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RESEARCH ARTICLE

NUMERICAL COMPUTATIONS OF GENERAL NON-LINEAR SECOND ORDER INITIAL VALUE PROBLEMS BY USING MODIFIED RUNGE-KUTTA METHOD

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ABSTRACT

Numerical solution of ordinary differential equations is the most important technique which is widely used for mathematical modelling in science and engineering. The differential equation that describes the problem is typically too complex to precisely solve in real-world circumstances. Since most ordinary differential equations are not solvable analytically, numerical computations are the only way to obtain information about the solution. Many different methods have been proposed and used is an attempt to solve accurately various types of ordinary differential equations. Among them, Runge-Kutta is a well-known and popular method because of their good efficiency. This paper contains an analysis for the computations of the modified Runge-Kutta method for nonlinear second order initial value problems. This method is wide quite efficient and practically well suited for solving linear and non-linear problems. In order to verify the accuracy, we compare numerical solution with the exact solution. We also compare the performance and the computational effort of this method. In order to achieve higher accuracy in the solution, the step size needs to be small. Finally, we take some examples of non-linear initial value problems (IVPs) to verify proposed method. The results of that example indicate that the convergence, stability analysis, and error analysis which are discussed to determine the efficiency of the method.

KEYWORDS

Nonlinear IVPs, Differential Equations, Runge-Kutta Method.

1. INTRODUCTION

There are many analytical methods for finding the solution of ordinary differential equations. But a few numbers of differential equations have analytic solutions where large numbers of differential equations have no analytic solutions. In this case we use the numerical methods to get an approximate solution of a differential equation. There are many types of numerical methods such as Eulers method, Runge-Kutta method etc. Runge-Kutta method is the powerful numerical technique to solve the initial value problems (IVPs). The use of the Euler method to solve IVPs numerically is less efficient since it requires being small for obtaining reasonable accuracy. But in Runge-Kutta method, the derivatives of higher order are required and they are designed to give greater accuracy with the advantage of requiring only the functional values at some selected points on the sub-interval. We observe that in the Runge-Kutta method extremely small step size converges to the analytical solution. In addition, Runge-Kutta method gives best results and it converges faster to an analytical solution and has less iteration to get accuracy in solution.

The literature contains several methods which have been proposed to solve initial value problems. Islam, M. A. discussed accuracy analysis of numerical solutions of initial value problems (IVP) for ordinary differential equations (ODE), and the author discussed accurate solutions of initial value problems for ordinary differential equations with fourth order Runge-kutta method (Islam, 2015). Ogunrinde, Fadugba, and Okunlola studied on some numerical methods for solving initial value problems in ordinary differential equations (Ogunrinde et al., 2012). Ahamed, N. and Charan, S. proposed a numerical accuracy of Runge-Kutta second, third and fourth order (Ahamed et al., 2015). Adomian, G. proposed the Adomian decomposition method (ADM) for solving nonlinear differential equations (Adomian, 1998). Adomian, G. proposed a modified approach to the Adomian polynomials which converges a little faster than the original Adomian polynomials and is favorable for computer generation was introduced (Adomian, 1996). Wazwaz, A.M. proposed the proper use of the ADM has made it possible to obtain an analytic solution of a singular initial value problem when it is homogeneous or inhomogeneous (Wazwaz, 2002).

Abdelrazec, A. H. M. proposed some of the merits of ADM method that are converge fast to the exact solution (Abdelrazec, 2008). Hasan, Y. Q., Zhu, L. M. proposed an efficient modification of the Adomian decomposition method for solving a singular initial value problem for a second-order ordinary differential equation (Hasan et al., 2008). Goeken, D., Johnson, O. proposed multi-step Runge–Kutta method can be thought of as replacing functional evaluations with higher order derivative approximations (Goeken et al., 2000). In Rabiei, F., and Ismail, F. presented fifth order improved Runge-Kutta method for solving ordinary differential equation (Rabiei et al., 2012). In Butcher, J. C. presented on fifth order Runge-Kutta methods (Butcher, 1995). In Butcher, J. C. proposed High order Runge-Kutta methods are capable of achieving highly accurate approximations of differential equations solutions at lower computational cost than low order Runge-Kutta methods (Butcher, 1996).

In this paper we apply modified fourth order Runge-Kutta method for solving initial value problem of second order ordinary differential

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equation. A more robust and intricate numerical technique is the modified Runge-Kutta method. Some examples of different kinds of ordinary differential equations are given to verify our proposed formulation. The results of each numerical example indicate the convergence and error analysis is discussed to illustrate the efficiency of the present method. This work is organized this work as follows: methodology of our proposed numerical schemes is available in section 2. In section 3, some nonlinear differential equations are solved numerically by our proposed method and compare them with the existing results. In the last section, the conclusion of the paper is inserted. Finally, all the relevant references are included. We use MATLAB R2019a to get the numerical results as well as figures.

2. METHODOLOGY

2.1 Derivation of Modified Runge-Kutta Method for Solutions of General Second Order Initial Value Problems

Consider the second order IVP of the form

$$\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=g(t), y(d)=\alpha\,, y'(d)=\beta, d\le t\le e \tag{1}$$

where $p(t), q(t),\ g(t)$ are all continuous and differentiable functions of t defined on the interval $[d,e]$.

Let

$$\frac{dy}{dt}=z=f(t,y,z)\,,\ \frac{dz}{dt}=g(t,y,z)\,, y(d)=\alpha\,, y'(d)=\beta, d\le t\le e \tag{2}$$

Let $[d,e]$ be the interval over which we want to find the solution of equation (1). In actually we will not find a differentiable function that satisfies the initial value problem. Instead of, a set points (t_n,y_n,z_n) is generated which are used for approximation. Here means for convenience we subdivide the interval $[d,e]$ into M equal subintervals and select mesh points as follows:

$$t_n=d+hn, \text{for } n=0,1,2,3\ldots\ldots\ldots\ldots\ldots, M \text{ where } h=\frac{(d-e)}{M}$$

From the Taylor series expansion, we get

$$y_{n+1}=y_n+\frac{y_n'(t_{n+1}-t_n)}{1!}+\frac{y_n''(t_{n+1}-t_n)^2}{2!}+\frac{y_n'''(t_{n+1}-t_n)^3}{3!}+\cdots \tag{3}$$

Assuming that $t=t_{n+1}=t_n+h$, then the equation (3) becomes

$$y_{n+1}=y_n+\frac{y_n'h}{1!}+\frac{y_n''h^2}{2!}+\frac{y_n'''h^3}{3!}+\cdots \tag{4}$$

Let $\Delta y=y_{n+1}-y_n$ and $\Delta z=z_{n+1}-z_n$ then

$$\Delta \mathrm{y}=\frac{y_n'h}{1!}+\frac{y_n''h^2}{2!}+\frac{y_n'''h^3}{3!}+\cdots \tag{5}$$

Similarly, we can write

$$\Delta \mathrm{z}=\frac{z_n'h}{1!}+\frac{z_n''h^2}{2!}+\frac{z_n'''h^3}{3!}+\cdots \tag{6}$$

We note that

$$y'=f(t,y,z) \tag{7}$$

$$z'=g(t,y,z) \tag{8}$$

By using chain rule, we have

$$y''=f'=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial t}=f_t+f_yf+f_zg \tag{9}$$

$$z''=g'=\frac{\partial g}{\partial t}+\frac{\partial g}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial g}{\partial z}\frac{\partial z}{\partial t}=g_t+g_yf+g_zg \tag{10}$$

$$y'''=f''=\frac{\partial f'}{\partial t}+\frac{\partial f'}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial f'}{\partial z}\frac{\partial z}{\partial t}$$

$$=(f_{tt}+f_{ty}f+f_{tz}g)+f(f_{ty}+f_{yy}f+f_{yz}g)+f_y(f_t+f_yf+f_zg)+ f_z(g_t+g_yf+g_zg)+g(f_{tz}+f_{yz}f+f_{zz}g) \tag{11}$$

$$z'''=g''=\frac{\partial g'}{\partial t}+\frac{\partial g'}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial g'}{\partial z}\frac{\partial z}{\partial t}$$

$$=(g_{tt}+g_{ty}f+g_{tz}g)+f(g_{ty}+g_{yy}f+g_{yz}g)+g_y(f_t+f_yf+f_zg)+ g_z(g_t+g_yf+g_zg)+g(g_{tz}+g_{yz}f+g_{zz}g) \tag{12}$$

where the subscripts t, y and z designate the differentiation with respect to t, y and z respectively. From equation (5) and (6) we get

$$\Delta y=hf_n+\frac{h^2}{2!}[f_t+f_yf+f_zg]_n+\frac{h^3}{3!}[f_{tt}+f_{ty}f+f_{tz}g+f(f_{ty}+f_{yy}f+ f_{yz}g)+f_y(f_t+f_yf+f_zg)+f_z(g_t+g_yf+g_zg)+g(f_{tz}+f_{yz}f+ f_{zz}g)]_n+\cdots \tag{13}$$

$$\Delta z=hg_n+\frac{h^2}{2!}[g_t+g_yf+g_zg]_n+\frac{h^3}{3!}[g_{tt}+g_{ty}f+g_{tz}g+f(g_{ty}+ g_{yy}f+g_{yz}g)+g_y(f_t+f_yf+f_zg)+g_z(g_t+g_yf+g_zg)+g(g_{tz}+ g_{yz}f+g_{zz}g)]_n+\cdots \tag{14}$$

where the subscript n denotes that the functions to be evaluated at the point (t_n,y_n,z_n). Runge-Kutta was the first to point out that it was possible to avoid the successive differentiation in the Taylor series while preserving the accuracy. The new feature is to set up a problem with undetermined parameters and make the result as higher order as possible by using evaluations of $f(t,y,z)$ and $g(t,y,z)$ within the interval (t_n,y_n,z_n) and $(t_{n+1},y_{n+1},z_{n+1})$. In the other words, the derivatives in the Taylor series are passed by requiring $f(t,y,z)$ and $g(t,y,z)$. Thus, we set up the general single-step equations as follows:

$$y_{n+1}=y_n+\sum_{i=1}^{p}w_i\,k_i$$

$$\therefore \Delta y=\sum_{i=1}^{p}w_i\,k_i \tag{15}$$

$$z_{n+1}=z_n+\sum_{i=1}^{p}w_i\,l_i$$

$$\therefore \Delta z=\sum_{i=1}^{p}w_i\,l_i \tag{16}$$

with the w_i as weighting coefficients to be determined, p as the number of $f(t,y,z)$ and $g(t,y,z)$ substitutions, and the k_i, satisfying the explicit sequences.

$$k_1=hf(t_n,y_n,z_n)$$

$$l_1=hg(t_n,y_n,z_n)$$

$$k_2=hf(t_n+a_1h,y_n+b_1k_1,z_n+c_1l_1)$$

$$l_2=hg(t_n+a_1h,y_n+b_1k_1,z_n+c_1l_1)$$

$$k_3=\ hf(t_n+a_2h,y_n+b_2k_1+b_3k_2,z_n+c_2l_1+c_3l_2)$$

$$l_3=\ hg(t_n+a_2h,y_n+b_2k_1+b_3k_2,z_n+c_2l_1+c_3l_2)$$

$$k_4=\ hf(t_n+a_3h,y_n+b_4k_1+b_5k_2+b_6k_3,z_n+c_4l_1+c_5l_2+c_6l_3)$$

$$l_4=\ hg(t_n+a_3h,y_n+b_4k_1+b_5k_2+b_6k_3,z_n+c_4l_1+c_5l_2+c_6l_3)$$

$$k_5=\ hf(t_n+a_4h,y_n+b_7k_1+b_8k_2+b_9k_3+b_{10}k_4,z_n+c_7l_1+c_8l_2 +c_9l_3+c_{10}l_4)$$

$$l_5=\ hg(t_n+a_4h,y_n+b_7k_1+b_8k_2+b_9k_3+b_{10}k_4,z_n+c_7l_1+c_8l_2 +c_9l_3+c_{10}l_4)$$

$$k_6=\ hf(t_n+a_5h,y_n+b_{11}k_1+b_{12}k_2+b_{13}k_3+b_{14}k_4+b_{15}k_5,z_n+c_{11}l_1 +c_{12}l_2+b_{13}l_3+c_{14}l_4+c_{15}l_5)$$

It can be seen that there are parameters $w_1,w_2,\ldots\ldots\ldots,w_p$ and $l_1,l_2,\ldots\ldots,l_p$ in (16) and $a_1,a_2,a_3,a_4,a_5,\ldots\ldots\ldots, b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8,b_9,b_{10},b_{11},b_{12}$ $, b_{13},b_{14},b_{15},\ldots\ldots\ldots\ldots\ldots\ldots$ and $c_1,c_2,c_3,c_4,c_5,c_6,c_7,c_8,c_9,c_{10},c_{11},c_{12},c_{13},$ $, c_{14},c_{15},\ldots\ldots\ldots$ that must be determined. Each set of parameters when determined will specify the points $f(t,y,z)$ and $g(t,y,z)$ at which $f(t,y,z)$ and $g(t,y,z)$ is to be evaluated. Thus, while the overall calculation yields y_{n+1} and z_{n+1} and it is necessary to evaluate $f(t,y,z)$ and $g(t,y,z)$.

2.2 Derivation of Second Order Runge-Kutta Method for Solution of Second Order Initial Value Problems

We start the simple derivation of order two since it is easier to understand and illustrate the principle involved.

If we take p =2, then equation (15) becomes

$$\Delta y=w_1k_1+w_2k_2 \tag{17}$$

where

$$k_1=hf(t_n,y_n,z_n)$$

$$l_1=hg(t_n,y_n,z_n)$$

$$k_2=hf(t_n+a_1h,y_n+b_1k_1,z_n+c_1l_1) \tag{18}$$

$$l_2=hg(t_n+a_1h,y_n+b_1k_1,z_n+c_1l_1) \tag{19}$$

Since the equation (17) consists of two terms, it is often referred to as a second formula. We now seek to determine the following constants: w_1,w_2,a_1,b_1,c_1

The Taylor series for three independent variables t, y & z about the point (a, b, c) is

$$f(t,y,z) = f(\mathrm{a+h,b+k,c+l}) = f(\mathrm{a,b,c}) + \mathrm{h}f_t(\mathrm{a,b,c}) + \mathrm{k}f_y(\mathrm{a,b,c}) + lf_z(\mathrm{a,b,c}) + \frac{1}{2!}\left[\mathrm{h}^2 f_{tt}(a,b,c) + 2hkf_{ty}(a,b,c) + 2hlf_{tz}(a,b,c) + 2klf_{yz}(a,b,c) + k^2 f_{yy}(a,b,c) + l^2 f_{zz}(a,b,c)\right] + \cdots \quad (20)$$

$$g(t,y,z) = g(\mathrm{a+h,b+k,c+l}) = g(\mathrm{a,b,c}) + \mathrm{h}g_t(\mathrm{a,b,c}) + \mathrm{k}g_y(\mathrm{a,b,c}) + lg_z(\mathrm{a,b,c}) + \frac{1}{2!}\left[\mathrm{h}^2 g_{tt}(a,b,c) + 2hkg_{ty}(a,b,c) + 2hlg_{tz}(a,b,c) + 2klg_{yz}(a,b,c) + k^2 g_{yy}(a,b,c) + l^2 g_{zz}(a,b,c)\right] + \cdots \quad (21)$$

This can be symbolically written as

$$f(t,y,z) = f(\mathrm{a+h,b+k,c+l}) = f(\mathrm{a,b,c}) + \left(\mathrm{h}\frac{\partial}{\partial t} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right) f(a,b,c) + \frac{1}{2!}\left(\mathrm{h}\frac{\partial}{\partial t} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^2 f(a,b,c) + \cdots \quad (22)$$

and $g(t,y,z) = g(\mathrm{a+h,b+k,c+l}) = g(\mathrm{a,b,c}) + \left(\mathrm{h}\frac{\partial}{\partial t} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right) g(a,b,c) + \frac{1}{2!}\left(\mathrm{h}\frac{\partial}{\partial t} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^2 g(a,b,c) + \cdots \quad (23)$

Now

$k_1 = hf_n$

$l_1 = hg_n$

Using (22) into (18) we get,

$$k_2 = hf(t_n + a_1 h, y_n + b_1 k_1, z_n + c_1 l_1)$$

$$= h\left[f(t_n, y_n, z_n) + (a_1 h\frac{\partial}{\partial t} + b_1 k_1\frac{\partial}{\partial y} + c_1 l_1\frac{\partial}{\partial z})f(t_n, y_n, z_n) + \ \dots\dots\dots\right]$$

$$= h\left[f_n + \left(a_1 h\frac{\partial}{\partial t} + b_1 k_1\frac{\partial}{\partial y} + c_1 l_1\frac{\partial}{\partial z}\right)f_n + \ \dots\dots\dots\right]$$

$$= h\left[f_n + \left(a_1 h\frac{\partial}{\partial t} + b_1 hf_n\frac{\partial}{\partial y} + c_1 hg_n\frac{\partial}{\partial z}\right)f_n + \ \dots\dots\dots\right]$$

$$= hf_n + h^2(a_1 f_t + b_1 f_n f_y + c_1 g_n f_z) + 0(h^3)$$

Similarly,

$$l_2 = h\left[g_n + \left(a_1 h\frac{\partial}{\partial t} + b_1 k_1\frac{\partial}{\partial y} + c_1 l_1\frac{\partial}{\partial z}\right)g_n\right] + 0(h^3)$$

$$= hg_n + h^2(a_1 g_t + b_1 f_n g_y + c_1 g_n g_z) + 0(h^3)$$

Introducing the above values into equation (17) we get,

$$\Delta y = h(w_1 + w_2)f_n + h^2\left[a_1 w_2 f_t + b_1 w_2 f f_y + c_1 w_2 g f_z\right]_n + 0(h^3) \quad (24)$$

Also from equation (13) we have,

$$\Delta y = hf_n + \frac{h^2}{2!}\left[f_t + f_y f + g f_z\right]_n + 0(h^3) \quad (25)$$

Equating the corresponding terms between equation (24) and (25) yield

f: $w_1 + w_2 = 1$

f_t: $a_1 w_2 = \frac{1}{2}$

ff_y: $b_1 w_2 = \frac{1}{2}$

gf_z: $c_1 w_2 = \frac{1}{2}$

Here only three equations with four unknowns. If we take w_2 as a free variable, choose $w_2 = \frac{1}{2}$ then it leads to $a_1 = 1, b_1 = 1, c_1 = 1$ and $w_1 = \frac{1}{2}$

Putting the values of $w_1 and w_2$ into (17), we get

$$\Delta y = \frac{1}{2}(k_1 + k_2)$$

$$\Rightarrow y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

where,

$k_1 = hf(t_n, y_n, z_n)$

$l_1 = hg(t_n, y_n, z_n)$

$k_2 = hf(t_n + h, y_n + k_1, z_n + l_1)$

$l_2 = hg(t_n + h, y_n + k_1, z_n + l_1)$

This is the required Runge-Kutta second order method.

2.3 Derivation of Fourth Order Runge-Kutta Method for Solution of Second Order Initial Value Problems

If we take $\mathrm{p} = 4$, then equation (15) becomes

$$\Delta y = w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4 \quad (26)$$

where

$k_1 = hf(t_n, y_n, z_n)$

$l_1 = hg(t_n, y_n, z_n)$

$k_2 = hf(t_n + a_1 h, y_n + b_1 k_1, z_n + c_1 l_1)$

$l_2 = hg(t_n + a_1 h, y_n + b_1 k_1, z_n + c_1 l_1)$

$k_3 = hf(t_n + a_2 h, y_n + b_2 k_1 + b_3 k_2, z_n + c_2 l_1 + c_3 l_2)$

$l_3 = hg(t_n + a_2 h, y_n + b_2 k_1 + b_3 k_2, z_n + c_2 l_1 + c_3 l_2)$

$k_4 = hf(t_n + a_3 h, y_n + b_4 k_1 + b_5 k_2 + b_6 k_3, z_n + c_4 l_1 + c_5 l_2 + c_6 l_3)$

$l_4 = hg(t_n + a_3 h, y_n + b_4 k_1 + b_5 k_2 + b_6 k_3, z_n + c_4 l_1 + c_5 l_2 + c_6 l_3)$

Since the equation consists of four terms, it is often referred to as a fourth order formula. We now seek to determine the following constants: $w_1, w_2, w_3, w_4, a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, b_6$ and $c_1, c_2, c_3, c_4, c_5, c_6$.

The most useful choice is $a_1 = \frac{1}{2}$ and $b_1 = 0,\ c_1 = 0$. Then the solutions for the remaining variables are as follows:

$a_2 = \frac{1}{2},\ a_3 = 1,\ b_2 = 0,\ b_3 = \frac{1}{2},\ b_4 = 0,\ b_5 = 0,\ b_6 = 1,\ c_2 = 0,\ c_3 = \frac{1}{2},\ c_4 = 0,\ c_5 = 0,\ c_6 = 1$

$$w_1 = \frac{1}{6},\ w_2 = \frac{1}{3},\ w_3 = \frac{1}{3},\ w_4 = \frac{1}{6}$$

Substituting the values into the equation (26), we obtain

$$\Delta y = \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\Rightarrow y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$k_1 = hf(t_n, y_n, z_n)$

$l_1 = hg(t_n, y_n, z_n)$

$$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}, z_n + \frac{l_1}{2}\right)$$

$$l_2 = hg(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}, z_n + \frac{l_1}{2})$$

$$k_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}, z_n + \frac{l_2}{2}\right)$$

$$l_3 = hg(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}, z_n + \frac{l_2}{2})$$

$k_4 = hf(t_n + h, y_n + k_3, z_n + l_3)$

$l_4 = hg(t_n + h, y_n + k_3, z_n + l_3)$

This is the required Runge-Kutta fourth order method.

3. Results and Discussion

In this section, we consider four nonlinear initial value problems to verify the proposed formulation which are available in the existing literature. For this we give the results in brief depending on prescribed boundary.

Example 1

We consider the following non-linear initial value problem for equating our algorithm with the current algorithm (Al-khaled and Anwar, 2007)

$$\frac{d^2y}{dx^2} = \frac{8y^2}{1+2x},\ y(0) = 1,\ y'(0) = -2,\ 0 < x < 1 \quad (27)$$

whose exact solution is given by $y(x) = \frac{1}{1+2x}$; using the method discussed in section two, the results are summarized in Table 1.

Table 1: Numerical Results of Example 1

x	Exact values	Modified RK-fourth order	
		Approximate values	Relative error
0.0	1.0000000000000000	1.0000005152061686	5.15206E-07
1/20	0.9090909090909091	0.9090914242970778	5.66727E-07
2/20	0.8333333333333334	0.8333338485395021	6.18247E-07
3/20	0.7692307692307692	0.7692312844369379	6.69768E-07
4/20	0.7142857142857143	0.7142862294918830	7.21289E-07
5/20	0.6666666666666666	0.6666671818728354	7.72809E-07
6/20	0.6250000000000000	0.6250005152061687	8.24330E-07
7/20	0.5882352941176470	0.5882358093238157	8.75850E-07
8/20	0.5555555555555556	0.5555560707617243	9.27371E-07
9/20	0.5263157894736842	0.5263163046798529	9.78892E-07
10/20	0.5000000000000000	0.5000005152061687	1.03041E-06
11/20	0.4761904761904762	0.4761909913966449	1.08193E-06
12/20	0.4545454545454545	0.4545459697516233	1.13345E-06
13/20	0.4347826086956522	0.4347831239018209	1.18497E-06
14/20	0.4166666666666667	0.4166671818728354	1.23649E-06
15/20	0.4000000000000000	0.4000005152061688	1.28802E-06
16/20	0.3846153846153846	0.3846158998215533	1.33954E-06
17/20	0.3703703703703704	0.3703708855765391	1.39106E-06
18/20	0.3571428571428572	0.3571433723490259	1.44258E-06
19/20	0.3448275862068966	0.3448281014130653	1.49410E-06
1.0	0.3333333333333333	0.3333338485395020	1.54562E-06

It is seen that in Table 1, maximum relative error by our proposed method is 1.49×10^{-6} whereas by Al-khaled and Anwar, 2007 is 1.40×10^{-4}.

Example 2

We consider the following non-linear initial value problem for equating our algorithm with the current algorithm (Al-khaled and Anwar, 2007)

$$\frac{d^2y}{dx^2} = -e^{-2y},\ y(0) = 1, y'(0) = \frac{1}{e},\ 0 < x < 1 \quad (28)$$

whose exact solution is given by $y(x) = \ln(x + e)$; using the method discussed in section two, the results are summarized in Table 2.

Table 2: Numerical Results of Example 2

x	Exact values	Modified RK-fourth order	
		Approximate values	Relative error
0.0	1.0000000000000000	1.0000000078938826	7.89388E-09
1/20	1.0182268492122895	1.0182268471442120	2.03105E-09
2/20	1.0361274185748344	1.0361274165434475	1.96056E-09
3/20	1.0537131844573173	1.0537131824613417	1.89423E-09
4/20	1.0709950281849114	1.0709950262231334	1.83174E-09
5/20	1.0879832764774002	1.0879832745486679	1.77277E-09
6/20	1.1046877385103153	1.1046877366135339	1.71703E-09
7/20	1.1211177399311396	1.1211177380652677	1.66429E-09
8/20	1.1372821541259368	1.1372821522899832	1.61433E-09
9/20	1.1531894309988706	1.1531894291918909	1.56694E-09
10/20	1.1688476234983058	1.1688476217193999	1.52194E-09
11/20	1.1842644120979606	1.1842644103462694	1.47915E-09
12/20	1.1994471274194201	1.1994471256941230	1.43841E-09
13/20	1.2144027711628125	1.2144027694631268	1.39961E-09
14/20	1.2291380354952466	1.2291380338204223	1.36260E-09
15/20	1.2436593210313902	1.2436593193807108	1.32728E-09
16/20	1.2579727535271203	1.2579727518998995	1.29353E-09
17/20	1.2720841993952090	1.2720841977907895	1.26125E-09
18/20	1.2859992801414097	1.2859992785591612	1.23036E-09
19/20	1.2997233858098522	1.2997233842491702	1.20078E-09
1.0	1.3132616875182230	1.3132616859785275	1.17242E-09

It is seen that in Table 2, maximum relative error by our proposed method is 2.03×10^{-9} whereas by Al-khaled and Anwar, 2007 is 2.20×10^{-6}.

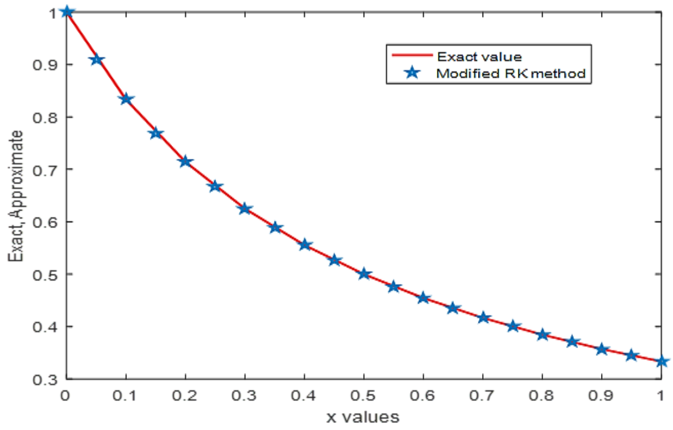

Figure 1: Approximate vs analytical solutions for Example 1

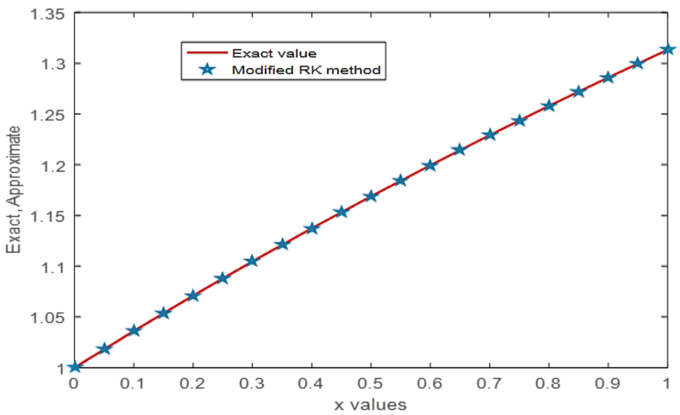

Figure 2: Approximate vs Analytical Solutions for Example 2.

Example 3

We consider the following non-linear initial value problem for equating our algorithm with the current algorithm (Abdelraze, 2008)

$$\frac{d^2y}{dx^2} + e^{-2x}y^3 = 2e^x, y(0) = y'(0) = 1, 0 < x < 1 \tag{29}$$

whose exact solution is given by $y(x) = e^x$; using the method discussed in section two, the results are summarized in Table 3.

Table 3: Numerical Results of Example 3

x	Exact values	Modified RK-fourth order	
		Approximate values	Relative error
0.0	1.0000000000000000	1.0000000078938826	5.46001E-11
1/20	1.0182268492122895	1.0182268471442120	5.46006E-11
2/20	1.0361274185748344	1.0361274165434475	5.46071E-11
3/20	1.0537131844573173	1.0537131824613417	5.46034E-11
4/20	1.0709950281849114	1.0709950262231334	5.46093E-11
5/20	1.0879832764774002	1.0879832745486679	5.46017E-11
6/20	1.1046877385103153	1.1046877366135339	5.46057E-11
7/20	1.1211177399311396	1.1211177380652677	5.46063E-11
8/20	1.1372821541259368	1.1372821522899832	5.46044E-11
9/20	1.1531894309988706	1.1531894291918909	5.46064E-11
10/20	1.1688476234983058	1.1688476217193999	5.46059E-11
11/20	1.1842644120979606	1.1842644103462694	5.46026E-11
12/20	1.1994471274194201	1.1994471256941230	5.46067E-11
13/20	1.2144027711628125	1.2144027694631268	5.46060E-11
14/20	1.2291380354952466	1.2291380338204223	5.46045E-11
15/20	1.2436593210313902	1.2436593193807108	5.46056E-11
16/20	1.2579727535271203	1.2579727518998995	5.46071E-11
17/20	1.2720841993952090	1.2720841977907895	5.46024E-11
18/20	1.2859992801414097	1.2859992785591612	5.46063E-11
19/20	1.2997233858098522	1.2997233842491702	5.46039E-11
1.0	1.3132616875182230	1.3132616859785275	5.46042E-11

It is seen that in Table 3, maximum relative error by our proposed method is 5.46093×10^{-11} whereas by Abdelraze, 2008 is 4.792×10^{-3}.

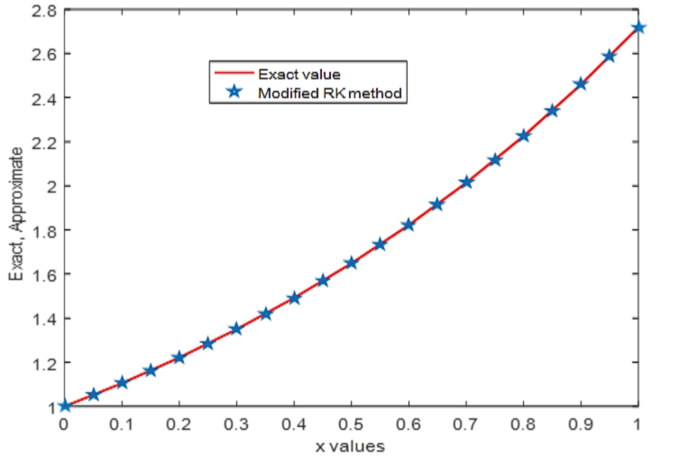

Figure 3: Approximate vs analytical solutions for Example 3

Example 4

We consider the following non-linear initial value problem for equating our algorithm with the current algorithm (Abdelraze, 2008)

$$-\frac{d^2y}{dx^2}+\left(1-\frac{3}{\cosh^2(x)}\right)y+y^3=0,\ \ y(0)=1,\ \ y'(0)=0, 0<x<1 \qquad (30)$$

whose exact solution is given by $y(x)=\text{sech}(x)$; using the method discussed in section two, the results are summarized in Table 4.

Table 4: Numerical Results of Example 4

x	Exact values	Modified RK-fourth order	
		Approximate values	Relative error
0.0	1.0000000000000000	0.9999999688884305	3.11116E-08
1/20	0.9987513007608890	0.9987512696493195	3.11505E-08
2/20	0.9950207489532266	0.9950207178416570	3.12673E-08
3/20	0.9888545124349606	0.9888544813233910	3.14622E-08
4/20	0.9803279976447253	0.9803279665331558	3.17359E-08
5/20	0.9695436291402145	0.9695435980286450	3.20889E-08
6/20	0.9566279119002483	0.9566278807886788	3.25221E-08
7/20	0.9417279294851757	0.9417278983736062	3.30367E-08
8/20	0.9250074519057549	0.9250074207941854	3.36339E-08
9/20	0.9066428345104007	0.9066428033988312	3.43151E-08
10/20	0.8868188839700740	0.8868188528585045	3.50822E-08
11/20	0.8657248513182940	0.8657248202067245	3.59370E-08
12/20	0.8435506876218067	0.8435506565102372	3.68817E-08
13/20	0.8204836682568648	0.8204836371452953	3.79186E-08
14/20	0.7967054599928750	0.7967054288813055	3.90503E-08
15/20	0.7723896738572644	0.7723896427456949	4.02796E-08
16/20	0.7476999182374196	0.7476998871258500	4.16097E-08
17/20	0.7227883423692113	0.7227883112576418	4.30438E-08
18/20	0.6977946411003322	0.6977946099887626	4.45856E-08
19/20	0.6728454778385035	0.6728454467269339	4.62388E-08
1.0	0.6480542736638854	0.6480542425523158	4.80077E-08

It is seen that in Table 4, maximum relative error by our proposed method is 4.62388×10^{-8} whereas by Abdelraze, 2008 is 8.0752×10^{-3}.

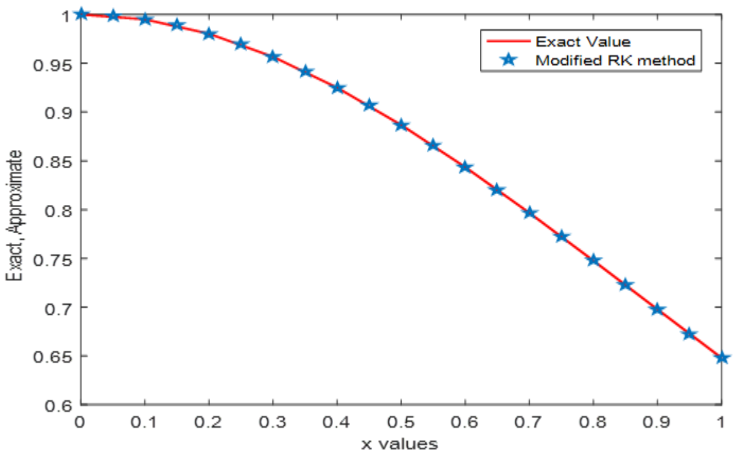

Figure 4: Approximate vs analytical solutions for Example 4.

4. Conclusions

In this work, we have discussed second and fourth order modified Runge-Kutta method for solving second order initial value problems that provides efficient solutions. To achieve the desired accuracy of the numerical solution it is necessary to take step size small. From the tables and figures, we can see that accuracy of the method obtained for decreasing the step size h. It may be concluded that the modified Rung-Kutta method is powerful and more efficient in finding numerical solutions of second order initial value problems. The results obtained in the literature are in excellent agreement with exact solution compared to the existing methods. Our research will be helpful in many scientific areas where numerical computations are needed.

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