Matrix Science Mathematic (MSMK)

DOI: http://doi.org/10.26480/msmk.02.2023.114.121

ZIBELINE INTERNATIONAL™ PUBLISHING

ISSN: 2521-0831 (Print)
ISSN: 2521-084X(Online)
CODEN: MSMAD

CrossMark

RESEARCH ARTICLE

# COMPLEMENTAL BINARY OPERATIONS OF SETS AND THEIR APPLICATION TO GROUP THEORY

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**ARTICLE DETAILS**

***Article History:***

Received 05 July 2023
Revised 19 September 2023
Accepted 03 October 2023
Available online 16 October 2023

**ABSTRACT**

Set theory is considered as the foundation of all mathematics since many mathematical concepts cannot be defined precisely without using set-theoretical concepts. In this study, we define new complemental binary operations, called union complements, intersection left complement, and union right complement and investigate their properties in detail. We contribute to the literature of sets by illuminating the relationships between these complemental binary operations and inclusive\exclusive complements via researching the distribution rules. Moreover, we show that the set of all the sets together with these new complemental binary operations form some algebraic structures. Finally, with the inspiration of these novel concepts, we give an application to group theory as regards subgroups by defining new type of subgroups in order to prompt the reader to think via interesting questions. Since the concept of operations of soft set theory, one of the most popular theory for uncertainty modeling in the past twenty four years, is the crucial notion for developing the theory and since all the types of soft set operations are based on the classical set operations, generation of new complemental binary operations on sets, and thus on soft sets and derivation of their algebraic properties will provide new perspectives for solving problems related to parametric data.

**KEYWORDS**

Sets, Soft Sets, Conditional Complements, Complemental binary operations, Intersection left complement subgroup.

## 1. INTRODUCTION

Set theory is the branch of mathematical logic that studies sets and is one of the greatest achievements of modern mathematics. This branch of mathematics forms a foundation for other topics. All kinds of objects can be grouped into sets, but set theory as a branch of mathematics deals mainly with objects related to mathematics as a whole. Modern research in set theory began in the 1870s with German mathematicians Richard Dedekind and Georg Cantor. Georg Cantor, in particular, is generally regarded as the founder of set theory. Sets are the building blocks of mathematics. The language of sets can be used to describe all mathematical concepts. Linear algebra, graph theory, calculus, and number theory are all areas of mathematics built around sets. Set theory also plays an important role in computer programming. Boolean sets and logic are used in many programming languages. In principle, all mathematical concepts, methods and results can be expressed in axiomatic set theory. Set theory is thus very special in systematizing modern mathematics and in dealing in a unified form with all the fundamental problems of admissible mathematical discourse.

For example, various mathematical structures such as graphs, manifolds, rings, vector spaces, relational algebras, etc. can all be defined as sets satisfying various (axiomatic) properties. Equivalence and order relations are ubiquitous in mathematics, and the theory of mathematical relations can be explained in set theory. So, set theory can be regarded as the basis of mathematical analysis, topology, abstract algebra, and discrete mathematics. Real problems are essentially connected uncertainty and inaccuracy. Especially these problem categories arise in economy, technology environmental sciences, medical sciences, social sciences etc. Over time many mathematical theories such as probability theory, fuzzy set theory, approximate set theory, interval mathematical theory, fuzzy set theory, etc., which are formulated only a partial solution to such problems was found successful. The main reason for the aforementioned difficulties is that the theories are due to the incompleteness of these parameters tools.

To overcome these difficulties, Molodtsov presented the theory of soft sets in a completely new way as mathematical tool with sufficient parameters for processing with uncertainties (Molodtsov, 1999). In addition, this prototype works by Molodtsov reveals rich possibilities for applications of soft set theory, contained directions for further research, especially discoveries new operations with soft sets and their properties. In the past few years, the fundamentals of soft set theory have been studied by various researchers especially in decision making as (Maji et al., 2002; Chen et al., 2003; Chen et al., 2005; Xiao et al., 2005; Mushrif et al., 2006; Herawan and Deris, 2009; Herawan and Deris, 2010; Herawan, 2010; Çağman and Enginoğlu, 2010; Gong et al., 2010; Xiao et al., 2010; Feng et al., 2012; Feng and Zhou, 2014; Dauda et al., 2015; Inthumathi et al., 2017; Atagün et al., 2018; Kamacı et al., 2018; Yang and Yao, 2020; Zorlutuna, 2021). A group researchers provided a detailed theoretical study of soft sets involving the subset and soft set superset, soft set equality, operations on soft sets such as intersection, AND-product and OR-product and

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Cite The Article: Aslıhan Sezgin, Naim Çağman, Akın Osman Atagün, Fitnat Nur Aybek (2023). Complemental Binary Operations of Sets and Their Application to Group Theory. *Matrix Science Mathematic, 7(2): 114-121.*

discussed the main characteristics of these operations (Maji et al., 2003). Pei and Miao redefined the intersection of subset and Ali et al. introduced some new operations as restricted union, restricted intersection, restricted difference and extended intersection of two soft sets and discussed their basic characteristics (Pei and Miao, 2005; Ali et al., 2009).

After then, the authors pointed out several gaps as regards basic operations in the existing literature also received new results by clarifying some conceptual misunderstandings of the fundamentals of soft set theory (Yang, 2008; Feng et al., 2010; Jiang et al., 2010; Ali et al., 2011; Sezgin and Atagün, 2011; Neog and Sut, 2011; Fu, 2011; Ge and Yang, 2011; Tridiv and Dusmanta, 2011; Singh and Onyeozili, 2012; Singh and Onyeozili, 2012; Singh and Onyeozili, 2012; Zhu and Wen, 2013; Feng and Li, 2013; Sen, 2014; Onyeozili et al., 2014; Husain and Shivani, 2018). In the recent years, studies on soft sets have been progreesing rapidly. Sezgin et al. introduced extended difference of soft sets and Stojanovic defined extended symmetric difference of soft sets and obtained its basic properties (Sezgin et al., 2019; Stojanovic, 2021). Since soft set operations are key concepts in soft set theory, soft set operations have been widely studied since 2003. Ali et al. defined a new complement for soft set theory, called relative complement and Neog and Sut showed that the laws of exclusion and contradiction, involution, De Morgan inclusions and De Morgan laws are valid for soft sets with respect to the definition of complement (Ali et al., 2009; Neog and Sut, 2011). They argued that the definition of the relative complement proposed by Ali et al actually should be definition of the complement of a soft set (Ali et al., 2009).

In a study, author defined two conditional complements of a set as new concepts of set theory, i.e., inclusive complement and exclusive complement were defined, the relationships between the two were investigated and complement subgroup of a group was introduced and interesting two questions about it were given to reader (Çağman, 2021). Since soft set operations, the crucial concept of soft set theory, are based on the classical soft set operations, by being inspired with the study, Sezgin and Sarıalioğlu, Sezgin and Demircioğlu, Sezgin and Çağman, Sezgin and Aybek, Sezgin et al., and Yavuz, defined a new type of soft set operation, called complementary soft piecewise operation for soft sets and introduced its basic properties (Çağman, 2021; Sezgin and Sarialioglu, 2023; Sezgin and Demirci, 2023; Sezgin and Çağman, 2023; Sezgin and Aybek, 2023; Sezgin et al., 2023; Sezgin and Yavuz, 2023).

In this paper, we define several complemental binary operations of sets as novel concepts in set theory. Our goal is to contribute to the set literature by elucidating the relationships between these complemental binary operations and others previously defined as conditional complements and our ulitimate aim is to contribute to the soft theory to introduce new soft set operations with the inspiration of these new complemental binary operations. We also show that the set of all sets together with complemental binary operations form some algebraic structures. Furthermore, we give an application by the aforesaid new complements to group theory as regards subgroup perspective by defining intersection left complement subgroup and symmetric difference subgroup of groups, inviting the reader to consider interesting questions. Note that in set theory, complement is a unary operation on the set of sets; however the complements defined in this paper are all binary operations on the set of sets like inclusive complement and exclusive complement.

This paper is arranged in the following manner. In Section 2, we recall preliminary concepts in set theory together with some algebraic structures. In Section 3, we remind the conditional complements defined in Çağman, 2021 and define several complemental binary operations of sets. Particularly, in Section 4, more on the operation of union left complement of sets (called as "inclusive complement"); in Section 5, more on the operation of intersection complements of sets (called as "exclusive complement"); in Section 6, operation of union complements and its properties; in Section 7, operation of intersection left complement and its properties; in Section 8, operation of union right complement and its properties are investigated in detail and we show which algebraic structures they form in the set of sets. In Section 9 and Section 10, we give some applications of these complemental binary operations in group theory and define intersection left complement of subgroup and symmetric difference subgroup, respectively. In the last section, we propose four interesting questions as regards these aforesaid new subgroups to prompt the reader to think via interesting questions. In the conclusion section, we put into focus the meaning of the study's findings and its potential influence on the field.

## 2. Preliminaries

In set theory, a universal set is a set that contains all objects, including itself. Throughout this paper, U is the universal set. The complement of a set A is the set of elements not contained in A. In other words, let U be the set containing all the elements under consideration. A′ denotes the complement of A, hence $A'=U\setminus A$ or formally $A'=\{x \in U : x \notin A\}$. For the sake of illustration, let $\mathcal{H}$ denote the set of sets that we consider in set theory. Namely, the elements of $\mathcal{H}$ are all sets. We can comfortably say that complement operation is a unary operation on $\mathcal{H}$.

In abstract algebra, a magma or rarely groupoid is a basic kind of algebraic structure (Bergman, 2011). Specifically, a magma consists of a set equipped with a single binary operation that must be closed by definition. No other properties are imposed. Let (S, •) be an algebraic structure. If (S, •) has more than one left identity, then it has no right identity.

Let (S,•) be a groupoid. An element e∈S is called a left identity element if e•s=s for all s∈S. Similiarly, e is a right identity element is s•e=x for all e∈S. An element which is both a left and a right identity is an identity element. A groupoid may have more than one left identify element: in fact the operation defined by x*y=y for all x, y∈S defines a groupoid (in fact, a semigroup) on any set S, and every element is a left identity. But as soon as a groupoid has both a left and a right identity, they are necessarily unique and equal. For if e is a left identity and f is a right identity, then f=e•f=e (Howie, 1995).

Let (S,•) be a groupoid. An absorbing (annihilating/zero) element is an element z such that for all s in S, z • s=s • z=z. This notion can be refined to the notions of left absorbing, where one requires only that z • s=z, and right absorbing, where s • z=z. As a similar case for identity element, if a groupoid has both a left absorbing element z and a right absorbing element z′, then it has an absorbing element, since z=z • z′=z′ (Kilp et al., 2001).

## 3. Conditional Complements and Complemental BINARY Operations of Sets

**Definition 3.1.** Let A and L be two subsets of U. Then, L-inclusive complement of A is defined by $A^{+L}=A'\cup L$ (Çağman,2021).

For the sake of illustration, we prefer to use A+L rather than of $A^{+L}$ and "Union left complement of A and L " rather than "L-inclusive complement of A". Hence,

$A^{+L}=A+L=(A\setminus L)'=A'\cup L$

**Definition 3.2.** Let A and L be two subsets of U. Then, L-exclusive complement of A is defined by $A^{-L}=A'\cap L'$ (Çağman,2021).

For the sake of illustration, we prefer to use AθL rather than $A^{-L}$ and "Intersection left and right complement of A and L " rather than "L-exlusive complement of A". Hence,

$A^{-L}=A\theta L=(A\cup L)'=A'\cap L'$

For the ease of writing, throughout the paper, we use "Intersection complements of A and L " instead of "Intersection left and right complement of A and L ".

**Definition 3.3.** Let A and L be two subsets of U. "Union left and right complemet of A and L ", denoted by A ∗ L, is defined by

$A * L=(A\cap L)'=A'\cup L'$

If B={x}, then we will use A ∗ x instead of A ∗ {x}.

For the ease of writing, throughout the paper we use "Union complements of A and L " instead of "Union left and right complement of A and L ".

**Definition 3.4.** Let A and L be two subsets of U. "Union right complement of A and L ", denoted by AλL, is defined by

$A\lambda L=(L\setminus A)'=A\cup L'$

If B={x}, then we will use Aλx instead of Aλ{x}.

**Definition 3.5.** Let A and L be two subsets of U. "Intersection left complemet of A and L ", denoted by AɣL, is defined by

$A\gamma L=(A\lambda L)'=A'\cap L$

If B={x}, then we will use Aλx instead of Aλ{x}.

## 4. More on the Operation of Union Left Complement of Sets

**Proposition 4.1.** The set $\mathcal{H}$ is closed under the operation of union left complement. That is, when A and L are two sets, then so is A+L.

**Proof:** It is clear that the operation of union left complement is a well-defined binary operation in $\mathcal{H}$.

$+ : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$

$(A, L) \to A + L = A' \cup L$

Here, note that, complement of a set is a unary operation on $\mathcal{H}$; however the operation of union left complement is a binary operation on $\mathcal{H}$.

**Proposition 4.2.** The operation of union left complement is not associative on the set $\mathcal{H}$.

**Proposition 4.3.** (Çağman, 2021)

i) A+A=U ii) U+A=A iii) A'+A=A iv) ∅+A=U v) A+U=U

vi) A+∅=A' vii) A+A'=A'

**Corollary 4.4.** By Proposition 4.3. (i), the operation of union left complement does not have idempotency property on the set $\mathcal{H}$. By Proposition 4.3. (ii), U is the left-identity and by Proposition 4.3. (iii), complement of a set is its own left-identity element for the operation of union left complement on the set $\mathcal{H}$ . Moreover by Proposition 4.3. (v), U is the right-absorbing element and by Proposition 4.3. (vii), complement of a set is its own right-absorbing element for the operation of union left complement on the set $\mathcal{H}$. Hence, $(\mathcal{H}, +)$ is a magma (groupoid) with left identies.

**Proposition 4.5.** L⊆A+L

**Corollary 4.6** i) (A+L)∩L=L iii) (A+L)∪L=A+L ii) L\(A+L)=∅

**Proposition 4.7.** (A+L)∩(L+A)=(A△L)'

**Proof:**(A+L)∩(L+A)=(A'∪L)∩(L'∪A)=(A∩L')'∩(L∩A')'=[(A∩L')∪(L∩A')]'=[(A\L)∪(L\A)]'=(A△L)'

**Proposition 4.8**. A+ L≠L+A but A'+L=L'+A.

### 4.1 Distribution Laws

#### *4.1.1 Distributions of Intersection, Union and Difference Operations Over the Operation of Union Left Complement.*

i. A∩(L+C)=(A\L)∪(A∩C) and (A+L)∩C=(AγL)∪(L∩C)

ii. A∪(L+C)=(AλL)∪(A∪C) and (A+L)∪C=(A+C)∪(L∪C)

iii. A\(L+C)=(A∩L)∩(A\C) and (A+L)\C=(AθC)∪(L\C)

**Proof:** We give the proofs for left distributions; right distributions can be shown similarly.

i. A∩(L+C)=A∩(L'∪C)=(A∩L')∪(A∩C)=(A\L)∪(A∩C)

ii. A∪(L+C)=A∪(L'∪C)=(A∪L')∪(A∪C)=(AλL)∪(A∪C)

iii. A\(L+C)=A∩(L+C)'=A∩(L'∪C)'=A∩(L∩C')=(A∩L)∩(A∩C')=(A∩L)∩(A\C)

#### *4.1.2 Distributions of The Operation of Union Left Complement Over Other Set Operations:*

i. A+(L∩C)=(A+L)∩(A+C) and (A∩L)+C=(A+C)∪(L+C) [Çağman, 2021] (left distributive holds)

ii. A+(L∪C)=(A+L)∪(A+C) and (A∪L)+C=(A+C)∩(L+C) [Çağman, 2021]. (left distibutive holds)

iii. A+(L+C)=(A∗L)∪(A+C) and (A+L)+C=(A∪C)∩(L+C)

iv. A+(L\C)=(A+L)∩(A∗C) and (A\L)+C=(A+C)∪(L∪C)

v. A+(LθC)=(A∗L)∩(A∗C) and (AθL)+C=(A∪C)∪(L∪C)

vi. A+(L∗C)=(A∗L)∪(A∗C) and (A∗L)+C=(A∪C)∩(L∪C)

vii. A+(LɣC)=(A∗L)∩(A+C) and (AɣL)+C=(A∪C)∪(L+C)

viii. A+(LλC)=(A+L)∪(A∗C) and (AλL)+C=(A+C)∩(L∪C)

**Proof:** We give the proofs for left distributions; right distributions can be shown similarly.

iii. A+(L+C)=A+ (L'∪C)=A'∪(L'∪C)=(A'∪L')∪(A'∪C)=(A∗L)∪(A+C)

iv. A+(L\C)=A'∪(L\C)=A'∪(L∩C')=(A'∪L)∩(A'∪C')=(A+L)∩(A∗C)

v. A+(LθC)=A'∪(L'∩C')=(A'∪L')∩(A'∪C')=(A∗L)∩(A∗C)

vi. A+(L∗C)=A'∪(L'∪C')=(A'∪L')∪ (A'∪C')=(A∗L)∪ (A∗C)

vii. A+(LɣC)=A'∪(L'∩C)=(A'∪L')∩(A'∪C)=(A∗L)∩(A+C)

viii. A+(LλC)=A'∪(L∪C')=(A'∪L)∪ (A'∪C')=(A+L)∪ (A∗C)

## 5. More on the Operation of Intersection Complements of Sets

**Proposition 5.1.** The set $\mathcal{H}$ is closed under the operation of intersection complements. That is, when A and L are two sets, then so is AθL.

**Proof:** It is clear that the operation of intersection complements is a well-defined binary operation in $\mathcal{H}$. Hence,

$\theta: \mathcal{H} \times \mathcal{H} \to \mathcal{H}$

$(A,L) \to A\theta L = A' \cap L'$

Thus, the set $\mathcal{H}$ is closed under the operation of intersection complements.

**Proposition 5.2.** The operation of intersection complements is not associative on the set $\mathcal{H}$.

**Proposition 5.3.** (Çağman, 2021)

i) AθA=A' ii) Aθ∅=∅θA=A' iii) UθA=AθU=∅ iv) AθA'=A'θA=∅ v) AθL=LθA.

**Corollary 5.4.** By Proposition 5.3 (i) the operation of intersection complements does not have idempotency property on the set $\mathcal{H}$. Since, the operation of intersection complements is closed and commutative; $(\mathcal{H}, \theta)$ is a commutative magma (gropoid).

**Proposition 5.5.** AθL=A'\L=L'\A, (AθL)∩L=∅ and so (AθL)\L=AθL

**Corollary 5.6.** (AθL)\(LθA)=∅ and (AθL)△(LθA)=∅. (By Proposition 5.3. (v))

**Proposition 5.7**. Let A be a subset of U. Then,

i. $x \in A \Rightarrow x\theta A = A\theta x = A'$

ii. $x \notin A \Rightarrow x\theta A = A\theta x = A' - \{x\}$

**Distribution Laws**

#### *5.1.1 Distributions of Intersection, Union and Difference Operations Over the Operation of Intersection Complements*

i. A∩(LθC)=(A\L)∩(A\C) and (AθL)∩C=(AɣC)∩(LɣC)

ii. A∪(LθC)=(AλL)∩(AλC) and (AθL)∪C=(A+C)∩(L+C)

iii. A\(LθC)=(A∩L)∪(A∩C) and (AθL)\C=(AθC)∩(LθC)

**Proof:** We give the proofs for left distributions; right distributions can be shown similarly.

i. A∩(LθC)=A∩(L'∩C')=(A∩L')∩(A∩C')=(A\L)∩(A\C)

ii. A∪(LθC)=A∪(L'∩C')=(A∪L')∩(A∪C')=(AλL)∩(AλC)

iii. A\(LθC)=A∩(LθC)'=A∩(L'∩C')'=A∩(L∪C)=(A∩L)∪(A∩C)

#### *5.1.2 Distributions of The Operation of Intersection Complements Over Other Set Operations*

i. Aθ(L∩C)=(AθL)∪ (AθC) and (A∩L)θC=(AθC)∪(LθC) (Çağman, 2021).

ii. Aθ(L∪C)=(AθL)∩(AθC) and (A∪L)θC=(AθC)∩(LθC) (Çağman, 2021).

iii. Aθ(L+C)=(AɣL)∩(AθC) and (A+L)θC=(A\C)∩(LθC)

iv. Aθ(L\C)=(AθL)∪ (AɣC) and (A\L)θC=(AθC)∪(L\C)

v. Aθ(LθC)=(AɣL)∪ (AɣC) and (AθL)θC=(A\C)∪(L\C)

vi. Aθ(L∗C)=(AɣL)∩(AɣC) and (A∗L)θC=(A\C)∩(L\C)

vii. Aθ(LɣC)=(AɣL)∪(AθC) and (AɣL)θC=(A\C)∪(LθC)

viii. Aθ(LλC)=(AθL)∩(AɣC) and (AλL)θC=(AθC)∩(L\C)

**Proof:** We give the proofs for left distributions; right distributions can be shown similarly.

iii. $A\theta(L+C)=A'\cap(L+C)'=A'\cap(L'\cup C)'=A'\cap(L\cap C')=(A'\cap L)\cap(A'\cap C')=(A\gamma L)\cap(A\theta C)$

iv. $A\theta(L\setminus C)=A'\cap(L\setminus C)'=A'\cap(L\cap C')'=A'\cap(L'\cup C)=(A'\cap L')\cup(A'\cap C)=(A\theta L)\cup(A\gamma C)$

v. $A\theta(L\theta C)=A'\cap(L\theta C)'=A'\cap(L'\cap C')'=A'\cap(L\cup C)=(A'\cap L)\cup(A'\cap C)=(A\gamma L)\cup(A\gamma C)$

vi. $A\theta(L\ast C)=A'\cup(L\ast C)'=A'\cup(L'\cup C')'=A'\cup(L\cap C)=(A'\cap L)\cap(A'\cap C)=(A\gamma L)\cap(A\gamma C)$

vii. $A\theta(L\gamma C)=A'\cap(L\gamma C)'=A'\cap(L'\cap C)'=A'\cap(L\cup C')=(A'\cap L)\cup(A'\cap C')=(A\gamma L)\cup(A\theta C)$

viii. $A\theta(L\lambda C)=A'\cup(L\lambda C)'=A'\cup(L\cup C')'=A'\cup(L'\cap C)=(A'\cap L')\cap(A'\cap C)=(A\theta L)\cap(A\gamma C)$

## 6. Operation of Union Complements and Its Properties

**Proposition 6.1.** The set $\mathcal{H}$ is closed under the operation of union complements. That is, when A and L are two sets, then so is $A\ast L$.

**Proof:** It is clear that $\ast$ is a well-defined binary operation in $\mathcal{H}$. Hence,

$\ast:\mathcal{H}\text{ x }\mathcal{H}\rightarrow\mathcal{H}$

$(A, L)\rightarrow A\ast L=A'\cup L'$

**Proposition 6.2.** The operation of union complements is not associative on the set $\mathcal{H}$, but it is commutative.

**Corollary 6.3. i)** $(A\ast L)\setminus(L\ast A)=\emptyset$ **ii)** $(A\ast L)\triangle(L\ast A)=\emptyset$ **iii)** $A\ast(L\ast C)=A\ast(C\ast L)$ **iv)** $(A\ast L)\ast C=(L\ast A)\ast C$

**Proposition 6.4.**

i) $A\ast A=A'$  ii) $U\ast A=A\ast U=A'$  iii) $A\ast\emptyset=\emptyset\ast A=U$  iv) $A\ast A'=A'\ast A=U$

**Corollary 6.5.** By Proposition 6.4. (i), the operation of union complements does not have idempotency property on the set $\mathcal{H}$. Since, the operation of union complements is closed and commutative; $(\mathcal{H},\ast)$ is a commutative magma (groupoid).

**Proposition 6.6**. Let A be a subset of U. Then,

i. $x\in A\Rightarrow x\ast A=A\ast x=\{x\}'$

ii. $x\notin A\Rightarrow x\ast A=A\ast x=U$

**Proposition 6.7.** i) $(A\ast L)\ast C=(A\lambda C)\cap(L\lambda C)$ ii) $(A\ast C)\ast L=(A\lambda L)\cap(C\lambda L)$

**Remark 6.8.** $(A\ast L)\ast C\neq(A\ast C)\ast L$

### 6.1 Distribution Laws

#### *6.1.1 Distributions of Intersection, Union and Difference Operations Over the Operation of Union Complements*

i. $A\cap(L\ast C)=(A\setminus L)\cup(A\setminus C)$ and $(A\ast L)\cap C=(A\gamma C)\cup(L\gamma C)$

ii. $A\cup(L\ast C)=(A\lambda L)\cup(A\lambda C)$ and $(A\ast L)\cup C=(A+C)\cup(L+C)$

iii. $A\setminus(L\ast C)=(A\cap L)\cap(A\cap C)$ and $(A\ast L)\setminus C=(A\theta C)\cup(L\theta C)$

**Proof:** We give the proofs for left distributions; right distributions can be shown similarly.

i. $A\cap(L\ast C)=A\cap(L'\cup C')=(A\cap L')\cup(A\cap C')=(A\setminus C)\cup(L\setminus C)$

ii. $A\cup(L\ast C)=A\cup(L'\cup C')=(A\cup L')\cup(A\cup C')=(A\lambda C)\cup(L\lambda C)$

iv. $A\setminus(L\ast C)=A\cap(L'\cup C')'=A\cap(L\cap C)=(A\cap L)\cap(A\cap C)$

#### *6.1.2 Distributions of The Operation of Union Complements Over Other Set Operations*

i. $A\ast(L\cap C)=(A\ast L)\cup(A\ast C)$ and $(A\cap L)\ast C=(A\ast C)\cup(L\ast C)$

ii. $A\ast(L\cup C)=(A\ast L)\cap(A\ast C)$ and $(A\cup L)\ast C=(A\ast C)\cap(L\ast C)$

iii. $A\ast(L\setminus C)=(A\ast L)\cup(A+C)$ and $(A\setminus L)\ast C=(A\ast C)\cup(L\lambda C)$

iv. $A\ast(L+C)=(A+L)\cap(A\ast C)$ and $(A+L)\ast C=(A\lambda C)\cap(L\ast C)$

v. $A\ast(L\theta C)=(A+L)\cup(A+C)$ and $(A\theta L)\ast C=(A\lambda C)\cup(L\lambda C)$

vi. $A\ast(L\ast C)=(A+L)\cap(A+C)$ and $(A\ast L)\ast C=(A\lambda C)\cap(L\lambda C)$

vii. $A\ast(L\gamma C)=(A+L)\cup(A\ast C)$ and $(A\gamma L)\ast C=(A\lambda C)\cap(L\ast C)$

viii. $A\ast(L\lambda C)=(A\ast L)\cap(A+C)$ and $(A\lambda L)\ast C=(A\ast C)\cap(L\lambda C)$

**Proof:** We give the proofs for left distributions; right distributions can be shown similarly.

i. $A\ast(L\cap C)=A'\cup(L\cap C)'=A'\cup(L'\cup C')=(A'\cup L')\cup(A'\cup C')=(A\ast L)\cup(A\ast C)$

ii. $A\ast(L\cup C)=A'\cup(L\cup C)'=A'\cup(L'\cap C')=(A'\cup L')\cap(A'\cup C')=(A\ast L)\cap(A\ast C)$

iii. $A\ast(L\setminus C)=A'\cup(L\cap C')'=A'\cup(L'\cup C)=(A'\cup L')\cup(A'\cup C)=(A\ast L)\cup(A+C)$

iv. $A\ast(L+C)=A'\cup(L'\cup C)'=A'\cup(L\cap C')=(A'\cup L)\cap(A'\cup C')=(A+L)\cap(A\ast C)$

v. $A\ast(L\theta C)=A'\cup(L'\cap C')'=A'\cup(L\cup C)=(A'\cup L)\cup(A'\cup C)=(A+L)\cup(A+C)$

vi. $A\ast(L\ast C)=A'\cup(L'\cup C')'=A'\cup(L\cap C)=(A'\cup L)\cap(A'\cup C)=(A+L)\cap(A\ast C)$

vii. $A\ast(L\gamma C)=A'\cup(L'\cap C)'=A'\cup(L\cup C')=(A'\cup L)\cup(A'\cup C')=(A+L)\cup(A+C)$

viii. $A\ast(L\lambda C)=A'\cup(L\cup C')'=A'\cup(L'\cap C)=(A'\cup L')\cap(A'\cup C)=(A\ast L)\cap(A+C)$

## 7. Operation of Intersection Left Complement and its Properties

**Proposition 7.1.** The set $\mathcal{H}$ is closed under the operation of intersection left complement. That is, when A and L are two sets, then so is $A\gamma L$.

**Proof:** It is clear that the operation of intersection left complement is a well-defined binary operation in $\mathcal{H}$. Hence,

$\gamma:\mathcal{H}\text{ x }\mathcal{H}\rightarrow\mathcal{H}$

$(A, L)\rightarrow A\gamma L=A'\cap L$

Thus, the set $\mathcal{H}$ is closed under the operation of intersection left complement.

**Proposition 7.2.** The operation of intersection left complement is not associative on the set $\mathcal{H}$.

**Proposition 7.3.**

i) $A\gamma A=\emptyset$  ii) $\emptyset\gamma A=A$  iii) $A'\gamma A=A$  iv) $A\gamma U=A'$

v) $A\gamma\emptyset=\emptyset$  vi) $A\gamma A'=A'$  vii) $U\gamma A=\emptyset$

**Corollary 7.4.** By Proposition 7.3.(i), the operation of intersection left complement does not have idempotency property on the set $\mathcal{H}$. By Proposition 7.3 (ii); $\emptyset$ is the left-identity and by Proposition 7.3 (iii) complement of a set is its own left-identity element for the operation of intersection left complement on the set $\mathcal{H}$. Moreover, by Proposition 7.3 (v), $\emptyset$ is the right-absorbing element and by Proposition 7.3 (vi), complement of a set is its own right-absorbing element for the operation of intersection left complement on the set $\mathcal{H}$. $(\mathcal{H},\gamma)$ is a magma (groupoid) with left identies.

**Proposition 7.5.** $A\gamma L=L\setminus A$ and $A\gamma L\subseteq L$.

**Corollary 7.6** i) $(A\gamma L)\cap L=A\gamma L$  iii) $(A\gamma L)\cup L=L$  ii) $(A\gamma L)\setminus L=\emptyset$

**Proposition 7.7** $(A\gamma L)\cap L'=\emptyset$, $(A\gamma L)'=A'+L'$, $A\gamma L\neq L\gamma A$ but $A'\gamma L=L'\gamma A$.

**Proposition 7.8**. Let A be a subset of U. Then,

i. $x\in A\Rightarrow x\gamma A=A-\{x\}$

ii. $x\notin A\Rightarrow x\gamma A=A$

iii. $x\in A\Rightarrow A\gamma x=\emptyset$

iv. $x\notin A\Rightarrow A\gamma x=\{x\}$

**Proposition 7.9** $A\gamma(L\gamma C)\neq A\gamma(C\gamma L)$ and $(A\gamma L)\gamma C\neq(A\gamma C)\gamma L$

### 7.1 Distribution Laws

#### *7.1.1 Distributions of Intersection, Union And Difference Operations Over The Operation Of Intersection Left Complement*

i. A∩(LɣC)=(A∩L)ɣ(A∩C) and (AɣL)∩C=(A∩C)ɣ(L∩C) (complete distribution holds)

ii. A∪(LɣC)=(AλL)∩(A∪C) and (AɣL)∪C=(A+C)∩(L∪C)

iii. A\(LɣC)=(A∩L)∪(A\C) and (AɣL)\C=(AθC)∩(L\C)

**Proof:** We give the proofs for left distributions; right distributions can be shown similarly.

i. A∩(LɣC)=A∩(C\L)=(A∩C)\(A∩L)=(A∩L)ɣ(A∩C)

ii. A∪(LɣC)=A∪(L'∩C)=(A∪L')∩(A∪C)=(AλL)∩(A∪C)

iii. A\(LɣC)=A∩(LɣC)'=A∩(L'∩C)'=A∩(L∪C')=(A∩L)∪(A∩C')=(A∩L)∪(A\C)

#### *7.1.2 Distributions of The Operation of Intersection Left Complement Over Other Set Operations*

i. Aɣ(L∩C)=(AɣL)∩(AɣC) and (A∩L)ɣC=(AɣC)∪(LɣC) (left distribution holds)

ii. Aɣ(L∪C)=(AɣL)∪(AɣC) and (A∪L)ɣC=(AɣC)∩(LɣC) (left distribution holds)

iii. Aɣ(L\C)=(AɣL)\(AɣC) and (A\L)ɣC=(AɣC)∪(L∩C) (left disribution holds)

iv. Aɣ(L+C)=(AθL)∪(AɣC) and (A+L)ɣC=(A∩C)∩(LɣC)

v. Aɣ(LθC)=(AθL)∩(AθC) and (AθL)ɣC=(A∩L)∪(L∩C)

vi. Aɣ(L ∗ C)=(AθL)∪(AθC) and (A ∗ L)ɣC=(A∩C)∩(L∩C)

vii. Aɣ(LɣC)=(AθL)∩(AɣC) and (AɣL)ɣC=(A∩C)∪(LɣC)

viii. Aɣ(LλC)=(AɣL)∪(AθC) and (AλL)ɣC=(AɣC)∩(L∩C)

**Proof:** We give the proofs for left distributions; right distributions can be shown similarly.

i. Aɣ(L∩C)=A'∩(L∩C)=(A'∩L)∩(A'∩C)=(AɣL)∩(AɣC)

ii. Aɣ(L∪C)=A'∩(L∪C)=(A'∩L)∪(A'∩C)=(AɣL)∪(AɣC)

iii. Aɣ(L\C)=A'∩(L\C)=(A'∩L)\(A'∩C)=(AɣL)\(AɣC)

iv. Aɣ(L+C)=A'∩(L′∪C)=(A'∩L')∪(A'∩C)=(AθL)∪(AɣC)

v. Aɣ(LθC)=A'∩(L′∩C′)=(A'∩L')∩(A'∩C′)=(AθL)∩(AθC)

vi. Aɣ(L ∗ C)=A'∩(L′∪C′)=(A'∩L')∪(A'∩C′)=(AθL)∪(AθC)

vii. Aɣ(LɣC)=A'∩(L′∩C)=(A'∩L')∩(A'∩C)=(AθL)∩(AɣC)

viii. Aɣ(LλC)=A'∩(L∪C′)=(A'∩L)∪(A'∩C′)=(AɣL)∪(AθC)

## 8. Operation of Union Right Complement and its Properties

**Proposition 8.1.** The set 𝓗 is closed under the operation of union right complement. That is, when A and L are two sets, then so is AλL.

**Proof:** It is clear that the operation of union right complement is a well-defined binary operation in 𝓗. Hence,

λ: 𝓗 x 𝓗 → 𝓗

(A, L) → AλL=A∪L'

**Proposition 8.2.** The operation of union right complement is not associative on the set 𝓗.

**Proposition 8.3.**

i) AλA=U ii) AλU=A iii) AλA'=A iv) ∅λA=A′

v) Aλ∅=U vi) UλA=U vii) A'λA=A'

**Corollary 8.4** By Proposition 8.3. (i), the operation of union right complement does not have idempotency property on the set 𝓗. By Proposition 8.3. (ii), U is the right-identity and by Proposition 8.3. (iii), complement of a set is its own right-identity element for the operation of union right complement on the set 𝓗. Moreover, by Proposition 8.3. (iv), ∅ is the left-absorbing element and by Proposition 8.3. (vi), complement of a set is its own left-absorbing element for the operation of union right complement on the set 𝓗. (𝓗, λ) is a magma (groupoid) with right identies.

**Proposition 8.5**. AλL=A∪L'=L'∪A=L+A, Aλ L≠Lλ A but A'λ L=L'λ A.

**Proposition 8.6.** Aλ(LλC)≠ Aλ(CλL) and (AλL)λ C≠(AλC)λ L

**Proposition 8.7**. Let A be a subset of U. Then,

i. x ∉ A ⇒ x λA = A′

ii. x∈ A ⇒ Aλx = U

iii. x ∉ A ⇒ A λx = {x}′

### 8.1 Distribution Laws

#### *8.1.1 Distributions of Intersection, Union and Difference Operations Over The Operation of Union Right Complement*

i. A∩(LλC)=(A∩L)∪(A\C) and (AλL)∩C=(A∩C)∪(LɣC)

ii. A∪(LλC)=(A∪L)∪(AλC) and (AλL)∪C=(A∪L)∪(L+C)

iii. A\(LλC)=(A\L)∩(A∩C) and (AλL)\C=(A\C)∪(LθC)

**Proof:** We give the proofs for left distributions; right distributions can be shown similarly.

i. A∩(LλC)=A∩(L∪C')=(A∩L)∪(A∩C')=(A∩L)∪(A\C)

ii. A∪(LλC)=A∪(L∪C')=(A∪L)∪(A∪C')=(A∪L)∪(AλC)

iii. A\(LλC)=A∩(L∪C')'=A∩(L'∩C)=(A∩L')∩(A∩C)=(A\L)∩(A∩C)

#### *8.1.2 Distributions of The Operation of Union Right Complement Over Other Set Operations:*

i. Aλ(L∩C)=(AλL)∪(AλC) and (A∩L)λC=(AλC)∩(LλC) (right distribution holds)

ii. Aλ(L∪C)=(AλL)∩(AλC) and (A∪L)λC=(AλC)∪(LλC) (right distribution holds)

iii. Aλ(L\C)=(AλL)∪(A∪C) and (A\L)λ C=(AλC)∩(L ∗ C)

iv. Aλ(LλC)=(AλL)∩(A∪C) and (AλL)λ C=(AλC)∪(L ∗ C)

v. Aλ(L+C)=(A∪L)∩(AλC) and (A+L)λ C=(A ∗ C)∪(LλC)

vi. Aλ(LθC)=(A∪L)∪(A∪C) and (AθL)λ C=(A ∗ C)∩(L ∗ C)

vii. Aλ(L ∗ C)=(A∪L)∩(A∪C) and (A ∗ L)λ C=(A ∗ C)∪(L ∗ C)

viii. Aλ(LɣC)=(A∪L)∪(AλC) and (AɣL)λ C=(A ∗ C)∩(LλC)

**Proof:** We give the proofs for left distributions; right distributions can be shown similarly.

i. Aλ(L∩C)=A∪(L∩C)'=A∪(L'∪C')=(A∪L')∪(A∪C')=(AλL)∪(AλC)

ii. Aλ(L∪C)=A∪(L∪C)'=A∪(L'∩C')=(A∪L')∩(A∪C')=(AλL)∩(AλC)

iii. Aλ(L\C)=A∪(L∩C')'=A∪(L'∪C)=(A∪L')∪(A∪C)=(AλL)∪(A∪C)

iv. Aλ(LλC)=A∪(L∪C')'=A∪(L'∩C)=(A∪L')∩(A∪C)=(AλL)∩(A∪C)

v. Aλ(L+C)=A∪(L′∪C)'=A∪(L∩C')=(A∪L)∩(A∪C')=(A∪L)∩(AλC)

vi. Aλ(LθC)=A∪(L′∩C′)'=A∪(L∪C)=(A∪L)∪(A∪C)

vii. Aλ(L ∗ C)=A∪(L′∪C′)'=A∪(L∩C)=(A∪L)∩(A∪C)

viii. Aλ(LɣC)=A∪(L′∩C)'=A∪(L∪C′)=(A∪L)∪(A∪C′)=((A∪L)∪(AλC)

## 9. Intersection Left Complement Subgroup

In abstract algebra, it is well-known that if H and K are subgroups of G, then H∩K is a subgroup of G, wheras H∪K needs not to be. In general, an arbitrary intersection of subgroups is a subgroup. It is also known that a group cannot be written as the union of two proper subgroups, where proper subgroup of a group G is a subgroup H which is a proper subset of G (that is, H ≠ G). In Bruckheimer, Bryan and Muir it is proved that a group is the union of three proper subgroups if and only if it has a quotient isomorphic to $V_4$. In this section, we define a new type of subgroup, which we call intersection left complement subgroup and then ask a question to readers to inspire them to think in through (Bruckheimer et al., 1970). For all undefined definitions, theorems, and basic properties of group theory, please see references (Dummit, 2004; Hungerford, 2003).

**Definition 9.1.** Let G be a group, H and K be two subgroups of G. Let

$H\ddot{\gamma}K = (H\gamma K) \cup \{e\} = (H' \cap K) \cup \{e\} = [(G\backslash H) \cap K]\ ) \cup \{e\}$

If $H\ddot{\gamma}K$ is a subgroup of G, then $H\ddot{\gamma}K$ is called an intersection left complement subgroup of G.

**Definition 9.2.** Let G be a group, H and K be two subgroups of G. If $H\ddot{\gamma}K = \{e\}$ or $H\ddot{\gamma}K = G$, then these intersection left complement subgroups are called trivial intersection left complement subgroups of G; otherwise they are called non-trivial intersection left complement subgroups of G.

**Proposition 9.3.** Let G be a group with identity e and H be a subgroup of G. Then,

i. $G\ddot{\gamma}\{e\}$ is a trivial intersection left complement subgroup of G.

ii. $\{e\}\ddot{\gamma}G$ is a trivial intersection left complement subgroup of G.

iii. $\{e\}\ddot{\gamma}\{e\}$ a trivial intersection left complement subgroup of G.

iv. $G\ddot{\gamma}G$ a trivial intersection left complement subgroup of G.

v. $H\ddot{\gamma}H$ a trivial intersection left complement subgroup of G.

vi. $H\ddot{\gamma}\{e\}$ a trivial intersection left complement subgroup G.

**Proof:**

i. $G\ddot{\gamma}\{e\} = [G' \cap \{e\}] \cup \{e\} = \emptyset \cup \{e\} = \{e\}$

ii. $\{e\}\ddot{\gamma}G = [\{e\}' \cap G] \cup \{e\} = G$

iii. $\{e\}\ddot{\gamma}\{e\} = [\{e\}' \cap \{e\}] \cup \{e\} = \emptyset \cup \{e\} = \{e\}$

iv. $G\ddot{\gamma}G = [G' \cap G] \cup \{e\} = \emptyset \cup \{e\} = \{e\}$

v. $H\ddot{\gamma}H = [H' \cap H] \cup \{e\} = \emptyset \cup \{e\} = \{e\}$

vi. $H\ddot{\gamma}\{e\} = [H' \cap \{e\}] \cup \{e\} = \emptyset \cup \{e\} = \{e\}$

**Proposition 9.4.** Let H and K be any subgroups of G such that $H \subseteq K$. Then, $K\ddot{\gamma}H$ is a trivial intersection left complement subgroup of G.

**Proof:** Since $K\ddot{\gamma}H = [K' \cap H] \cup \{e\} = \emptyset \cup \{e\} = \{e\}$

**Example 9.5** Let $G = (Z_{12},+)$ be the group, $H=\{\overline{0},\overline{6}\}$ and $K=\{\overline{0},\overline{3},\overline{6},\overline{9}\}$ be the subgroups of G such that $H \subseteq K$ Then since,

$K\ddot{\gamma}H = [K' \cap H] \cup \{\overline{0}\} = \emptyset \cup \{\overline{0}\} = \{\overline{0}\}$

$K\ddot{\gamma}H$ is a trivial intersection left complement subgroup G.

**Note 9.6.** Let H and K be any subgroups of G such that $H \subseteq K$. Then, $H\ddot{\gamma}K$ needs not to be an intersection left complement subgroup of G.

**Example 9.7.** Let $G = (Z_8,+)$ be the group $H=\{\overline{0},\overline{4}\}$ and $K=\{\overline{0},\overline{2},\overline{4},\overline{6}\}$ be the subgroups of G such that that $H \subseteq K$. Then, since

$H\ddot{\gamma}K = [H' \cap K] \cup \{\overline{0}\} = \{\overline{2},\overline{6}\} \cup \{\overline{0}\} = \{\overline{0},\overline{2},\overline{6}\}$

$H\ddot{\gamma}K$ is not a subgroup of G.

**Definition 9.8** Let G be a group, H and K be two subgroups of G. If $H\ddot{\gamma}K = H$ or $H\ddot{\gamma}K = K$, then these intersection left complement subgroup are called proper intersection left complement subgroups of G; otherwise (if there exist), they are called improper intersection left complement subgroup of G.

It is well known that if K and H be subgroups of G such that gcm(|K|,|H|)=1; then by Lagrange's Theorem, we have that $H \cap K \subseteq K$ and $H \cap K \subseteq H$; so $|H \cap K|$ devides |K|, and $|H \cap K|$ devides |H|; therefore $|H \cap K|$ devides gcm(|K|,|H|), where gcm(|K|,|H|) is greatest common devisor of the order of K and H. Since gcm(|K|,|H|)=1 by assumption, then $|H \cap K|=1$. Hence $H \cap K = \{e\}$.

**Proposition 9.9.** Let H and K be any subgroups of G such that $H \cap K = \{e\}$. Then, $H\ddot{\gamma}K$ and $K\ddot{\gamma}H$ are proper intersection left complement subgroupS of G.

**Proof:** Since $H\ddot{\gamma}K = [H\gamma K] \cup \{e\} = [H' \cap K] \cup \{e\} = K$ and $K\ddot{\gamma}H = [K\gamma H] \cup \{e\} = [K' \cap H] \cup \{e\} = H$.

**Corollary 9.10.** Let H be any subgroup of G. Then $\{e\}\ddot{\gamma}H$ is a proper intersection left complement subgroup of G.

**Proof:** Since $\{e\} \cap H = \{e\}$, by Proposition 9.9, $\{e\}\ddot{\gamma}H = H$.

**Example 9.11.** Let $G = (Z_{10},+)$, $H=\{\overline{0},\overline{5}\}$ and $K=\{\overline{0},\overline{2},\overline{4},\overline{6},\overline{8}\}$ be subgroups of $Z_{10}$. Then,

$H \cap K = \{\overline{0}\}$ and $H\ddot{\gamma}K = [H' \cap K] \cup \{\overline{0}\} = \{\overline{2},\overline{4},\overline{6},\overline{8}\} \cup \{\overline{0}\} = \{\overline{0},\overline{2},\overline{4},\overline{6},\overline{8}\} = K$

And thus, $H\ddot{\gamma}K$ is a proper intersection left complement subgroup of G. Moreover,

$K\ddot{\gamma}H = [K' \cap H] \cup \{\overline{0}\} = \{\overline{5}\} \cup \{\overline{0}\} = \{\overline{0},\overline{5}\} = H$

And thus, $K\ddot{\gamma}H$ is a proper intersection left complement subgroup of G.

## 10. Symmetric Difference Subgroup

**Definition 10.1.** Let G be a group, H and K be two subgroups of G and

$H\ \ddot{\triangle}\ K = (H \triangle K) \cup \{e\}$,

where $H \triangle K$ is the symmetric difference of H and K. If $H\ \ddot{\triangle}\ K$ is a subgroup of G, then it is called symmetric difference subgroup of G.

Note that, since symmetric difference operation is commutative, $H\ \ddot{\triangle}\ K = H\ \ddot{\triangle}\ K$ is satisfied throughout the paper.

**Definition 10.2.** Let G be a group, H and K be two subgroups of G. If $H\ \ddot{\triangle}\ K = \{e\}$ or $H\ \ddot{\triangle}\ K = G$, then these symmetric difference subgroups are called trivial symmetric difference subgroups of G; otherwise they are called non-trivial symmetric difference subgroups of G.

**Proposition 10.3.** Let G be a group e and H be a subgroup of G. Then,

i. $G\ \ddot{\triangle}\ \{e\}$ and $\{e\}\ \ddot{\triangle}\ G$ are trivial symmetric difference subgroups of G.

ii. $\{e\}\ \ddot{\triangle}\ \{e\}$, $G\ \ddot{\triangle}\ G$, $H\ \ddot{\triangle}\ H$ are trivial symmetric difference subgroups of G.

**Proof:** Let G be a group e and H be a subgroup of G. Since,

i. $G\ \ddot{\triangle}\ \{e\} = \{e\}\ \ddot{\triangle}\ G = G$

ii. $\{e\}\ \ddot{\triangle}\ \{e\} = G\ \ddot{\triangle}\ G = H\ \ddot{\triangle}\ H = \{e\}$

**Definition 10.4** Let G be a group, H and K be two subgroups of G. If $H\ \ddot{\triangle}\ K = H$ or $H\ \ddot{\triangle}\ K = K$, then these symmetric symmetric subgroups are called proper symmetric difference subgroups of G; otherwise (if there exist), they are called improper symmetric difference subgroup of G.

**Proposition 10.5.** Let H be any subgroup of G. Then, $H\ \ddot{\triangle}\ \{e\}$ is a proper symmetric difference subgroup of G.

**Proof:** Since $H\ \ddot{\triangle}\ \{e\} = \{e\}\ \ddot{\triangle}\ H = H$, $H\ \ddot{\triangle}\ \{e\}$ is a proper symmetric difference subgroup of G.

**Example 10.6** Let $G = (Z_{12},+)$ be the group, $H=\{\overline{0},\overline{6}\}$ and $K=\{\overline{0},\overline{3},\overline{6},\overline{9}\}$ be the subgroups of G such that $H \subseteq K$. Since $H\ \ddot{\triangle}\ K=\{\overline{0},\overline{3},\overline{9}\}$, $H\ \ddot{\triangle}\ K$ is not a symmetric difference subgroup of G.

**Example 10.7.** Let $G = (Z_{10},+)$, $H=\{\overline{0},\overline{5}\}$ and $K=\{\overline{0},\overline{2},\overline{4},\overline{6},\overline{8}\}$ be subgroups of $Z_{10}$ such that $H \cap K = \{\overline{0}\}$. Since $H\ \ddot{\triangle}\ K=\{\overline{0},\overline{2},\overline{4},\overline{5},\overline{6},\overline{8}\}$, $H\ \ddot{\triangle}\ K$ is not a symmetric difference subgroup of G.

**Example 10.8.** Let $G = (Z_{12},+)$, $H=\{\overline{0},\overline{3},\overline{6},\overline{9}\}$ and $K=\{\overline{0},\overline{2},\overline{4},\overline{6},\overline{8},\overline{10}\}$ be subgroups of $Z_{12}$. Then, $H \cap K \neq \{\overline{0}\}$ and neither $H \subseteq K$ nor $K \subseteq H$. Since $H\ \ddot{\triangle}\ K=\{\overline{0},\overline{2},\overline{3},\overline{4},\overline{8},\overline{9},\overline{10}\}$ is not a subgroup of G, hence $H\ \ddot{\triangle}\ K$ is not a symmetric difference subgroup of G.

**Example 10.9.** Let $(Z_8^*,.)$, $H=\{\overline{1},\overline{3}\}$ and $K=\{\overline{1},\overline{5}\}$ be subgroups of $Z_8^*$. Then, since $H\ \ddot{\triangle}\ K=\{\overline{1},\overline{3},\overline{5}\}$ is not a subgroup of G, hence $H\ \ddot{\triangle}\ K$ is not a symmetric difference subgroup of G.

## 11. Questions

**Question 1.** Assume that H and K be any subgroups of a group G. Then, is there any group G such that $H\ddot{\gamma}K$ is a non-trivial improper intersection left complement subgroup of G? If not, why?

**Question 2.** Assume that H and K be any subgroups of a group G. Then, is there any group G such that $H\ \ddot{\triangle}K$ is an improper symmetric difference subgroup of G except H ⩟ {e}? If not, why?

**Question 3.** Assume that H and K be any subgroups of a group G. Then, is there any group G such that H ⩟K is a non-trivial improper symmetric difference subgroup of G? If not, why?

**Question 4.** Assume that H and K be any subgroups of a group G. Then, is there any group G such that H ⩟K=G? If not, why?

## 12. Conclusion

In this study, we have defined some complemental binary operations as new concepts in set theory. Our goal is to contribute to the set literature by elucidating the relationships between these new complemental binary operations and others previously defined by obtaining the distribution rules. Furthermore, we have obtained that the set of all sets together with these new complementary binary operations forms some algebraic structures. Finally, by taking inspiration from these new concepts and defining new types of subgroups of groups, we have offered applications to group theory related to subgroups and invited readers to ponder interesting questions that encourage the readers to think through the subgroups together with the complemental binary operations defined in this paper. Since soft set theory is one of the most popular theory dealing with uncertainty and solving problems related to decision making and cripthology, and since the concept of operations of soft sets is the corner stone of the theory, new complementary binary operations defined in this paper will shedlight of soft set theory for proposing new soft set operations.

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