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RESEARCH ARTICLE

EXISTENCE RESULTS TO A CLASS OF FIRST-ORDER FUNCTIONAL INTEGRO-DIFFERENTIAL INCLUSIONS

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ABSTRACT

We establish the existence of continuous solutions to initial value problem for first order functional integro-differential inclusion. The study holds in the case when the set-valued function has Lipschitz selection, also we discuss the existence of maximal and minimal solutions. The continuous dependence and uniqueness of the solution will be proved. As an application, the initial value problem for the arbitrary-order differential inclusion will be studied.

KEYWORDS

Fixed point theorem, Caratheodory condition, Differential inclusion, Maximal and Minimal solutions, Lipschitz selection, Continuous dependence.

1. Introduction

In this paper, we are concerned with the existence of solutions to some classes of initial value problems (in short IVP) for the first-order functional integro-differential equation and inclusion. Precisely, we are interested to study the following problems:

$$\frac{dx}{dt} = f_1(t, x(t), I^{\alpha} f_2(t, x(\varphi(t)))),\ \alpha \in (0,1), t \in [0,T] \tag{1}$$

and

$$\frac{dx}{dt} \in F_1(t, x(t), I^{\alpha} f_2(t, x(\varphi(t)))),\ \alpha \in (0,1), t \in [0,T]. \tag{2}$$

As an application, we study Set-valued ordinary second-order functional differential equation

$$\frac{d^2x(t)}{dt^2} \in F_1(t, x'(t), D^{\beta} x(t)),\ a.e,\ t \in (0,T],\ \ \beta > 0,\ x'(0) = 0, \tag{3}$$

all equipped with condition

$$x(0) = x_{\circ}, \tag{4}$$

where $F_1 : [0,T] \times R^2 \to P(R)$ is a multivalued map, $P(R)$ is a family of all nonempty subsets of R, and $f_2 : [0,T] \times R \to R$ is a given function? The present work is motivated by a recent paper where the authors proved the integrable solution for problem (2),(4) with set-valued function F_1 has selection of L^1 −Carathèodory type (El-Sayed and Al-Issa, 2010). We establish our existence results by using Schauder's and contraction fixed point theorems, where the set-valued map F_1 has Lipschitz selection and by reformulating our problem into a coupled system. The theory of fractional differential equations has emerged as an important field of inquiry in recent years. Let us mention that this theory has many applications in the real world, which is often applicable to engineering, physics, chemistry and biology. Fractional differential equations are also considered in monographs (Kilbas et al., 2006; Lakshmikantham et al., 2009; Podlubny, 1999; Guerekata, 2009; Anguraj et al., 2009; Benchohra et al., 2008).

Recently, several problems in ordinary differential equations and partial differential equations leads to the concept of differential inclusion. Differential inclusion is a concept that generalizes the concept of the differential equations and is more suitable for studying the existence and qualitative properties of solutions of differential equations with discontinuous forcing functions, the existence of solutions for models of control systems, perturbed systems, stochastic models and generally, multi-valued dynamical systems. The nice monograph of Smirnov contains an excellent overview of the theory. Indeed, differential inclusions also serve to properly describe models in relation to differential equations, differential and integral inclusions have been extensively investigated by a number of authors and there are many interesting results concerning this problem (El-Sayed and Al-Issa, 2010; El-Sayed and Al-Issa, 2020a; El-Sayed and Al-Issa, 2020b; Smirnov, 2002; Al-Issa and El-Sayed, 2009; El-Sayed and Ibrahim, 1995; El-Sayed and Ibrahim, 2001; El-Sayed and Al-Issa, 2010; El-Sayed and Al-Issa, 2019; El-Sayed and Al-Issa, 2019; El-Sayed and Al-Issa, 2019; Ibrahim and El-Sayed, 1996).

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The paper is organized as follows. In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout this paper. While Section 3, contains the existence results for the single value problem (1),(4) and we discuss the maximal and minimal solutions. We also study the continuous dependence on initial data $x_\circ$ and functionf_2. Section 4, is devoted to prove the existence results for a set-valued problem (2),(4) with continuous dependence on the set S_{F_1}. Finally, the initial value problem (3)-(4) will be considered.

2. PRELIMINARIES

In this section, we provide some well-known definitions and mathematical concepts of fractional calculus.

Let $C([0,T],R) = \ C([0,T])$ denote the Banach space of all continuous functions from $I = [0,T]$ into R with the supremum norm:

$$\parallel x \parallel_C = \sup_{t\in[0,T]} |x(t)|.$$

Let X be the class of all ordered pair $u = (x,y)$, $x,y \in C(I)$ and define the Banach space $X = C(I) \times C(I)$ with the norm

$$\parallel (x,y) \parallel_X = \parallel x \parallel_C + \parallel y \parallel_C.$$

Definition 2.1. The Riemann-Liouville of fractional integral of the function $f \in L^1(I)$ of order $\alpha \in R^+$ is defined by (Podlubny and El-Sayed, 1996; Podlubny, 1999; Miller and Ross, 1993; Samko et al., 1987).

$$I_a^\alpha f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds.$$

Definition 2.2. The (Caputo) fractional order derivative D^α, $\alpha \in (0,1]$ of the absolutely continuous function g is defined as

$$D_a^\alpha g(t) = I_a^{1-\alpha} \frac{d}{dt} g(t).$$

More properties for fractional calculus (Caputo, 1967; Podlubny and El-Sayed, 1996; Podlubny, 1999; Samko et al., 1987).

Definition 2.3. Let X and Y be two nonempty sets, a set-valued (multivalued) map $F: X \to Y$ is a function that associates to any element $x \in X$ a subset $F(x)$ of Y, called the (image) valued of F at x.

Definition 2.4. Let F be a strict set-valued map (we say F is strict if the domain of F is X itself), f is called a selection of F if $f(x) \in F(x)$, for every $x \in X$, we denote by $S_F = \{f: f(x) \in F(x), x \in X\}$ the set of all selection of F (for the properties of the selection of F see (Cellina and Solimini, 1978; El-Sayed and Ibrahim, 1995; Kuratowski and Ryll-Nardzewski, 1965).

Definition 2.5. A set-valued map F from $I \times E$ to family of all nonempty closed subsets of E is called Lipschitzian if there exists $L > 0$ such that for all $t_1, t_2 \in I$ and all $x_1, x_2 \in E$, we have

$$H(F(t,x_1,y_1), F(t,x_2,y_2)) \le L(\parallel x_1 - x_2 \parallel + \parallel y_1 - y_2 \parallel) \tag{5}$$

Indeed, $H(A,B)$ denote the Hausdorff metric distance between the two subsets $A, B \in I \times E$ (Aubin and Cellina, 1984).

Denote $S_F = Lip(I,E)$ be the set of all Lipschitz selections of the set-valued function F with values in the Banach space $E = R^n$.

Theorem 2.1. Let M be a metric space and F be Lipschitzian set-valued function from M into the nonempty compact convex subsets of R^n (El-Sayed and Ibrahim, 1995). Assume, moreover, that for some$\lambda > 0$, $F(x) \subset \lambda B$ for all $x \in M$ where B is the unit ball on R^n. Then there exists a constant c and a single-valued function $f: M \to R^n$, $f(x) \in F(x)$ for $x \in M$; this function is Lipschitzian with constant c.

3. SINGLE-VALUED PROBLEM

Here we are deal with IVP (1),(4)

3.1 Existence of at least one continuous solution

Consider the following assumptions:

I. The function $\varphi: I \to I$ is a continuous.

II. The function $f_1: I \times R \to R$ is continuous and there exists positive constants k and f_1^*, s.t

$$|f_1(t,x,y)| \le f_1^* + k(|x| + |y|)$$

III. The Caratheodory function $f_2: I \times R \to R$ is measurable in t for any $x \in R$ and continuous in x for almost all $t \in [0,T]$, such that, there exist a measurable function $a(t)$ that is measurable bounded and a positive constant $b > 0$, with

$$|f_2(t,x)| \le \ a(t) + \ b|x|, \forall \ t \ \in I \text{ and } x \in R.$$

IV. $I_c^\gamma\, a(.) \le M \quad \forall\ \gamma \le \alpha, \quad c \ge 0\,.$

We start by proving the following lemma.

Lemma 3.1. If the solution of the problem (1),(4) exists. Then it represents by the integral equation

$$x(t) = x_\circ + \int_0^t f_1(s, x(s), I^\alpha f_2(s, x(\varphi(s))))ds,\ t \in I. \tag{6}$$

Proof. Let x is the solution of IVP (1),(4). We integrate each side of (1) and utilizing the condition (4), we obtain the desired result.

$$x(t) = x_\circ + \int_0^t f_1(s, x(s), I^\alpha f_2(s, x(\varphi(s))))ds,\ t \in I.$$

Now, by considering Eq. (6) and letting

$$y(t) = I^\alpha f_2(t, x(\varphi(t))), \quad t \in I. \tag{7}$$

Then Eq. (6) become

$$x(t) = x_\circ + \int_0^t f_1(s, x(s), y(s))ds, \quad t \in I. \tag{8}$$

So, Eq. (6) represented by the coupled system (7) and (8). Then, existence results for Eq. (6), it follows from existence results for the coupled system (7) and (8).

Theorem 3.2. Assume that conditions (I)-(IV) are valid. Then the coupled system (7), (8) have at least one continuous solution $u = (x,y)$, $x,y \in C([0,T])$.

Proof. Let Q_r is a set defined as

$$Q_r = \{u = (x,y): (x,y) \in R^2, \parallel u \parallel \le r_1 + r_2 = r\}$$

$$\text{where,}\ \ r = \frac{|x_\circ| + f_1^* T}{1-2kT} + \left(1 - \frac{b\, T^\alpha}{\Gamma(\alpha+1)}\right)^{-1} \frac{M\, T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}.$$

We see that Q_r is closed, bounded and nonempty convex set. Next, defined the operator A on $C([0,T]) \times C([0,T])$ by

$$Au(t) = A(x,y)(t) = (A_1 y(t), A_2 x(t))$$

$$A_1 y(t) \ = \ x_\circ + \int_0^t f_1\,(s, x(s), y(s))\, ds, \quad t \in I$$

and

$$A_2 x(t) \ = \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, x(\varphi(s)))\, ds \quad t \in I,$$

where for $u = (x,y) \in Q_r$, we have

$$\begin{aligned} |A_1 y(t)| &= |x_\circ + \int_0^t f_1(s, x(s), y(s))ds| \\ &\le |x_\circ| + \int_0^t |f_1(s, x(s), y(s))|ds \\ &\le |x_\circ| + [k(|x| + |y|) + f_1^*]T. \end{aligned}$$

This further implies that

$$\| A_1 y \| \quad \leq \quad |x_\circ| + 2kr_1T + f_1^*T, \quad r_1 = \frac{|x_\circ| + f_1^*T}{1-2kT}.$$

Also,

$$\begin{aligned}
| A_2 x(t) | &= | \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, x(\varphi(s)))ds | \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} | f_2(s, x(\varphi(s))) | ds \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [a(s) + b\,| x(\varphi(s)) |] ds \\
&\leq \int_0^t a(s) \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \int_0^t b\,| x(\varphi(s)) | \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\leq I^\alpha a(t) + br_2 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\leq I^{\alpha-\gamma} I^\gamma a(t) + br_2 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\leq M \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} ds + br_2 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\leq \frac{M\,T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + br_2 \frac{T^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

This further implies that

$$\| A_2 x \| \leq \frac{MT^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{br_2T^\alpha}{\Gamma(\alpha+1)}, \quad r_2 = \left(1 - \frac{bT^\alpha}{\Gamma(\alpha+1)}\right)^{-1} \frac{MT^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}.$$

Now

$$\begin{aligned}
\| Au \|_X &= \| A_1 y \|_c + \| A_2 x \|_c \leq r_1 + r_2 \\
&\leq \frac{|x_\circ| + f_1^* T}{1-2kT} + \left(1 - \frac{bT^\alpha}{\Gamma(\alpha+1)}\right)^{-1} \frac{M\,T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} = r.
\end{aligned}$$

Then $AQ_r \subset Q_r$, we note that$\{Au\}$, $u \in Q_r$ is uniformly bounded Set. Then, for $u = (x, y) \in Q_r$, $\forall\ \epsilon > 0, \delta > 0$ and for each $t_1, t_2 \in [0, T]$, $t_1 < t_2$, s.t $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
|A_1y(t_2) - A_1y(t_1)| &= | x_\circ + \int_0^{t_2} f_1(s, x(s), y(s))ds - x_\circ - \int_0^{t_1} f_1(s, x(s), y(s))ds | \\
&\leq \int_{t_1}^{t_2} |f_1(s, x(s), y(s))| ds \\
&\leq (k(|x| + |y|) + f_1^*) \int_{t_1}^{t_2} ds \\
&\leq (2kr_1 + f_1^*)(t_2 - t_1),
\end{aligned}$$

and

$$\begin{aligned}
| A_2x(t_2) - A_2x(t_1) | &= \Big| \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, x(\varphi(s)))ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, x(\varphi(s)))ds \Big| \\
&= \Big| \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, x(\varphi(s)))ds - \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, x(\varphi(s)))ds \\
&\quad + \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, x(\varphi(s)))ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, x(\varphi(s)))ds \Big| \\
&\leq \Big| \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, x(\varphi(s)))ds \Big| + \Big| \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, x(\varphi(s)))ds \\
&\quad - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, x(\varphi(s)))ds \Big| \\
&\leq \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| f_2(s, x(\varphi(s))) \Big| ds + \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| f_2(s, x(\varphi(s))) \Big| ds \\
&\quad + \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| f_2(s, x(\varphi(s))) \Big| ds \\
&\leq \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} [| a(s) | + b\,| x(\varphi(s)) |] ds \\
&\quad + \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} [| a(s) | + b\,| x(\varphi(s)) |] ds \\
&\leq (\| a \| + br_2) \left[\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right] \\
&\leq (\| a \| + br_2) \frac{| t_2^\alpha - t_1^\alpha - 2(t_2 - t_1)^\alpha |}{\Gamma(\alpha+1)}.
\end{aligned}$$

For $u \in Q_r$,

$$\begin{aligned}
Au(t_2) - Au(t_1) &= A(x, y)(t_2) - A(x, y)(t_1) \\
&= (A_2x(t_2), A_1y(t_2)) - (A_2x(t_1), A_1y(t_1)) \\
&= (A_2x(t_2) - A_2x(t_1), A_1y(t_2) - A_1y(t_1)).
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
|Au(t_2) - Au(t_1)| &= | A(x, y)(t_2) - A(x, y)(t_1)| \\
&= |A_1y(t_2) - A_1y(t_1)| + |A_2y(t_2) - A_2x(t_1)| \\
&\leq (2kr_1 + f_1^*)(t_2 - t_1) + (a + br_2) \frac{| t_2^\alpha - t_1^\alpha - 2(t_2 - t_1)^\alpha |}{\Gamma(\alpha+1)}.
\end{aligned}$$

Thus, $\{Au\}$ is equi-continuous on Q_r, and A is compact operator by the Arzela-Ascoli Theorem (Curtain and Pritchard, 1977). To show $A: Q_r \to Q_r$ is continuous. Consider the sequence $u_n = (x_n, y_n)$ in Q_r with $x_n \to x$, and $y_n \to y$, from continuity of $f_2(t, x(t))$ in $C([0, T]) \times R$ this implies that $f_2(t, x_n(t)) \to f_2(t, x(t))$, and $f_2(t, x_n(\varphi(t))) \to f_2(t, x(\varphi(t)))$ as $n \to \infty$, from (III)-(IV), using Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} A_1 y_n(t) &= x_\circ + \int_0^t \lim_{n\to\infty} f_1(s, x(s), y_n(s))ds \\
&= x_\circ + \int_0^t f_1(s, x(s), y(s))ds = A_1 y(t) \\
\lim_{n\to\infty} A_2 x_n(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \lim_{n\to\infty} f_2(s, x_n(\varphi(s)))ds \\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, x(\varphi(s)))ds = A_2 x(t).
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{n\to\infty} Au_n(t) &= \lim_{n\to\infty} (A_1 y_n(t), A_2 x_n(t)) \\
&= (\lim_{n\to\infty} A_1 y_n(t), \lim_{n\to\infty} A_2 x_n(t)) = (A_1 y(t), A_2 x(t)) = Au(t).
\end{aligned}$$

Hence $Au_n \to Au$ as $n \to \infty$, therfore operator A is continuous. So, conditions of the Schauder fixed-point theorem satisfy this give that A has a fixed point $u \in Q_r$, so, there exists at least one continuous solution $u = (x, y) \in Q_r, x; y \in C([0, T])$ for the system (7), (8).

Thus, Eq. (1) has at least one solution $x \in C([0, T])$ (Lakshmikantham et al., 2009).

Also, differentiating Eq. (6), we obtain

$$\begin{aligned}
\frac{dx}{dt} &= \frac{d}{dt} \{x_\circ + \int_0^t f_1(s, x(s), I^\alpha f_2(s, x(s), x(\varphi(s))))ds\} \\
&= f_1(t, x(t), I^\alpha f_2(t, x(\varphi(t)))),
\end{aligned}$$

and at $t = 0$ we get

$$x(0) = x_\circ$$

Hence, IVP (1),(4) has at least one solution $x \in C([0, T])$. This completes the proof.

3.2 Maximal and minimal solutions

Here, we study the maximal and minimal solutions for Eq. (6).

Definition 3.1. Let $m(t)$ be a solution of Eq. (6), then $m(t)$ is a maximal solution of Eq. (6) if for every solution x of Eq. (6) existing on $[0, T]$ the inequality $x(t) \leq m(t),\ \ t \in [0, T]$ holds. The minimal solution $s(t)$ may be defined similarly by reversing the last inequality i.e., $x(t) > s(t),\ \ \forall\, t \in [0, T]$.

Consider the following lemma

Lemma 3.3. Assume $p(t)$, $f_1(t, x, y)$ $f_2(t, x)$ and $\varphi(t)$ as in Theorem 3.2 and for $x(t), y(t)$ be two continuous functions on $[0, T]$ satisfying

$$x(t) \quad \leq \quad x_\circ + \int_0^t f_1\,(s, x(s),\ I^\alpha f_2(s, x(\varphi(s))))\, ds, \qquad t \in [0, T]$$

$$y(t) \quad \geq \quad x_\circ + \int_0^t f_1\,(s, y(s),\ I^\alpha f_2(s, y(\varphi(s))))\, ds, \qquad t \in [0, T]$$

where one of them is strict.

Let functions f_1 and f_2 are monotonic non decreasing in x, then

$$x(t) < y(t), \quad t > 0. \tag{9}$$

Proof. Let's assume the conclusion (9) not true, then there exists t_1 with

$$x(t_1) = y(t_1), \ t_1 > 0$$

and

$$x(t) < y(t), \ 0 < t < t_1, \ t \in [0, T].$$

For f_1 and f_2 are monotonic functions in x, then

$$\begin{aligned} x(t_1) &= x_\circ + \int_0^{t_1} f_1(s, x(s), \ I^\alpha f_2(s, x(\varphi(s)))) \, ds \\ &< x_\circ + \int_0^{t_1} f_1(s, y(s), \ I^\alpha f_2(s, y(\varphi(s)))) \, ds \\ x(t_1) &< y(t_1). \end{aligned}$$

This contrasts with the fact that $x(t_1) = y(t_1)$.

Hence

$$x(t) < y(t).$$

This completes the proof.

For the existence of the continuous maximal and minimal solutions for Eq. (6), we have the following theorem.

Theorem 3.4. Let assumptions (I)-(III) of Theorem 3.2 be hold. Moreover, if f_1 and f_2 are monotonic nondecreasing functions in x for each $t \in [0, T]$. Then both maximum and minimum solutions for Eq. (6) exists.

Proof. First, we must demonstrate the existence of the maximal solution of Eq. (6).

For given $\epsilon > 0$ s.t $0 < \epsilon < \frac{T}{2}$, and consider the integral equation

$$x_\epsilon(t) = x_\circ + \int_0^t f_{1_\epsilon}(s, x_\epsilon(s), I^\alpha f_{2_\epsilon}(s, x_\epsilon(\varphi(s))) \, ds, \quad t \in [0, T]$$

where

$$\begin{aligned} f_{2_\epsilon}(t, x_\epsilon(\varphi(t))) &= f_2(t, x_\epsilon(\varphi(t))) + \epsilon \\ \int_0^t f_{1_\epsilon}(s, x_\epsilon(s), I^\alpha f_{2_\epsilon}(s, x_\epsilon(\varphi(s)))ds &= \int_0^t (f_1(s, x_\epsilon(s), I^\alpha f_{2_\epsilon}(s, x_\epsilon(\varphi(s))) + \epsilon)ds \\ \int_0^t f_{1_\epsilon}(s, x_\epsilon(s), I^\alpha f_{2_\epsilon}(s, x_\epsilon(\varphi(s)))ds &= \int_0^t f_1(s, x_\epsilon(s), I^\alpha f_{2_\epsilon}(s, x_\epsilon(\varphi(s))) + \epsilon T. \end{aligned}$$

Clearly the functions $f_{1_\epsilon}(t, x_\epsilon(t), I^\alpha f_{2_\epsilon}(t, x_\epsilon(\varphi(t)))$ and $f_{2_\epsilon}(t, x_\epsilon(\varphi(t)))$ satisfy conditions of Theorem 3.2 hence, Eq. (6) exists a continuous solution x_ϵ, with respect to Theorem 3.2. Let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$. Then

$$x_{\epsilon_2}(t) = x_\circ + \int_0^t f_{1_{\epsilon_2}}(s, x_{\epsilon_2}(s), I^\alpha f_{2_{\epsilon_2}}(s, x_{\epsilon_2}(\varphi(s))) \, ds$$

$$x_{\epsilon_2}(t) = x_\circ + \int_0^t f_1\left(s, x_{\epsilon_2}(s), I^\alpha f_2\left(s, x_{\epsilon_2}(\varphi(s))\right) + I^\alpha \epsilon_2\right) ds + \int_0^t \epsilon_2 \, ds. \tag{10}$$

Also,

$$\begin{aligned} x_{\epsilon_1}(t) &= x_\circ + \int_0^t f_{1_{\epsilon_1}}(s, x_{\epsilon_1}(s), I^\alpha f_{2_{\epsilon_1}}(s, x_{\epsilon_1}(\varphi(s))) \\ x_{\epsilon_1}(s) \, ds &= x_\circ + \int_0^t f_1\left(s, x_{\epsilon_1}(s), I^\alpha f_2\left(s, x_{\epsilon_1}(\varphi(s))\right) + I^\alpha \epsilon_1\right) + \int_0^t \epsilon_1 \, ds \end{aligned}$$

$$x_{\epsilon_1}(t) > x_\circ + \int_0^t f_1\left(s, x_{\epsilon_1}(s), I^\alpha f_2\left(s, x_{\epsilon_1}(\varphi(s))\right) + I^\alpha \epsilon_2\right) ds + \int_0^t \epsilon_2 \, ds. \tag{11}$$

Applying Lemma 3.3, on (10) and (11), we get

$$x_{\epsilon_2}(t) < x_{\epsilon_1}(t), \qquad t \in [0, T].$$

As previously explained in the proof of Theorem 3.2, the family of functions $x_\epsilon(t)$ defined by Eq. (6) is uniformly bounded and equi-continuous functions. Hence by the Arzela- Ascoli Theorem, there exists a decreasing sequence ϵ_n such that $\epsilon_n \to 0$ as $n \to \infty$, and $\lim_{n\to\infty} x_{\epsilon_n}(t)$ exists uniformly in $[0, T]$ and we denote this limit by $m(t)$. From the continuity of the functions $f_{i_{\epsilon_n}}$ for $i = 1,2$, we get

$$\begin{aligned} f_{2_{\epsilon_n}}(t, x_{\epsilon_n}(\varphi(t))) &\to f_2(t, x(\varphi(t))), \ as \ n \to \infty \\ f_{1_{\epsilon_n}}(t, x_{\epsilon_n}(t), I^\alpha f_{2_{\epsilon_n}}(t, x_{\epsilon_n}(\varphi(t))) &\to f_1(t, x(t), I^\alpha f_2(t, x(\varphi(t))), \ as \ n \to \infty \end{aligned}$$

And

$$m(t) = \lim_{n\to\infty} x_{\epsilon_n} = x_\circ + \int_0^t f_1(s, x(s), I^\alpha f_2(s, x(\varphi(s))) \, ds$$

Then, Eq. (6) has a solution $m(t)$.

At the last step, we will clear that $m(t)$ is the maximal solution of Eq. (6). For this assume that $x(t)$ be any solution of Eq. (6), then

$$x(t) = x_\circ + \int_0^t f_1(s, x(s), I^\alpha f_2(s, x(\varphi(s))) \, ds. \tag{12}$$

Also,

$$\begin{aligned} x_\epsilon(t) &= x_\circ + \int_0^t f_{1_\epsilon}(s, x_\epsilon(s), I^\alpha f_{2_\epsilon}(s, x_\epsilon(\varphi(s))) \, ds \\ x_\epsilon(t) &= x_\circ + \int_0^t f_1(s, x_\epsilon(s), I^\alpha f(s, x_\epsilon(\varphi(s))) + I^\alpha \epsilon) \, ds + \int_0^t \epsilon \, ds \end{aligned}$$

$$x_\epsilon(t) > x_\circ + \int_0^t f_1(s, x_\epsilon(s), I^\alpha f_2(s, x_\epsilon(\varphi(s))) \, ds \tag{13}$$

Applying Lemma 3.3 on (12) and (13) we get

$$x(t) < x_\epsilon(t), \ \ for \ t \in [0, T].$$

Since the maximal solution is uniqueness, it is normal that $x_\epsilon(t)$ approach to $m(t)$ uniformly in $[0, T]$ as $\epsilon \to 0$ (Lakshmikantham et al., 1969).

In the same way, we will demonstrate the existence of the minimum solution. We put

$$\begin{aligned} f_{1_\epsilon}(t, x_\epsilon(t), I^\alpha f_{2_\epsilon}(t, x_\epsilon(\varphi(t))) &= f_1(t, x_\epsilon(t), I^\alpha f_{2_\epsilon}(t, x_\epsilon(\varphi(t))) - I^\alpha \epsilon) - \epsilon \\ f_{2_\epsilon}(t, x_\epsilon(\varphi(t))) &= f_2(t, x_\epsilon(\varphi(t))) - \epsilon \end{aligned}$$

This proves existence of a minimal solution. This completes the proof.

3.3 Existence of a unique continuous solution

Here, we give the sufficient condition for the uniqueness of the solution for IVP (1)-(4), let us replace assumptions (II) and (III) by:

(II)*. Let $f_1: I \times I \times R \times R \to R$ is continuous and satisfies the Lipschitz condition, which there exists a positive constant k such that

$$|f_1(t, x_1, y_1) - f_1(t, x_2, y_2)| \le k(|x_1 - x_2| + |y_1 - y_2|).$$

(III)*. Let $f_2: I \times I \times R \to R$ is continuous and and satisfies the Lipschitz condition, which there exists a positive constant b such that

$$|f_2(t, x) - f_2(t, y)| \le b|x - y|.$$

From this assumption (II)*, we have

$$|f_1(t, x, y)| - |f_1(t, 0,0)| \le |f_1(t, x, y) - f_1(t, 0,0)| \le k(|x| + |y|),$$

then

$$|f_1(t, x, y)| \le k(|x| + |y|) + |f_1(t, 0,0)|$$

and

$$|f_1(t, x, y)| \le k(|x| + |y|) + f_1^*,$$

where $f_1^* = \sup_{t\in I} |f_1(t, 0,0)|$ and from assumption (III)*, we have

$$|f_2(t,x)| - |f_2(t,0)| \le |f_2(t,x) - f_2(t,0)| \le b|x|,$$

then

$$|f_2(t,x)| \le b|x| + |f_2(t,0)|$$

and

$$|f_2(t,x)| \le b|x| + f_2^*,$$

where $f_2^* = \sup_{t\in I}|f_2(t,0)|$.

Theorem 3.5. Consider assumptions of Theorem 3.2 with replace (II), (III) by (II)* ,(III)*. If $(T + \frac{b\,T^{\alpha+1}}{\Gamma(\alpha+1)})k < 1$, then the IVP (1),(4) has a unique solution on $[0,T]$.

Proof. Let $x_1(t)$ and $x_2(t)$ be solutions of the IVP (1),(4), then

$$\begin{aligned}
&|x_1(t) - x_2(t)| \\
&\le \left|\int_0^t f_1(s, x_1(s), I^\alpha f_2(s, x_1(\varphi(s))))ds - f_1(s, x_2(s), I^\alpha f_2(s, x_2(\varphi(s))))ds\right| \\
&\le \int_0^t \left|f_1\left(s, x_1(s), I^\alpha f_2\left(s, x_1(\varphi(s))\right)\right) - f_1\left(s, x_2(s), I^\alpha f_2\left(s, x_2(\varphi(s))\right)\right)\right| ds.
\end{aligned}$$

From assumption (II)*, we have

$$\begin{aligned}
&|x_1(t) - x_2(t)| \\
&\le k\int_0^t |x_1(s) - x_2(s)|\,ds + k\int_0^t |I^\alpha f_2(s, x_1(\varphi(s))) - I^\alpha f_2(s, x_2(\varphi(s)))|\,ds \\
&\le k \parallel x_1 - x_2 \parallel \int_0^t ds + k\int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left|f_2\left(\tau, x_1(\varphi(\tau))\right) - f_2\left(\tau, x_2(\varphi(\tau))\right)\right| d\tau ds.
\end{aligned}$$

From assumption (III)*, we have

$$\begin{aligned}
|x_1(t) - x_2(t)| &\le k \parallel x_1 - x_2 \parallel T + k\,b\int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|x_1(\varphi(\tau)) - x_2(\varphi(\tau))|\,d\tau\,ds \\
&\le k \parallel x_1 - x_2 \parallel T + k\,b \parallel x_1 - x_2 \parallel \int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\,d\tau\,ds,
\end{aligned}$$

which, on taking the norm for $t \in [0,T]$, yields

$$\parallel x_1 - x_2 \parallel \quad\le\quad (T + \frac{b\,T^{\alpha+1}}{\Gamma(\alpha+1)})k \parallel x_1 - x_2 \parallel.$$

Hence

$$\left(1 - (T + \frac{b\,T^{\alpha+1}}{\Gamma(\alpha+1)})k\right) \parallel x_1 - x_2 \parallel \quad\le\quad 0.$$

Since $(T + \frac{b\,T^{\alpha+1}}{\Gamma(\alpha+1)})k < 1$, then $x_1(t) = x_2(t)$. Then there exists a unique continuous solution for the IVP (1),(4). This completes the proof.

3.4 Continuous dependence of the solution

Here, we discuss the continuous dependence of unique solution for the IVP (1),(4).

3.4.1 Continuous dependence on the initial data $x_\circ$

Definition 3.2. The solutions of the IVP (1)-(4), depends continuously on initial data $x_\circ$, if $\forall\, \epsilon > 0$, $\exists\, \delta > 0$, s.t

$$|x_\circ - x_\circ^*| \le \delta \;\Rightarrow \parallel x - x^* \parallel \le \epsilon.$$

Corollary 3.5.1. Assume conditions of Theorem 3.5 be satisfied. Then the solution of the IVP (1),(4) is continuous dependence on initial data $x_\circ$.

Proof. For $x(t)$ and $x^*(t)$ be solutions of (1),(4), then

$$\begin{aligned}
|x(t) - x^*(t)| &\le |x_\circ - x_\circ^*| + \int_0^t |f_1(s, x(s), I^\alpha f_2(s, x(\varphi(s))) - f_1(s, I^\alpha f_2(s, x^*(s), x^*(\varphi(s)))|\,ds \\
&\le \delta + k\int_0^t |x(s) - x^*(s)|\,ds + k\int_0^t |I^\alpha f_2(s, x(\varphi(s))) - I^\alpha f_2(s, x^*(\varphi(s)))|\,ds \\
&\le \delta + k \parallel x - x^* \parallel T + k\int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|f_2(\tau, x(\varphi(\tau))) - f_2(\tau, x^*(\varphi(\tau)))|\,d\tau ds \\
&\le \delta + k \parallel x - x^* \parallel T + kb\int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|x(\varphi(\tau)) - x^*(\varphi(\tau))|\,d\tau \\
&\le \delta + k \parallel x - x^* \parallel T + k\,b \parallel x - x^* \parallel \int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\,d\tau\,ds \\
&\le \delta + (k\,T + \frac{T^{\alpha+1}\,k\,b}{\Gamma(\alpha+1)}) \parallel x - x^* \parallel,
\end{aligned}$$

which, on taking supremum over $t \in [0,T]$, yields

$$\parallel x - x^* \parallel \quad\le\quad (1 - (T + \frac{T^{\alpha+1}\;b}{\Gamma(\alpha+1)})k)^{-1}\delta = \epsilon.$$

Hence,

$$\parallel x - x^* \parallel \le \epsilon(\delta).$$

Then solution of the IVP (1)-(4) depends continuously on $x_\circ$. This completes the proof.

3.4.2 Continuous dependence on the function f_2

Definition 3.3. The solutions of the IVP (1),(4), depends continuously on the function f_2, if $\forall\, \epsilon > 0$, $\exists\, \delta > 0$, such that

$$|f_2 - f_2^*| \le \delta \;\Rightarrow \parallel x - x^* \parallel \le \epsilon.$$

Corollary 3.5.2. Assume conditions of Theorem 3.5 be satisfied. Then we have continuous dependence of solutions for IVP (1),(4) on Lipschitz function f_2.

Proof. For $x(t)$ and $x^*(t)$ be solutions of problem (1),(4), then

$$\begin{aligned}
|x(t) - x^*(t)| &\le \int_0^t |f_1(s, x(s), I^\alpha f_2(s, x(\varphi(s))) - f_1(s, x^*(s), I^\alpha f_2^*(s, x^*(\varphi(s)))|\,ds \\
&\le k\int_0^t |x(s) - x^*(s)|\,ds + k\int_0^t |I^\alpha f_2(s, x(\varphi(s))) - I^\alpha f_2^*(s, x^*(\varphi(s)))|\,ds \\
&\le k \parallel x - x^* \parallel T + k\,[\int_0^t |I^\alpha f_2(s, x(\varphi(s))) - I^\alpha f_2(s, x^*(\varphi(s)))|\,ds \\
&+ \int_0^t |I^\alpha f_2(s, x^*(\varphi(s))) - I^\alpha f_2^*(s, x^*(\varphi(s)))|\,ds] \\
&\le k \parallel x - x^* \parallel T + k[|\int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|f_2(\tau, x(\varphi(\tau))) - f_2(\tau, x^*(\varphi(\tau)))|\,d\tau ds \\
&+ \int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|f_2(\tau, x^*(\varphi(\tau))) - f_2^*(\tau, x^*(\varphi(\tau)))|\,d\tau\,ds] \\
&\le k \parallel x - x^* \parallel T + k\,b\int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|x(\varphi(\tau)) - x^*(\varphi(\tau))|\,d\tau\,ds \\
&+ k\,\delta\int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\,d\tau\,ds \\
&\le k \parallel x - x^* \parallel T + \left[kb \parallel x - x^* \parallel + k\delta\right]\int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\,d\tau ds \\
&\le \left(kT + \frac{T^{\alpha+1}\,k\,b}{\Gamma(\alpha+1)}\right) \parallel x - x^* \parallel + \frac{T^{\alpha+1}\,k\,\delta}{\Gamma(\alpha+1)},
\end{aligned}$$

which, on taking supremum over $t \in [0,T]$, yields

$$\parallel x - x^* \parallel \quad\le\quad (1 - (Tk + \frac{T^{\alpha+1}k\,b}{\Gamma(\alpha+1)}))^{-1}\frac{T^{\alpha+1}\;k\;\delta}{\Gamma(\alpha+1)} = \epsilon.$$

Hence,

$$\parallel x - x^* \parallel \le \epsilon(\delta).$$

Then solution of the IVP (1),(4) depends continuously on the function f_2. This completes the proof.

4. SET-VALUED PROBLEM

In this section we study the existence of at least one continuous solution of the IVP of inclusion (2),(4) with the following assumptions:

I. The function $\varphi: I \to I$ is a continuous function.

II**. $F_1: I \times R \to 2^{R^+}$ is a Lipschitzian set-valued map with a nonempty compact convex subset of 2^{R^+}.

III. The Caratheodory function $f_2: I \times R \to R$ is measurable in t for any $x \in R$ and continuous in x for almost all $t \in [0,T]$, such that, there exist a measurable function $a(t)$ that is measurable bounded and a positive constant $b > 0$, with

$$|f_2(t,x)| \le\; a(t) +\; b|x|, \forall\; t\; \in I \text{ and } x \in R.$$

IV. $I_c^\gamma\; a(.) \le M \quad \forall\; \gamma \le \alpha, \quad c \ge 0\,.$

Theorem 4.1. Assume that condition (I)-(II)**-(III)-and (IV) be valid. Then the IVP of functional integral inclusion (2),(4) have at least one solution $x \in C([0,T])$.

Proof. The Theorem 2.1 and condition (II)**, implies that the set S_{F_1} is non empty. Then, the solution of the single-valued problem (1),(4) where $f_1 \in S_{F_1}$, is a solution to the set-valued problem (2),(4).

So, the Lipschitz selection $f_1: [0,T] \times R \times R \to R$, it meets the Lipschitz requirement

$$|f_1(t,x_1,y_1) - f_1(t,x_2,y_2)| \leq k(|x_1 - x_2| + |y_1 - y_2|).$$

From this condition with $f_1^* = \sup_{t \in I}|f_1(t,0,0)|$, we have

$$|f_1(t,x,y)| - |f_1(t,0,0)| \leq |f_1(t,x,y) - f_1(t,0,0)| \leq k(|x| + |y|),$$

then

$$|f_1(t,x,y)| \leq k(|x| + |y|) + |f_1(t,0,0)|$$

and

$$|f_1(t,x,y)| \leq k_1(|x| + |y|) + f_1^*,$$

i.e., assumption (II) of Theorem 3.2 satisfies. So, all conditions of Theorem 3.2 hold.

Note that if $x \in C(I,R)$ is a solution of single-valued problem (1),(4), then x is a solution to the set-valued problem (2),(4). This completes the proof.

4.1 Continuous dependence on the selection of set S_{F_1}

Definition 4.1. The solutions of initial-valued problem (2),(4) depends continuously on selections of set S_{F_1}. If $\forall\, \epsilon > 0$, $\exists\, \delta > 0$, s.t

$$|f_1(t,x,y) - f_1^*(t,x,y)| \quad < \quad \delta, \quad f_1,\ f_1^* \in S_{F_1}, \quad t \in [0,T],$$

then

$$\| x - x^* \| < \epsilon.$$

Corollary 4.1.1. The solution of IVP (2),(4) depends continuously on selections of set S_{F_1}

Proof. Let $x(t)$ and $x^*(t)$ be solution of problem (2),(4). Then

$$\begin{aligned}
x(t) - x^*(t) &= \int_0^t f_1(s,x(s),I^\alpha f_2(s,x(\varphi(s))) - f_1^*(s,x^*(s),I^\alpha f_2(s,x^*(\varphi(s)))ds \\
|x(t) - x^*(t)| &\leq \int_0^t |f_1(s,x(s),I^\alpha f_2(s,x(\varphi(s))) - f_1^*(s,x^*(s),I^\alpha f_2(s,x^*(\varphi(s)))|\,ds \\
&\leq \int_0^t |f_1(s,x(s),I^\alpha f_2(s,x(\varphi(s))) - f_1(s,x(s),I^\alpha f_2(s,x^*(\varphi(s)))|\,ds \\
&+ \int_0^t |\, f_1(s,x^*(s),I^\alpha f_2(s,x^*(\varphi(s))) - f_1^*(s,x^*(s),I^\alpha f_2(s,x^*(\varphi(s)))|\,ds \\
&\leq \int_0^t (|\, f_1(s,x(s),I^\alpha f_2(s,x(\varphi(s))) - f_1(s,x^*(s),I^\alpha f_2(s,x^*(\varphi(s)))| + \delta)ds \\
&\leq k\int_0^t |x(s) - x^*(s)| + |I^\alpha f_2(s,x(\varphi(s))) - I^\alpha f_2(s,x^*(\varphi(s)))|\,ds + \delta T \\
&\leq k\| x - x^* \| T + k\int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |\, f_2(\tau,x(\varphi(\tau))) - f_2(\tau,x^*(\varphi(\tau)))|\,d\tau\, ds + \delta T \\
&\leq k\| x - x^* \| T + kc\int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |x(\varphi(\tau)) - x^*(\varphi(\tau))|\,d\tau ds + \delta T \\
&\leq k\| x - x^* \| T + kc\| x - x^* \| \int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} d\tau\, ds + \delta T \\
&\leq \left(T + \frac{cT^{\alpha+1}}{\Gamma(\alpha+1)} \right) k\| x - x^* \| + \delta T,
\end{aligned}$$

which, on taking the norm for $t \in [0,T]$, yields

$$\| x - x^* \| \quad \leq \quad \left(1 - (T + \frac{c\, T^{\alpha+1}}{\Gamma(\alpha+1)})k \right)^{-1} \delta\, T = \epsilon(\delta).$$

Hence,

$$\| x - x^* \| \leq \epsilon(\delta).$$

This mean that the solution of the initial value problem (2),(4) depends continuously on the set S_{F_1} of all Lipschitzian selection of F_1. This completes the proof.

4.2 Set-valued fractional-order integro-differential equations

Here, we consider the IVP (3) - (4).

Theorem 4.2. Let the conditions of Theorem 4.1 be holed. Then problem (3) - (4) has at least one solution $x \in C([0,T])$.

Proof. Put $u(t) = \frac{dx(t)}{dt}$, and x\`(0)=0 $\alpha = 1 - \beta$ then the inclusion (3), will be

$$\frac{du(t)}{dt} \in F_1(t,u(t),I^{1-\beta}u(t)). \tag{14}$$

By applying Theorem 4.1 on the Eq. (2), and taking $f_2(t,x(t)) = x(t)$ and $\varphi(t) = t$, then Eq. (14) exists a continuous solution $u \in C([0,T])$, we deduce that there exists a solution $u \in C([0,T])$ of the functional inclusion (14).

This implies that there exists at least one solution $x \in C([0,T])$

$$x(t) \;=\; x_\circ \;+\; \int_0^t u\,(s)ds,$$

of the IVP (3)-(4). This completes the proof.

5. Conclusion

In our current research, we study an initial value problem for a class of first-order functional integro-differential equation and inclusions. In the process, we obtain the existence and uniqueness of solutions of this problem using Schauder and contraction fixed point theorems, also, we discuss the existence of maximal and minimal solutions of the proposed problem, and furthermore, we discuss the existence results for a set-valued problem (2),(4). Finally, the initial value problem for the arbitrary-order differential inclusion will be studied. Note that the fractional differential equation (2) includes the ordinary derivative $\frac{dx}{dt}$ of order 1 on its left-hand side. In the expected future, we suggest that investigate the possibility of extending our results to other higher-order derivatives such as

$$\frac{d^2x}{dt^2}, \frac{d^3x}{dt^3}, \frac{d^4x}{dt^4}, \ldots$$

and also, instead of initial condition, we can study our problem with new BCs such as integrated BCs and / or m-point BCs.

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