x as a function of β by curve fitting for Ex (1).

$u(0)$	$u(0.01)$	$u(0.02)$
0	0.009 β+ 0.00004	0.019β+0.00004
u(0.03)	$u(0.04)$	$u(0.05)$
0.029 β+ 0.00004	0.038 β+ 0.00004	0.047β
$u(0.06)$	$u(0.07)$	$u(0.08)$
0.056 β+0.00004	0.065 β −0.00004	0.073β −0.00004
$u(0.09)$	$u(0.1)$	
0.081 β+0.00004	0.09 β	

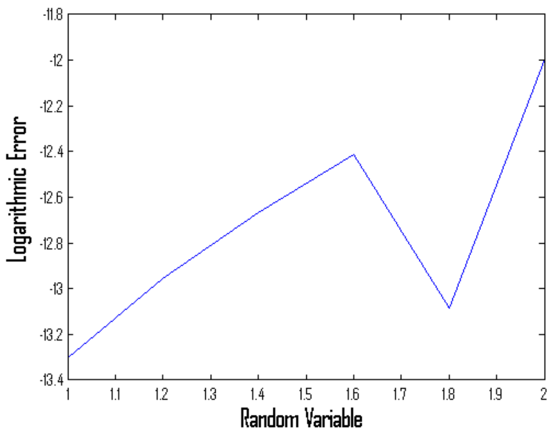

Figure 2: Maximum error at selected value of random variable of Ex (1).

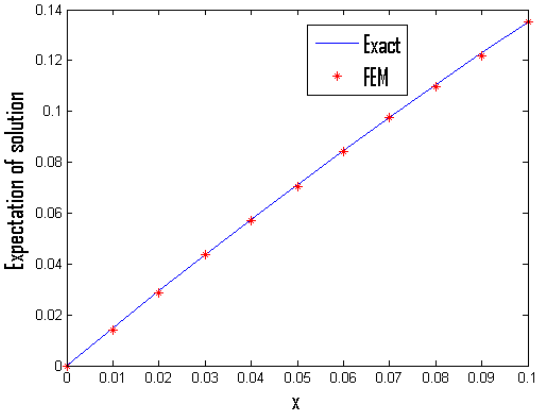

Figure 3: Expectation of finite element and exact solutions of Ex (1).

Table 2: Solution $u(x,\beta)$ at each element.

For $0 \le x \le 0.01$	u = 0.9 β x + 0.09765625 x
for $0.01 \le x \le 0.02$	u = β x – 0.001 β – 0.1953125 x + 0.0029296
for $0.02 \le x \le 0.03$	u = β x – 0.001 β – 0.0009765
for $0.03 \le x \le 0.04$	u = 0.9 β x + 0.002 β + 0.1953125 x – 0.0068359
for $0.04 \le x \le 0.05$	u = 0.9 β x + 0.002 β – 0.09765625 x + 0.0048828125
for $0.05 \le x \le 0.06$	u = 0.9 β x – 0.28 β – 0.09765625 x + 0.2868828
for $0.06 \le x \le 0.07$	u = 0.9 β x + 0.002 β – 0.09765625 x + 0.126953
for $0.07 \le x \le 0.08$	u = 0.8 β x + 0.009 β – 0.0009765625
for $0.08 \le x \le 0.09$	u = 0.8 β x + 0.009 + 0.1953125 x – 0.01660156
for $0.09 \le x \le 0.1$	u = 0.9 β x – 0.09765625x + 0.00965625

Example 2

$$\frac{d^2u}{dx^2} + u^3 + u = f(x,\beta), \qquad 0 < x < 1, \qquad (12a)$$

where $f(x,\beta)$ is chosen such that the exact solution is $u = \beta x^2$ with boundary conditions,

$$u(0) = 0, \qquad u(1) = \beta, \qquad (12b)$$

where β is a second order random variable with a range [0.2,1.2] and expectation $E(\beta)$=0.7.

Figure 4 illustrates finite element and exact solutions of nonlinear differential equation (12a) and (12b) at selected values of the random variable. Figure 5 shows the maximum difference between the finite element and exact solutions at certain values of random variable. Table 3 presents the finite element solution at every node as a function of the random variable. Table 4 illustrates the finite element solution at every element as a function of the random variable and independent variable. Figure 6 shows the expectation of finite element and exact solutions.

Table 3: The nodal value of u at some values of x as a function of β by curve fitting for Ex (2).

u (0)	u (0.1)	u (0.2)
0	0.01 β – 0.00001	0.04 β – 0.00005
u (0.3)	u (0.4)	u (0.5)
0.09 β – 0.00009	0.16 β – 0.00006	0.25 β – 0.00009
u (0.6)	u (0.7)	u (0.8)
0.36 β – 0.00009	0.49 β	0.64 β – 0.00009
u (0.9)	u (1)	
0.81 β – 0.00006	β	

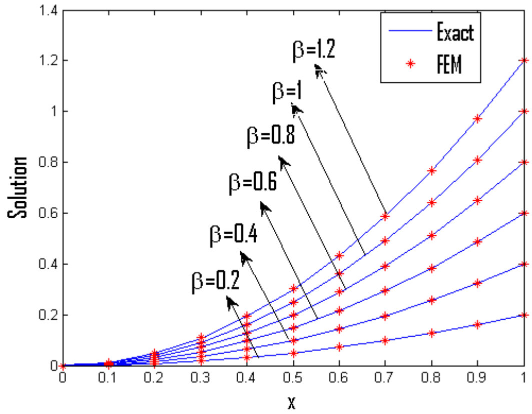

Figure 4: Finite element and exact solutions of Ex (2).

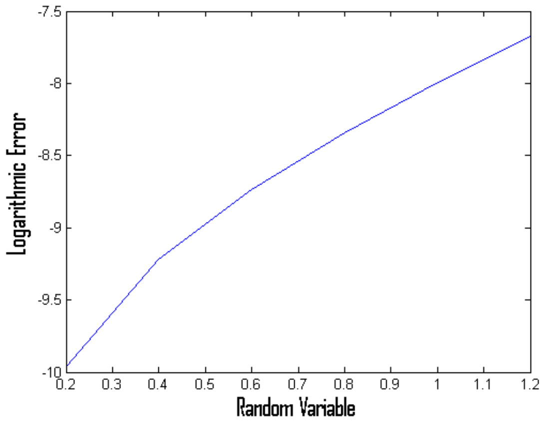

Figure 5: Maximum error at selected value of random variable of Ex (2).

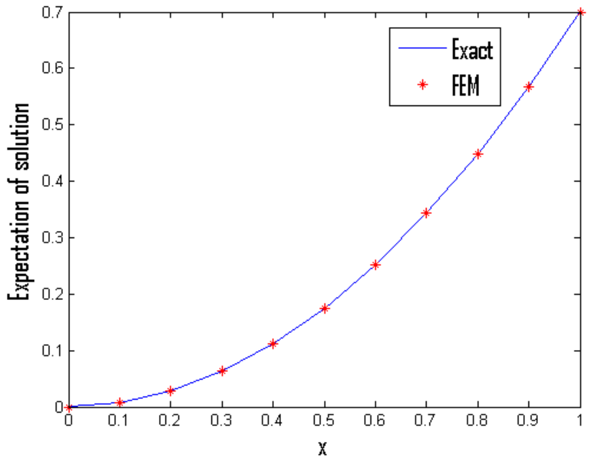

Figure 6: Expectation of finite element and exact solutions of Ex (2).

Table 4: Solution u(x,β) at each element.

for $0 \le x \le 0.1$	u = 0.01βx
for $0.1 \le x \le 0.2$	u = 0.3 βx – 0.02β
for $0.2 \le x \le 0.3$	u = 0.5 βx – 0.06β
for $0.3 \le x \le 0.4$	u = 0.7 βx – 0.12β
for $0.4 \le x \le 0.5$	u = 0.2 βx – 0.2β
for $0.5 \le x \le 0.6$	u = 1.1βx – 0.3β
for $0.6 \le x \le 0.7$	u = 1.3 βx – 0.42β
for $0.7 \le x \le 0.8$	u = 1.5 βx – 0.56β
for $0.8 \le x \le 0.9$	u = 1.7 βx – 0.72β
for $0.9 \le x \le 1$	u = 1.9 βx – 0.9 β

Example 3

$$\frac{d^2u}{dx^2} + u\frac{du}{dx} - u^2 = f(x,\beta), \qquad 0 < x < 1, \qquad (13a)$$

where $f(x,\beta)$ is chosen such that the exact solution is $u = e^{\beta x}(x-1)$ with boundary conditions

$$u(0) = -1, \qquad u(1) = 0, \qquad (13b)$$

where β is a second order random variable with a range [0,1] and expectation $E(\beta)$=0.5 and $E(\beta^2)$=1/3.

Figure 7 illustrates finite element and exact solutions of nonlinear differential equation (13a) and (13b) at selected values of the random variable. Figure 8 shows the maximum difference between the finite element and exact solutions at certain values of the random variable. Table 5 presents the finite element solution at every node as a function of the random variable. Table 6 illustrates the finite element solution at every element as a function of the random variable and independent variable. Figure 9 shows the expectation of finite element and exact solutions.

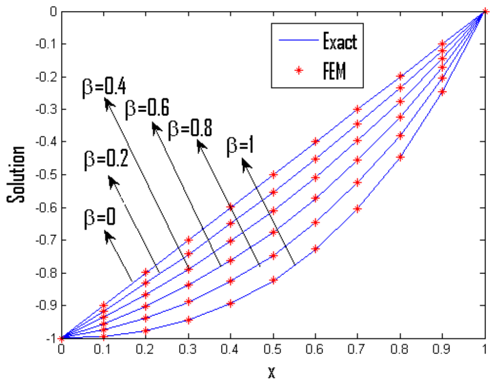

Figure 7: Finite element and exact solutions of Ex (3).

Table 5: The nodal value of u at some values of x as a function of β by curve fitting for Ex (3).

$u(0)$	$u(0.1)$	$u(0.2)$
-1	$-0.094\beta - 0.899$	$-0.176\beta - 0.797$
$u(0.3)$	$u(0.4)$	$u(0.5)$
$-0.244\beta - 0.695$	$-0.058\beta^2 - 0.236\beta - 0.6$	$-0.08\beta^2 - 0.243\beta - 0.5$
$u(0.6)$	$u(0.7)$	$u(0.8)$
$-0.097\beta^2 - 0.23\beta - 0.4$	$-0.105\beta^2 - 0.197\beta - 0.3$	$-0.096\beta^2 - 0.147\beta - 0.2$
$u(0.9)$	$u(1)$	
$-0.064\beta^2 - 0.08\beta - 0.1$	0	

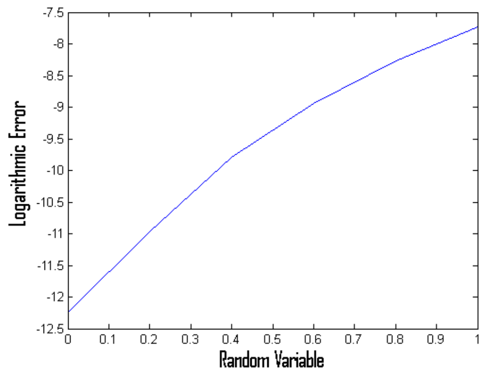

Figure 8: Maximum error at selected value of random variable of Ex (3).

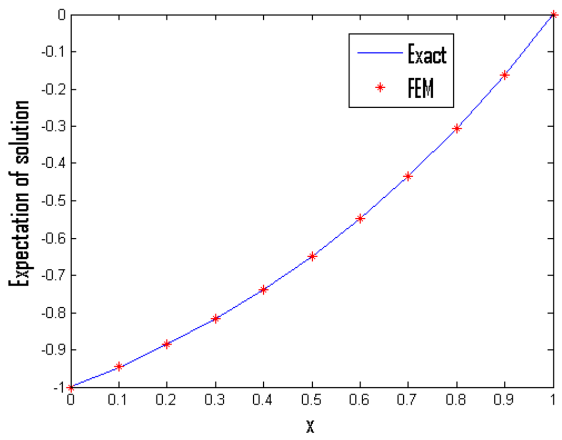

Figure 9: Expectation of finite element and exact solutions of Ex (3).

Table 6: Solution u(x,β) at each element.

for $0 \le x \le 0.1$	u = $-0.94\,\beta x + 1.01x - 1$
for $0.1 \le x \le 0.2$	u = $-0.82\,\beta x - 0.012\,\beta + 1.02\,x - 1.001$
for $0.2 \le x \le 0.3$	u = $-0.68\,\beta x - 0.04\,\beta + 1.02\,x - 1.001$
for $0.3 \le x \le 0.4$	u = $-0.58\,\beta^2 x + 0.08\,\beta x + 0.95\,x + 0.174\,\beta^2 - 0.268\,\beta - 0.98$
for $0.4 \le x \le 0.5$	u = $-0.22\,\beta^2 x - 0.07\,\beta x + x + 0.03\,\beta^2 - 0.208\,\beta - 1$
for $0.5 \le x \le 0.6$	u = $-0.17\,\beta^2 x + 0.13\,\beta x + x + 0.005\,\beta^2 - 0.308\,\beta - 1$
for $0.6 \le x \le 0.7$	u = $-0.08\,\beta^2 x + 0.33\,\beta x + x - 0.049\,\beta^2 - 0.428\,\beta - 1$
for $0.7 \le x \le 0.8$	u = $0.09\,\beta^2 x + 0.5\,\beta x + x - 0.168\,\beta^2 - 0.547\,\beta - 1$
for $0.8 \le x \le 0.9$	u = $0.32\,\beta^2 x + 0.67\,\beta x + x - 0.224\,\beta^2 - 0.683\,\beta - 1$
for $0.9 \le x \le 1$	u = $0.64\,\beta^2 x + 0.8\,\beta x + x - 0.64\,\beta^2 - 0.8\,\beta - 1$

4. CONCLUSION

In this paper we propose a finite element technique for solving nonlinear random ODEs. In the finite element approximation, the nodal coefficients are proposed as functions in the random variable. Then the nonlinear term is approximated by using a mix between nodal values and their initial guess. For some selected values of the random variable, the system of nonlinear algebraic equations is solved for the nodal coefficients. Curve fitting is applied to formulate these nodal coefficients as functions of the random variable. The results obtained in the numerical experiments illustrate the efficiency of the proposed scheme as for each example, the approximate solution and the corresponding expectation agree with those of the exact solution.

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