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REVIEW ARTICLE

THE EFFECT OF NON-LOCALITY IN BOUNDARY CONDITIONS ON THE EIGENVALUES OF SOME FINITE DIFFERENCE SCHEMES

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ABSTRACT

In this paper, we analyze a new form of non-local boundary conditions for a two-dimensional elliptic partial differential equation model. Some relations for the existence of the different types of eigenvalues and their corresponding eigenfunctions are proved. The figures of the relations are also dragged to show the effect of the non-locality in boundary conditions on the eigenvalue problem.

KEYWORDS

Elliptic partial differential equation; Non-local boundary condition; Eigenvalues and eigenvectors problem; Finite difference method.

1. INTRODUCTION

A great attention has been paid to problems of differential equations with different types of boundary conditions during the last two decades. One of the pioneers who investigated the parabolic problems with integral boundary condition nowadays this condition is named non-local boundary condition (NBC) (Cannon, 1963). After that, a boundary value problem (BVP) for an elliptic equation with NBC was investigated and formulated by Samarskii and Bitsadze (Bitsadze and Sawarskii, 1969). A condition is classified as non-local if it associates the values of the unknown function and/or its derivatives at two or more different points of the problem domain. These points may be located on different boundaries of the domain or they may be some interior points inside the problem domain. Recently, non-local BVPs of different types of partial differential equations (PDE) are widely used in several fields of applications in mathematical models of various physical, chemical, or biological processes.

Therefore, a great interest in developing computational techniques for the numerical solutions of PDEs with various types of NBCs. So, we may see the non-local condition in the form of multi-point which illustrated or integral condition as shown in (Elsaid et al., 2015; El-Sayed et al., 2014; Jesevioiute and Sapagovas, 2008). Models with NBCs include elliptic equations parabolic equation, and hyperbolic equations (Sapagovas and Stikoniene, 2011; El-Mowafy et al., 2020; Wang, 2002; Gulin and Morozova, 2009; Gulin et al., 2001; Ciegis et al., 2002; Ashyrayev and Aggez, 2004; Ashyralyev and Yurtsever, 2001; Ashyralyev and Ozdemir, 2005). One of the most important problems is to find eigenvalues of the difference schemes constructed to solve.

differential equations with NBCs (Stikonas, 2014). We construct and analyze finite difference method for one- and two-dimensional elliptic equation with NBCs. Using the techniques and arguments which were illustrated before to be able to analyze similar problems, but with different non-local conditions (Sajavicius, 2010; Sapagovas, 2008; Sapagovas and Tikonas, 2005; Sapagovas, 2008; Sapagovas, 2002; Doschorts, 2012).

Let us analyze the elliptic PDE

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y), 0<x<1, 0<y<1 \tag{1}$$

with two classical BCs

$$u(x,0)=\mu_1(x), \quad 0<x<1\,, \tag{2}$$

$$u(x,1)=\mu_2(x), \quad 0<x<1, \tag{3}$$

and with another two-point NBCs

$$u(0,y)=0, \qquad 0<y<1\,, \tag{4}$$

$$u(1,y)=\gamma\frac{\partial u}{\partial x}(\xi,y), \quad 0<y<1, \tag{5}$$

where ξ is a point that lies between $x=0$ and $x=1$, $\gamma\in R$ and μ_1 , μ_2 are given functions.

To study the standard finite- difference schemes for the elliptic PDEs (1)-(5), we begin by constructing the two-dimensional discrete grid $\Omega_{h\times k}$on the domain $(0,1)\times(0,1)$, which is defined by

$$\Omega_h = \{x_i : x_i = ih, i = 0,1,\ldots,M\},$$

$$\Omega_k = \{y_j : y_j = jk, j = 0,1,\ldots,N\},$$

where M and N are positive numbers that limit the grid dimensions. This grid $\Omega_{h\times k}$ is defined to merge conditions (2-5) into the difference equations system, we take the grid step sizes h and k which are defined by

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$h = \frac{1}{M}$ and $k = \frac{1}{N}$, so that ξ is a point on the grid $\Omega_{h\times k}$ i.e. $\xi = sh$, for positive integers, $s < N$.

At the beginning, we deal with the following non-local difference eigenvalue problem in one- dimensional

$$\frac{U_{i+1}-2U_i+U_{i-1}}{h^2} + \lambda U_i = 0, \qquad (6)$$

$$U_0 = 0, \qquad (7)$$

$$U_M = \frac{\gamma}{h}(U_s - U_{s-1}). \qquad (8)$$

Then, we use the obtained results from this one-dimensional problem to investigate the two-dimensional difference eigenvalue problem:

$$\frac{U_{i+1}^j - 2U_i^j + U_{i-1}^j}{h^2} + \frac{U_i^{j+1} - 2U_i^j + U_i^{j-1}}{k^2} + \lambda U_i^j = 0, \qquad (9)$$

$$U_i^0 = 0, \qquad (10)$$

$$U_i^M = 0, \qquad (11)$$

$$U_0^j = 0, \qquad (12)$$

$$U_M^j = \frac{\gamma}{h}\left(U_s^j - U_{s-1}^j\right). \qquad (13)$$

The value of λ that yields a nontrivial solution for problem $(6-8)$ or problem $(9-13)$ is called a difference eigenvalue of the corresponding problem, and the spectrum of the problem is the group of all eigenvalues. The objective of this work is to analyze the effect of the suggested NBCs on the existence of distinct types of eigenvalues.

2. The Eigenvalues and Eigenvectors of The one-Dimensional Problem

Here, we deal with the uniform discrete grid defined on the interval (0,1), $M = N$. So, equations (6-8) produces $(M-1)\times(M-1)$ linear system of equations. Therefore, matrix A of order $(M-1)$ is defined as

$$A = \frac{1}{h^2}(C), \qquad (14)$$

Where
$$C = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\gamma}{h} & -\frac{\gamma}{h} & \cdots & -1 & 2 \end{pmatrix},$$

Where $\frac{\gamma}{h}$ and $\frac{\gamma}{h}$ occupy the last row of column $M-s-1$ and $M-s$, respectively.

So, the finite-difference eigenvalue problem (6-8) is equivalent to the matrix eigenvalue problem

$$AU = \lambda U. \qquad (15)$$

Because of the non-locality of the boundary conditions, matrix A is non-symmetric. Thus, it may have zero, positive, negative, or complex eigenvalues. Equation (6) can be written as

$$U_{i+1} + 2\left(1 - \frac{\lambda}{h^2}\right)U_i + U_{i-1} = 0, \qquad (16)$$

this form is helpful in generating the subsequent arguments.

Lemma 2.1. The problem (6-8) has zero eigenvalue $\lambda = 0$, provided that it occurs, with a corresponding eigenvector $U_i = c(ih)$, where c is an arbitrary constant, if the subsequent condition is fulfilled
$\gamma = 1$.

Proof. In the case of zero eigenvalue, the problem $(6-8)$ has a general solution

$$U_i = c_1 + c_2 ih.$$

By substituting the condition (7) into the above equation, we find that $U_0 = 0$, which leads to $c_1 = 0$, then the corresponding eigenvector takes the form

$$U_i = c(ih),$$

where c is an arbitrary constant, using condition (8), we get
$c(Nh) = \frac{\gamma}{h}c(h)$,
for which the following condition $\gamma = 1$, is satisfied.

Lemma 2.2. The difference problem $(6-8)$ has a unique negative eigenvalue $\lambda < 0$, provided that it occurs, if the subsequent condition is fulfilled

$$\gamma = \frac{h\sinh(\alpha)}{\sinh(\xi\alpha) - \sinh(\xi\alpha - h\alpha)}, \qquad (17)$$

where the eigenvalue and the corresponding difference eigenvector are given by $\lambda = \frac{-4}{h^2}\sinh^2\left(\frac{\alpha h}{2}\right)$ and $U_i = c\sinh(\alpha ih)$,respectively, where c is an arbitrary constant.

proof. If $\lambda < 0$, we have

$$1 - \frac{\lambda h^2}{2} > 1.$$

Indicate

$$1 - \frac{\lambda h^2}{2} = \cosh(\alpha h).$$

The equation (16) can be written as

$$U_{i+1} - 2\cosh(\alpha h)U_i + U_{i-1} = 0.$$

Then the general solution can be in the form

$$U_i = c_1\cosh(i\alpha h) + c_2\sinh(i\alpha h).$$

By substituting the condition (7) into the above equation, we find that $U_0 = 0$, which leads to $c_1 = 0$, then the corresponding eigenvector be in the form $U_i = c\,\sinh(i\alpha\, h)$,

using condition (8) we get

$$h\sinh(N\alpha h) - \gamma(\sinh(s\alpha h) - \sinh(s\alpha h - \alpha h)) = 0.$$

The qualitative behavior of negative eigenvalues of the problem $(6-8)$ with different values of γ is shown in Figure 1.

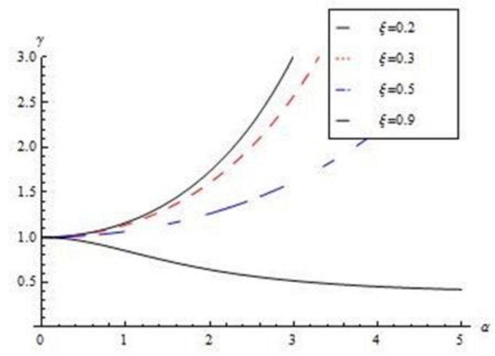

Figure 1: The variations in ξ ,γ and α according to (17).

Figure 1 confirms that for each combination of γ and ξ there exists a unique value of α, hence a unique negative eigenvalue.

Lemma 2. 3. The positive eigenvalues which are defined in the range $0 < \lambda < \frac{4}{h^2}$ for the difference eigenvalue problem $(6-8)$ provided that they occur, if the subsequent equation is proved

$$\gamma = \frac{\sin(\alpha)h}{\sin(\xi\alpha) - \sin(\xi h - \alpha h)} \tag{18}$$

where the eigenvalue and the corresponding difference eigenvector are given by $\lambda = \frac{4}{h^2}\sin^2\left(\frac{\alpha h}{2}\right)$ and $U_i = c\sin(\alpha i h)$, respectively, where c is an arbitrary constant.

Proof. For the eigenvalues located in the interval $0 < \lambda < \frac{4}{h^2}$ we find

$$\left|1 - \frac{\lambda h^2}{2}\right| < 1.$$

Indicate

$$1 - \frac{\lambda h^2}{2} = \cos(h\alpha),$$

so that, the equation (16) can be rewritten as

$$U_{i+1} - 2\cos(h\alpha)U_i + U_{i-1} = 0.$$

Then the solution of the previous equation is

$$U_i = c_1\cos(\alpha h i) + c_2\sin(\alpha h i),$$

where c_1 and c_2 are arbitrary constants.

By substituting the condition (7) into the above equation, we find that $U_0 = 0$, which leads to $c_1 = 0$, then the corresponding eigenvector takes the form

$$U_i = c\,sin(i\,\alpha\,h),$$

using condition (8), we get

$$h\,sin(N\alpha\,h) - \gamma\,(sin(s\alpha\,h) - sin(s\,\alpha\,h - \alpha h)) = 0.$$

The qualitative behavior of eigenvalues which are defined in the range $0 < \lambda < \frac{4}{h^2}$ of the problem (6-8) with different values of γ is shown in Figure 2.

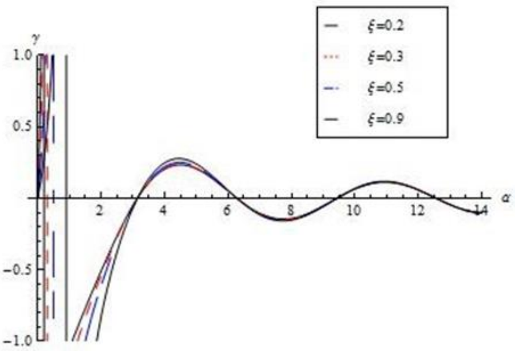

Figure 2. The Variations In ξ, γ And α According To (18).

Figure **2** illustrates that there exists an infinite number of values for ξ that satisfy (18). Due to the periodicity of the sine function, these values of ξ yields many positive eigenvalues.

***Lemma* 2.4**. For the problem (6-8), in the case of $\lambda = \frac{4}{h^2}$ occurs if the following equation is satisfied

$$\gamma = (-1)^{N-s}\frac{1}{2s-1}, \tag{19}$$

and the corresponding eigenvector is given by $U_i = c(-1)^i(ih)$.

Proof. If $\lambda = \frac{4}{h^2}$, in this case the finite-difference equation (16) can be written in the form

$$U_{i-1} + 2U_i + U_{i+1} = 0.$$

Then the previous equation can be written in the form

$$U_i = (-1)^i(c_1 + c_2(ih)).$$

By substituting condition (7) into the above equation, we find that $U_0 = 0$, which yields $c_1 = 0$, then the corresponding eigenvector takes the form

$$U_i = c(-1)^i(ih).$$

By using condition (8), we get

$$(-1)^N(Nh^2) - \gamma((-1)^s(sh) - (-1)^{s-1}(s-1)h) = 0.$$

***Lemma* 2.5.** For real positive eigenvalues which are in the range $\lambda > \frac{4}{h^2}$ for the difference eigenvalue problem (6-8), provided that they exist, if and only if the following condition is fulfilled

$$\gamma = \frac{(-1)^{N-S}\sinh(\alpha)h}{\sinh(sh\alpha) + \sinh((s-1))h\alpha)}, \tag{20}$$

where the eigenvalue and the corresponding difference eigenvector are defined as $\lambda = \frac{4}{h^2}\cosh^2\left(\frac{\alpha h}{2}\right)$, and $U_i = (-1)^i(c\sinh(\alpha i h))$, where c is an arbitrary constant.

Proof. If $\lambda > \frac{4}{h^2}$, indicate

$$\cosh(\alpha h) = -\left(1 - \frac{\lambda h^2}{2}\right),$$

so, the equation (16) can be written as

$$U_{i+1} + 2\cosh(\alpha h)\,U_i + U_{i-1} = 0.$$

The general solution of the previous equation is in the form

$$U_i = (-1)^i(c_1\cosh(i\alpha h) + c_2\sinh(\alpha h i)).$$

By substituting the condition (7) into the above equation, we find that $U_0 = 0$, which leads to $c_1 = 0$, then the corresponding eigenvector takes the form

$$U_i = c(-1)^i\sinh(i\alpha h),$$

using condition (8), we get

$$h(-1)^N\sinh(N\alpha h) - \gamma(-1)^s(\sinh(s\alpha h) - \sinh((s-1)\alpha h)) = 0.$$

Lemma 2.6. The complex eigenvalues $\lambda \in \mathbb{C}$ for the problem (6-8), occur if the following statement is fulfilled

$$\gamma = \frac{h\sinh(q)}{\sinh(shq) - \sinh((s-1)qh)}, \tag{21}$$

where $q = \phi + \iota\psi, \iota^2 = -1$, and $\phi \neq 0, \psi \neq 0$. The corresponding eigenvector is $U_i = 2c\sinh(iqh)$, where c is an arbitrary constant.

Proof. By considering that $\phi \neq 0, \psi \neq 0$, indicate that $1 - \frac{\lambda h^2}{2} = \cosh(iqh)$, as

$\iota^2 = -1$. So, the finite-difference equation (16) can be rewritten as

$$U_{i+1} - 2\cosh(qh)U_i + U_{i-1} = 0.$$

The general solution of the previous equation is

$$U_i = c_1e^{qhi} + c_2e^{-qhi}.$$

By substituting the condition (7) into the above equation, we find that $U_0 = 0$, which leads to $c_1 = -c_2$, then the corresponding eigenvector takes the form

$$U_i = c(e^{iqh} - e^{-iqh}) = 2c\sinh(iqh),$$

where c is an arbitrary constant. Using condition (8), we get

$$h(e^{Nqh} - e^{-Nqh}) - \gamma\left(e^{sqh} - e^{-sqh} - \left(e^{(s-1)qh} - e^{-(s-1)qh}\right)\right) = 0.$$

If $\phi = \psi = 0$, then this case is reduced to lemma (2.1), else $\phi \neq 0, \psi =$

0, or
$\phi = 0, \psi \neq 0$, this case matches with lemma (2.2) or lemma (2.3), respectively.

3. The Two-Dimensional Problem

We consider the two-dimensional difference eigenvalue problem (9-13). By using separation of variable method as illustrated in [3] and [21]. The solution of the problem (9-13) can be rewritten in the form.

$U_{ij} = G_i Q_j, i, j = 0,1,2,3, \cdots, M.$

After applying separation of variable method, the two one-dimensional difference eigenvalue problems are obtained as

$$\frac{G_{i+1} - 2G_i + G_{i-1}}{h^2} + \eta_i G_i = 0, G_0 = 0, G_M = \frac{\gamma}{h}(G_s - G_{s-1}), \qquad (22)$$

and

$$\frac{Q_{j+1} - 2Q_j + Q_{j-1}}{k^2} + \zeta_j Q_j = 0, \qquad Q_0 = Q_1 = 0. \qquad (23)$$

We note that problem (22) is exactly like the investigated problem in section 2, but problem (23) is familiar as a classic problem with homogeneous boundary conditions. The eigenvalues of the problem (23) are positive real and can be algebraically obtained by the formula [20]

$$\zeta_r = \frac{4}{h^2}\sin^2\left(\frac{h\pi r}{2}\right), r = 1,2,3 \cdots, M-1,$$

the corresponding eigenvectors are

$$Q_r = \sin\left(\frac{h\pi r}{2}\right), r = 1,2,3 \cdots, M-1,$$

we consider that the eigenvalues of problem (9-13) take the form

$\lambda = \lambda_{k,r} = \eta_k + \zeta_r.$

Hence, the value of $\lambda_{k,r}$ is completely dependent on the value of η_k. Therefore, the subsequent arguments are valid.

Corollary 3.1. The positive eigenvalues defined in the range $0 < \lambda_{k,r} < \frac{8}{h^2}$ of the problem (9-13), can be evaluated by the form

$$\lambda_{k,r} = \frac{4}{h^2}\left(\sin^2\left(\frac{\alpha_k \pi h}{2}\right) + \sin^2\left(\frac{\pi h r}{2}\right)\right), k = 1,2,3, \ldots\ldots, M-1$$

The corresponding eigenvectors of the previous eigenvalues can be obtained as

$$(U_{k,r})_{i,j} = c\sin(\alpha_k i h)\sin\left(\frac{h\pi r j}{2}\right), k = 1,2,3, \ldots\ldots, M-1,$$

where α_k are the roots of equation (18).

Corollary 3.2. For $\eta_k = \frac{4}{h^2}$, then the eigenvalues of problem (9-13) can be evaluated as

$$\lambda_r = \frac{4}{h^2}\left(1 + \sin^2\left(\frac{h\pi r}{2}\right)\right), r = 1,2,3, \ldots\ldots, M-1,$$

these occur if and only if the following condition is satisfied

$$\gamma = (-1)^{N-s}\frac{1}{2s-1}.$$

The corresponding eigenvectors in this case can be obtained as

$$(U_r)_{i,j} = c(-1)^i (ih)\sin\left(\frac{h\pi r j}{2}\right).$$

Corollary 3.3. In the case of $\eta_k > \frac{4}{h^2}$, then the eigenvalues of the problem (9-13) can be written in the form

$$\lambda_{k,r} = \frac{4}{h^2}\left(\cosh^2\left(\frac{\alpha_k h}{2}\right) + \sin^2\left(\frac{\pi h r}{2}\right)\right), r = 1,2,3, \ldots\ldots, M-1,$$

and this occur if and only if the following statement is achieved

$$\gamma = \frac{(-1)^{N-s}\sinh(\alpha)h}{\sinh(sh\alpha) + \sinh((s-1))h\alpha)}.$$

The corresponding eigenvectors in this case can be obtained as

$$(U_{k,r})_{i,j} = c(-1)^j \sinh(i\alpha_k h)\sin\left(\frac{h\pi r j}{2}\right).$$

Let us investigate negative and zero eigenvalues of problem (9-13). As illustrated before, there is a unique negative eigenvalue of problem (22) which was written in the form

$$\eta_{-1} = -\frac{4}{h^2}\sinh^2\left(\frac{\alpha_k h}{2}\right),$$

where α_k are the positive roots of equation (17). Then, since the numbers

$$\alpha^*_r = \frac{4}{h^2}\log\left(\sin\left(\frac{h\pi r}{2}\right) + \sqrt{\sin^2\left(\frac{\pi h r}{2}\right) + 1}\right), r = 1,2,3, \ldots\ldots, M-1,$$

are the positive roots of the equation (24) which yields

$$\sinh^2\left(\frac{\alpha h}{2}\right) = \sin^2\left(\frac{\pi h r}{2}\right), r = 1,2,3, \ldots\ldots, M-1,$$

the following statement is valid.

Corollary 3.4. The zero eigenvalue $\lambda_{k,r} = 0$, for problem (9-13) can be obtained in the condition of $\gamma = \frac{h\sinh(\alpha)}{\sinh(\xi\alpha) - \sinh(\xi\alpha - h\alpha)}$, and its eigenfunction takes the form $(U_{k,r})_{i,j} = c\sinh(i\alpha_k h)\sin\left(\frac{h\pi r j}{2}\right)$, where c is an arbitrary constant.

On the other hand, if the previous condition is achieved with α^*_n, where n is a positive number located in the range $(1 \leq n \leq M-1)$, then there are negative $n-1$ eigenvalues for problem (9-13) and they can be evaluated by

$$\lambda_{n,r} = -\frac{4}{h^2}\left(\sinh^2\left(\frac{\alpha^*_n h}{2}\right) - \sin^2\left(\frac{h\pi r}{2}\right)\right), r = 1,2,3, \ldots\ldots, n-1,$$

and the zero eigenvalue can be also simply evaluated when $r = n$, leads to $\lambda_{n,n} = 0$.

4. Conclusion

An elliptic PDE model proposed in this paper with a new NBC. We considered that the NBC is inside the domain which produces a shift in the tridiagonal elements which appear in the matrix for elliptic problems with classical conditions. The discussion of the eigenvalue problem investigates the effect of position of the NBC on the eigenvalue's value and type, therefore the corresponding eigenvectors. In the last, from the two-one dimensional problems with NBC, the eigenvalues, and eigenvectors of the two-dimensional problem is obtained.

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