ZIBELINE INTERNATIONAL™ PUBLISHING
ISSN: 2521-0831 (Print)
ISSN: 2521-084X(Online)
CODEN: MSMAD

Matrix Science Mathematic (MSMK)

DOI: http://doi.org/10.26480/msmk.02.2024.45.55

CrossMark

RESEARCH ARTICLE

A NEW TYPE OF OPERATION FOR SOFT SETS: SOFT BINARY PIECEWISE STAR OPERATION

Aslıhan Sezgin[a]*, Eda Yavuz[b]

[a]Department of Mathematics and Science Education, Faculty of Education, Amasya University, Amasya, Türkiye.
[b]Department of Mathematics, Graduate School of Natural and Applied Sciences, Amasya University, Amasya, Türkiye.
Corresponding Author Email: aslihan.sezgin@amasya.edu.tr

ARTICLE DETAILS

Article History:

Received 20 March 2024
Revised 14 April 2024
Accepted 30 Jue 2024
Available online

ABSTRACT

Soft set theory gained popularity as a cutting-edge approach to handling uncertainty-related problems and modeling uncertainty when it was introduced by Molodtsov in 1999. It may be applied in a variety of contexts, both theoretical and practical. This paper introduces a new soft set operation called the "soft binary piecewise star operation." Its basic algebraic characteristics are thoroughly examined. Moreover, this operation's distributions over various soft set operations are obtained. We prove that the soft binary piecewise star operation is a commutative semigroup under certain conditions and is also a right-left system. Furthermore, we show that the collection of soft sets over the universe, along with the soft binary piecewise star operation and some other types of soft sets, form many important algebraic structures, such as semirings and near-semirings, by considering the algebraic properties of the operation and its distribution rules together.

KEYWORDS

Soft sets, soft set operations, conditional complements, soft binary piecewise star operation

1. INTRODUCTION

Fuzzy set theory, interval mathematics, and probability theory are a few of the theories that may be used to explain uncertainty; yet, each of these theories has disadvantages of its own. Soft Set Theory is a unique approach to describing uncertainty and using it to solve issues related to uncertainty which was first described by (Molodtsov, 1999). This idea has been successfully applied to several mathematical fields since it was first introduced. Measurement theory, game theory, probability theory, Riemann integration, and Perron integration are a few of these disciplines that have been researched.

Soft set operations were first studied by (Maji et al., 2003 and Pei and Miao, 2005). A number of soft set operations, including restricted and extended soft set operations were proposed by (Ali et al., 2009). In their work on soft sets, Sezgin and Atagün developed and gave the characteristics of the restricted symmetric difference of soft set. Additionally, they covered the fundamentals of soft set operations and gave examples of how they connect to each other (Sezgin and Atagün, 2011), A thorough examination of the algebraic structures of soft sets was carried out by (Ali et al. , 2011). A number of academics were interested in soft set operations and studied the subject matter in depth (Yang, 2008; Neog and Sut, 2011, Fu, 2011; Ge and Yang, 2011; Singh and Onyeozili, 2012a; Singh and Onyeozili, 2012b; Singh and Onyeozili, 2012c; Singh and Onyeozili, 2012d; Ping, and Qiaoyan, 2013; Jayanta, 2014; Onyeozili and Gwary, 2014; Husain and Shamsham, 2018).

The idea of the soft binary piecewise difference operation in soft sets was proposed by (Eren and Çalışıcı, 2019). Also, Sezgin and Çalışıc carried out a thorough analysis of the soft binary piecewise difference operation (Sezgin and Çalışıcı, 2024). While the extended difference of soft sets was introduced by Sezgin et al, extended symmetric difference of soft sets was defined and investigated by Stojanovic (Sezgin et al., 2019; Stojanovic, 2021).

Two new complement operations were introduced to the literature by (Çağman, 2021). A group of researchers worked on these and many other new binary set operations were introduced by (Sezgin et al., 2023a). A significant number of additional restricted and extended soft set operations were proposed by Aybek via applying these new binary operations to soft sets (Aybek, 2024). The complementary extended soft set operations were the focus of their continuous attempt to modify the structure of extended operations in soft sets by (Akbulut, 2024; Demirci, 2024; Sarıalioğlu, 2024). The complementary soft binary piecewise operations were also examined by notably altering the form of the soft binary piecewise operation in soft sets by (Sezgin and Atagün, 2023; Sezgin and Aybek, 2023; Sezgin et al. 2023b; Sezgin et al. 2023c; Sezgin and Çagman, 2024; Sezgin and Demirci, 2023; Sezgin and Sarıalioğlu, 202; Sezgin and Yavuz, 2023b; Sezgin and Dagtoros, 2023), Two notable studies on soft binary piecewise operations were proposed by (Sezgin and Yavuz, 2023a; Yavuz, 2024). Studies concerning different types of soft equity are also crucial for the literature of soft sets (Jun and Yang, 2011; Liu et al., 2012; Feng and Li, 2013; Abbas et al., 2014; Abbas et al., 2017; Al-Shami, 2019; Alshasi and El-Shafei, 2020; Ali et al., 2020)

Algebraic structures, also referred to as mathematical systems or structures, have long piqued the curiosity of mathematicians. Sorting algebraic structures according to the properties of the operation given on a set is one of the most important algebraic mathematics problems. One of the best-known ideas in binary algebraic structures is the extension of rings: near-rings, semirings, and semifields. Scholars have been eager to learn more about this topic for a very long time. The first definition of the word semirings was provided by (Vandiver, 1934). Semirings have been the focus of extensive studies in more recent times, particularly concerning their applications (Vandiver, 1934). Semirings are important in geometry, but they are also crucial in pure mathematics and are needed

Quick Response Code	Access this article online	
	Website: www.matrixsmathematic.com	**DOI:** 10.26480/msmk.02.2024.45.55

to solve many problems in applied mathematics and the information sciences (Goodearl, 1979; Petrich, 1973; Reutenauer and Straubing, 1984; Glazek, 2002; Kolokoltsov and Maslov; 1997; Hopcroft and Ullman, 1979; Beasley and Pullman, 1988, Beasley and Pullman, 1992; Ghosh, 1996; Wechler, 1978; Golan, 1999; Hebisch, and Weinert, 1998, Mordeson and Malik, 2002). To sum up, semirings are important in pure mathematics as well as geometry. Hoorn and Rootselaar discussed the near-semiring (Hoorn and Rootselaar, 1967). More general than a near-ring or semiring, a seminearring is an algebraic structure in mathematics also referred to as a near-semiring. Finding near-semirings from functions on monoids is a simple task. Concepts of soft set operations for soft sets are fundamental, much as operations from classical set theory are to classical algebra. Thus, thinking about the algebraic structure of soft sets in terms of this point of view might help us better comprehend it.

We want to make a major contribution to the field of soft set theory by introducing the "soft binary piecewise star operation" and closely examining the algebraic structures associated with it as well as other soft set operations in the collection of soft sets over the universe. The structure of this study is as follows: The fundamental concepts of soft sets and various algebraic structures are reviewed in Section 2. In the third section, the algebraic characteristics of the newly proposed soft set operation are analyzed in detail. These characteristics enable us to demonstrate that, in addition to being a right-left system with the right identity empty soft set under specific circumstances, the soft binary piecewise star operation is also a commutative semigroup. Section 4 looks at how the soft binary piecewise star operation is distributed over several soft set operations, such as restricted, extended, and soft binary piecewise operations. Considering the distribution laws and the algebraic properties of the soft set operations, an extensive analysis of the algebraic structures formed by the set of soft sets with these operations is presented. It is demonstrated that a variety of significant algebraic structures, such as semirings and near-semirings, are constructed from the collection of soft sets over the universe using the soft binary piecewise star operation and other forms of soft sets. Section 5 discusses the significance of the study's results and how they could apply to the subject.

2. Preliminaries

Several algebraic structures and several fundamental ideas in soft set theory are provided in this section.

Definition 2.1. Let U be the universal set, E be the parameter set, P(U) be the power set of U, and let K ⊆ E. A pair (F, K) is called a soft set on U. Here, F is a function given by F: K → P(U) (Molodtsov, 1999).

The set of all soft sets over U is denoted by $S_E(U)$. Let K be a fixed subset of E, then the set of all soft sets over U with the fixed parameter set K is denoted by $S_K(U)$. In other words, in the collection $S_K(U)$, only soft sets with the parameter set K are included, while in the collection $S_E(U)$, soft sets over U with any parameter set can be included. Clearly, the set $S_K(U)$ is a subset of the set $S_E(U)$.

Definition 2.2. Let (F,K) be a soft set over U. If F(e)=∅ for all e∈K, then the soft set (F,K) is called a null soft set with respect to K, denoted by $\emptyset_K$. Similarly, let (F,E) be a soft set over U. If F(e)=∅ for all e∈E, then the soft set (F,E) is called a null soft set with respect to E, denoted by $\emptyset_E$ (Ali et al., 2009).

It is known that a function F: ∅ ⟶ K, where the domain is the empty set, is referred to as the empty function. Since the soft set is also a function, it is evident that by taking the domain as ∅, a soft set can be defined as F: ∅ ⟶ P(U), where U is a universal set. Such a soft set is called an empty soft set and is denoted as $\emptyset_\emptyset$. Thus, $\emptyset_\emptyset$ is the only soft set with an empty parameter set (Ali et al., 2011).

Definition 2.3. Let (F,K) be a soft set over U. If F(e)=U for all e∈K, then the soft set (F,K) is called an absolute soft set with respect to K, denoted by U_K. Similarly, let (F,E) be a soft set over U. If F(e)=U for all e∈E, then the soft set (F,E) is called an absolute soft set with respect to E, denoted by U_E (Ali et al., 2009).

Definition 2.4. Let (F,K) and (G,Y) be soft sets over U. If K⊆Y and for all e∈K, F(e) ⊆G(e), then (F,K) is said to be a soft subset of (G,Y), denoted by (F,K)$\tilde{\subseteq}$(G,Y) . If (G,Y) is a soft subset of (F,K), then (F,K) is said to be a soft superset of (G,Y), denoted by (F,K)$\tilde{\supseteq}$(G,Y). If (F,K)$\tilde{\subseteq}$(G,Y) and (G,Y)$\tilde{\subseteq}$(F,K), then (F,K) and (G,Y) are called soft equal sets (Pei and Miao, 2005).

Definition 2.5. Let (F,K) be a soft set over U. The soft complement of (F,K), denoted by $(F,K)^r$ =(F^r,K), is defined as follows: for all e∈K, F^r(e)=U-F(e) (Ali et al., 2009).

Two new complements as a novel concept in set theory were introduced (Çağman, 2021). For ease of representation, we denote these binary operations as + and θ, respectively. For two sets T and Y, these binary operations are defined as T+Y=T'∪Y and TθY=T'∩Y' (Sezgin et al., 2023a) investigated the relationship between these two operations and also introduced three new binary operations, examining their relationships with each other. For two sets T and Y, these new operations are defined as T*Y=K'∪Y', TγY= T'∩Y, T**λ**Y=T∪Y' (Sezgin et al., 2023a).

As a summary for soft set operations, we can categorize all types of soft set operations as follows: Let "⊗" be used to represent the set operations (i.e., here ⊗ can be ∩, ∪,\, Δ, +,θ, *, λ,γ), then all types of soft set operations are defined as follows:

Definition 2.6. Let (F, K) and (G, Y) be two soft sets over U. The restricted ⊗ operation of (F, K) and (G, Y) is the soft set (H, P), denoted by (F, K) $\otimes_\Re$ (G, Y)= (H, P), where P = K ∩ Y≠ ∅ and for all e ∈ P, H(e) = F(e)⊗G(e). Here, if P = K ∩ Y = ∅, then (F, K) $\otimes_\Re$(G, Y)= $\emptyset_\emptyset$ (Ali et al., 2009; Sezgin and Atagün, 2011; Ali et al., 2011; Aybek, 2024).

Definition 2.7. Let (F, K) and (G, Y) be two soft sets over U. The extended ⊗ operation (F, K) and (G,Y) is the soft set (H,P), denoted by (F, K) $\otimes_\varepsilon$(G, Y) = (H, P), where P = K ∪ Y, and for all e ∈ P,

$$H(e)=\begin{cases} F(e), & e\in K-Y \\ G(e), & e\in Y-K \\ F(e)\otimes G(e), & e\in K\cap Y \end{cases}$$

(Maji et al., 2003; Ali et al., 2009; Sezgin et al., 2019; Stojanovic, 2021; Aybek, 2024)

Definition 2.8. Let (F, K) and (G, Y) be two soft sets over U. The complementary extended ⊗ operation (F, K) and (G,Y) is the soft set (H,P), denoted by (F, K) $\overset{*}{\otimes}_\varepsilon$(G, Y) = (H, P), where P = K ∪ Y, and for all e ∈ P,

$$H(e)=\begin{cases} F'(e), & e\in K-Y \\ G'(e), & e\in Y-K \\ F(e)\otimes G(e), & e\in K\cap Y \end{cases}$$

(Akbulut, 2024; Demirci, 2024; Sarıalioğlu, 2024).

Definition 2.9. Let (F,K) and (G,Y) be two soft sets on U. The complementary soft binary piecewise ⊗ operation of (F,K) and (G,Y) is the soft set (H,K), denoted by $(F,K)\underset{\otimes}{\overset{*}{\sim}}(G,Y)=(H,K)$, where for all e ∈ K,

$$H(e)=\begin{cases} F'(e), & e\in K-Y \\ F(e)\otimes G(e), & e\in K\cap Y \end{cases}$$

(Sezgin and Atagün, 2023; Sezgin and Aybek, 2023; Sezgin et al., 2023b; Sezgin et al., 2023c; Sezgin and Çağman, 2024; Sezgin and Demirci, 2023; Sezgin and Sarıalioğlu, 2024; Sezgin and Yavuz, 2023b; Sezgin and Dagtoros, 2023)

Definition 2.10. Let (F,K) and (G,Y) be two soft sets on U. The soft binary piecewise ⊗ operation of (F,K) and (G,Y) is the soft set (H,K), denoted by $(F,K)\underset{\otimes}{\sim}(G,Y)=(H,K)$, where for all e ∈ K,

$$H(e)=\begin{cases} F(e), & e\in K-Y \\ F(e)\otimes G(e), & e\in K\cap Y \end{cases}$$

(Eren and Çalışıcı, 2019; Sezgin and Çalışıcı, 2024; Yavuz, 2024; Sezgin and Yavuz, 2023a).

For more about soft sets, we refer to the following (Çağman et al., 2012; Sezgin, 2016; Tunçay and Sezgin, 2016; Sezgin and Orbay, 2022; Mahmood et al., 2018; Jana et al., 2019; Muştuoğlu et al., 2016; Sezer et al., 2015; Sezer, 2014; Özlü and Sezgin, 2020; Atagün and Sezgin, 2018, Sezgin, 2018; Iftikhar and Mahmood, 2018; Sezgin et al., 2017; Mahmood et al., 2015; Sezgin et al., 2022).

Definition 2.11. Let (S, ⋆) be an algebraic structure. An element s ∈S is called idempotent, if s^2=s, for all s∈S. The algebraic structure (S,⋆) is said to be idempotent if all the elements of S are idempotent. An idempotent semigroup is called a band; an idempotent and commutative semigroup is called a semilattice; and an idempotent and commutative monoid is called a bounded semilattice (Clifford, 1954).

In a monoid, although the identity element is unique, a semigroup/groupoid can have one or more left identities; however, if it has more than one left identity, it does not have a right identity element, thus it does not have an identity element. Similarly, a semigroup/groupoid can have one or more right identities; however, if it has more than one right identity, it does not have a left identity element, thus it does not have an identity element (Kilp et al., 2001). Similarly, in a group, although each element has a unique inverse, in a monoid, an element can have one or more left inverses; however, if an element has more than one left inverse, it does not have a right inverse, thus it does not have an inverse. Similarly, in a monoid, an element can have one or more right inverses; however, if an element has more than one right inverse, it does not have a left inverse, thus it does not have an inverse (Kilp et al., 2001).

Definition 2.12. If a semigroup (S,*) has a left identity and every element has a right inverse, then the semigroup is called a left-right system and if the semigroup has a right identity and every element has a left inverse, then the semigroup is called a right-left system. The difference between the left-right system and the group is that a group has a left (resp., a right) identity, and every element has a left (resp., a right) inverse (Maan, 1994).

Definition 2.13. Let S be a non-empty set, and let "+" and "⋆" be two binary operations defined on S. If the algebraic structure (S, +, ⋆) satisfies the following properties, then it is called a semiring:

i. (S, +) is a semigroup.

ii. (S, ⋆) is a semigroup,

iii. For all x, y, z ∈S, x⋆(y + z) = x⋆y + x⋆z and (x +y) ⋆z = x⋆z + y⋆z

If x+y=y+z for all x,y∈S, then S is called an additive commutative semiring. If for all x,y∈S, x⋆y=y⋆x, then S is called a multiplicative commutative semiring. If there exists an element 1∈S such that x⋆1=1⋆x=x for all x∈S (multiplicative identity), then S is called semiring with unity. If there exists 0∈S such that for all x∈S, 0⋆x=x⋆0=0 and 0+x=x+0=x, then 0 is called the zero of S. A semiring with commutative addition, and a zero element, is called a hemiring (Vandiver, 1934).

Definition 2.14. Let S be a non-empty set, and let "+" and "⋆" be two binary operations defined on S. If the algebraic structure (S, +, ⋆) satisfies the following properties, then it is called a near-semiring (or seminearring):

i. (S,+) is a semigroup.

ii. (S, ⋆) is a semigroup.

iii. For all x,y,z∈ S, (x+y) ⋆z = x⋆z+y∗z (right distributivity)

If the additive zero element 0 of S (that is, for all x∈S, 0+x=0+x=x) satisfies that for all x∈S, 0⋆x=0 (left absorbing element), then (S, +, ⋆) is called a (right) near-semiring with zero. If (S, +, ⋆) additionally satisfies x⋆0=0 for all x∈S (right absorbing element), then it is called a zero symmetric near-semiring (Hoorn and Rootselaar, 1967).For possible applications of graphs and network research concerning soft sets, we refer to (Pant et al., 2024).

3. Soft Binary Piecewise Star Operation

A novel soft set operation called the soft binary piecewise star operation is presented in this section. It also looks at the distribution rules and algebraic structures of the operation form in $S_E(U)$, presents an example of the operation, and investigates its whole algebraic properties and relationships with other soft set operations.

Definition 3.1. Let (F, K) and (G, Y) be soft sets over U. The soft binary piecewise star of (F, K) and (G, Y) is the soft set (H, K), denoted by, $(F,K)\underset{*}{\sim}(G,Y) = (H,K)$, where for all ẟ∈K,

$$H(ẟ)=\begin{cases} F(ẟ), & ẟ\in K-Y \\ F'(ẟ)\cup G'(ẟ), & ẟ\in K\cap Y \end{cases}$$

Example 3.2. Let E={e_1,e_2,e_3,e_4} be the parameter set, K={e_1, e_4} and Y={e_2, e_3, e_4} be the subsets of E, and U={h_1,h_2,h_3,h_4,h_5, h_6} be the initial universe set. Assume that (F,K) and (G,Y) are the soft sets over U defined as following:

(F,K)={(e_1, {h_2, h_4,h_6}), (e_4,{h_1,h_2,h_5, h_6})}

(G,Y)={(e_2,{h_1 ,h_2}), (e_3,{h_2,h_3,h_4, h_5}),(e_4,{h_2, h_3,h_5})}

Let $(F,K)\underset{*}{\sim}(G,Y)$=(H,K), where for all ẟ∈K,

$$H(ẟ)=\begin{cases} F(ẟ), & ẟ\in K-Y \\ F'(ẟ)\cup G'(ẟ), & ẟ\in K\cap Y \end{cases}$$

Here since K={e_1,e_4} and K − Y={e_1}, for all ẟ ∈ K − Y={e_1}, H(ẟ)=F(ẟ) and so H(e_1)=F(e_1)={h_2, h_4,h_6}; for all ẟ ∈ K ∩ Y={e_4},H(ẟ)=F′(ẟ) ∪G′(ẟ), H(e_4)=F′(e_4) ∪G′(e_4)={h_3,h_4}∪{h_1,h_4, h_6}={h_1, h_3, h_4, h_6}}. Thus,

$(F,K)\underset{*}{\sim}(G,Y)$={(e_1,{h_2 ,h_4,h_6}),(e_4, {h_1, h_3, h_4, h_6})}.

Theorem 3.3. Algebraic Properties of the Operation

1) The set $S_E(U)$ is closed under $\underset{*}{\sim}$. That is, when (F,K) and (G,Y) are two soft sets over U, then so is $(F,K)\underset{*}{\sim}(G,Y)$.

Proof: It is clear that $\underset{*}{\sim}$ is a binary operation in $S_E(U)$. That is,

$\underset{*}{\sim}$: $S_E(U)$x $S_E(U)$→ $S_E(U)$

$((F,K), (G,Y)) \to (F,K)\underset{*}{\sim}(G,Y) = (H,K)$

Hence, the set $S_E(U)$ is closed under $\underset{*}{\sim}$. Similarly,

$\underset{*}{\sim}$: $S_K(U)$x $S_K(U)$→ $S_K(U)$

$((F,K), (G,K)) \to (F,K)\underset{*}{\sim}(G,K) =(H,K)$

That is, let K be a fixed subset of the set E, and (F,K) and (G,K) be elements of $S_K(U)$. Then so is $(F,K)\underset{*}{\sim}(G,K)$. Namely, $S_K(U)$ is closed under $\underset{*}{\sim}$.

2) If K∩Y′∩D=K∩Y∩D=∅, then $[(F,K)\underset{*}{\sim}(G,Y)]\underset{*}{\sim}(H,D)=(F,K)\underset{*}{\sim}[(G,Y)\underset{*}{\sim}(H,D)]$.

Proof: First, consider the left-hand side (LHS). Let $(F,K)\underset{*}{\sim}(G,Y)$=(T,K), where for all ẟ∈K,

$$T(ẟ)=\begin{cases} F(ẟ), & ẟ\in K-Y \\ F'(ẟ)\cup G'(ẟ), & ẟ\in K\cap Y \end{cases}$$

Let $(T,K)\underset{*}{\sim}(H,D)$=(M,K), where for all ẟ∈K,

$$M(ẟ)=\begin{cases} T(ẟ), & ẟ\in K-D \\ T'(ẟ)\cup H'(ẟ), & ẟ\in K\cap D \end{cases}$$

Thus,

$$M(ẟ)=\begin{cases} F(ẟ), & ẟ\in(K-Y)-D=K\cap Y'\cap D' \\ F'(ẟ)\cup G'(ẟ), & ẟ\in(K\cap Y)-D=K\cap Y\cap D' \\ F'(ẟ)\cup H'(ẟ), & ẟ\in(K-Y)\cap D=K\cap Y'\cap D \\ [F(ẟ)\cap G(ẟ)]\cup H'(ẟ), & ẟ\in(K\cap Y)\cap D=K\cap Y\cap D \end{cases}$$

Let $(G,Y)\tilde{*}(H,D)=(K,Y)$, where for all $\text{ծ}\in Y$,

$$K(\text{ծ})=\begin{cases} G(\text{ծ}), & \text{ծ}\in Y\text{-}D \\ G'(\text{ծ})\cup H'(\text{ծ}), & \text{ծ}\in Y\cap D \end{cases}$$

Let $(F,K)\tilde{*}(K,Y)=(S,K)$, where for all $\text{ծ}\in K$,

$$S(\text{ծ})=\begin{cases} F(\text{ծ}), & \text{ծ}\in K\text{-}Y \\ F'(\text{ծ})\cup K'(\text{ծ}), & \text{ծ}\in K\cap Y \end{cases}$$

Thus,

$$S(\text{ծ})=\begin{cases} F(\text{ծ}), & \text{ծ}\in K\text{-}Y \\ F'(\text{ծ})\cup G'(\text{ծ}), & \text{ծ}\in K\cap(Y\text{-}D)=K\cap Y\cap D' \\ F'(\text{ծ})\cup[G(\text{ծ})\cap H(\text{ծ})], & \text{ծ}\in K\cap(Y\cap D)=K\cap Y\cap D \end{cases}$$

Considering K-Y in the S function, since $K\text{-}Y=K\cap Y'$, if $\text{ծ}\in Y'$, then $\text{ծ}\in D\text{-}Y$ or $\text{ծ}\in(Y\cup D)'$. Thus, if $\text{ծ}\in K\text{-}Y$, then $\text{ծ}\in K\cap Y'\cap D'$ or $\text{ծ}\in K\cap Y'\cap D$. Thus, M=S, where $K\cap Y'\cap D= K\cap Y\cap D=\emptyset$. That is, under suitable conditions, the operation $\tilde{*}$ is associative $S_E(U)$.

3) $[(F,K)\tilde{*}(G,K)]\tilde{*}(H,K)\neq(F,K)\tilde{*}[(G,K)\tilde{*}(H,K)]$.

Proof: Consider first the LHS and let $(F,K)\tilde{*}(G,K)=(T,K)$, where for all $\text{ծ}\in K$;

$$T(\text{ծ})=\begin{cases} F(\text{ծ}), & \text{ծ}\in K\text{-}K=\emptyset \\ F'(\text{ծ})\cup G'(\text{ծ}), & \text{ծ}\in K\cap K=K \end{cases}$$

Let $(T,K)\tilde{*}(H,K)=(M,K)$, where for all $\text{ծ}\in K$;

$$M(\text{ծ})=\begin{cases} T(\text{ծ}), & \text{ծ}\in K\text{-}K=\emptyset \\ T'(\text{ծ})\cup H'(\text{ծ}), & \text{ծ}\in K\cap K=K \end{cases}$$

Thus,

$$M(\text{ծ})=\begin{cases} T(\text{ծ}), & \text{ծ}\in K\text{-}K=\emptyset \\ [F(\text{ծ})\cap G(\text{ծ})]\cup H'(\text{ծ}), & \text{ծ}\in K\cap K=K \end{cases}$$

Now consider the RHS. Let $(G,K)\tilde{*}(H,K)=(L,K)$, where for all $\text{ծ}\in K$;

$$L(\text{ծ})=\begin{cases} G(\text{ծ}), & \text{ծ}\in K\text{-}K=\emptyset \\ G'(\text{ծ})\cup H'(\text{ծ}), & \text{ծ}\in K\cap K=K \end{cases}$$

Let $(F,K)\tilde{*}(L,K)=(N,K)$, where for all $\text{ծ}\in K$;

$$N(\text{ծ})=\begin{cases} F(\text{ծ}), & \text{ծ}\in K\text{-}K=\emptyset \\ F'(\text{ծ})\cup L'(\text{ծ}), & \text{ծ}\in K\cap K=K \end{cases}$$

Thus,

$$N(\text{ծ})=\begin{cases} F(\text{ծ}), & \text{ծ}\in K\text{-}K=\emptyset \\ F'(\text{ծ})\cup[G(\text{ծ})\cap H(\text{ծ})], & \text{ծ}\in K\cap K=K \end{cases}$$

It is seen that M≠N. That is, for the soft sets whose parameter sets are the same, the operation $\tilde{*}$ is not associative.

4) $(F,K)\tilde{*}(G,Y)\neq(G,Y)\tilde{*}(F,K)$.

Proof: Let $(F,K)\tilde{*}(G,Y)=(H,K)$, where for all $\text{ծ}\in K$;

$$H(\text{ծ})=\begin{cases} F(\text{ծ}), & \text{ծ}\in K\text{-}Y \\ F'(\text{ծ})\cup G'(\text{ծ}), & \text{ծ}\in K\cap Y \end{cases}$$

Let $(G,Y)\tilde{*}(F,K)=(T,Y)$, where for all $\text{ծ}\in Y$;

$$T(\text{ծ})=\begin{cases} G(\text{ծ}), & \text{ծ}\in Y\text{-}K \\ G'(\text{ծ})\cup F'(\text{ծ}), & \text{ծ}\in Y\cap K \end{cases}$$

Here, while the parameter set of the soft set of the LHS is K; the parameter set of the soft set of the RHS is Y. Thus, by the definition of soft equality;

$(F,K)\tilde{*}(G,Y)\neq(G,Y)\tilde{*}(F,K)$.

But it is obvious that $(F,K)\tilde{*}(G,K)=(G,K)\tilde{*}(F,K)$. That is, while the operation $\tilde{*}$ is not commutative in $S_E(U)$, the operation $\tilde{*}$ is commutative in the set $S_K(U)$, where $K\subseteq E$ is a fixed parameter set. Namely,

$(F,K)\tilde{*}(G,K)=(G,K)\tilde{*}(F,K)$.

5) $(F,K)\tilde{*}(F,K)=(F,K)^r$.

Proof: Let $(F,K)\tilde{*}(F,K)=(H,K)$, where for all $\text{ծ}\in K$;

$$H(\text{ծ})=\begin{cases} F(\text{ծ}), & \text{ծ}\in K\text{-}K=\emptyset \\ F'(\text{ծ})\cup F'(\text{ծ}), & \text{ծ}\in K\cap K=K \end{cases}$$

where for all $\text{ծ}\in K$; $H(\text{ծ})=F'(\text{ծ})\cup F'(\text{ծ})=F'(\text{ծ})$, thus $(H,K)=(F,K)^r$. That is, the operation $\tilde{*}$ is not idempotent in $S_E(U)$.

Theorem 3.3.1. By Theorem 3.3 (1), (2) and (4), $(S_E(U),\tilde{*})$ is a commutative but not idempotent semigroup, under the condition $K\cap Y'\cap D=K\cap Y\cap D=\emptyset$, where (F,K), (G,Y) and (H,D) are elements of $S_E(U)$.

By Theorem 3.3 (3) since $\tilde{*}$ is not associative in $S_K(U)$, where $K\subseteq E$ is a fixed parameter set, $(S_K(U),\tilde{*})$ is not a semigrup; however, it is obvious that it is a commutative groupoid.

6) $(F,K)\tilde{*}\emptyset_K=\emptyset_K\tilde{*}(F,K)=U_K$.

Proof: Let $\emptyset_K=(S,K)$, where for all $\text{ծ}\in K$; $S(\text{ծ})=\emptyset$. Let $(F,K)\tilde{*}(S,K)=(H,K)$, where for all $\text{ծ}\in K$;

$$H(\text{ծ})=\begin{cases} F(\text{ծ}), & \text{ծ}\in K\text{-}K=\emptyset \\ F'(\text{ծ})\cup S'(\text{ծ}), & \text{ծ}\in K\cap K=K \end{cases}$$

Thus, $H(\text{ծ})=F'(\text{ծ})\cup S'(\text{ծ})=F'(\text{ծ})\cup U=U$, dor all $\text{ծ}\in K$. Hence, $(H,K)=U_K$.

7) $(F,K)\tilde{*}\emptyset_E=U_K$.

Proof: Let $\emptyset_E=(S,E)$, where for all $\text{ծ}\in E$; $S(\text{ծ})=\emptyset$. Let $(F,K)\tilde{*}(S,E)=(H,K)$. Thus, for all $\text{ծ}\in K$,

$$H(\text{ծ})=\begin{cases} F(\text{ծ}), & \text{ծ}\in K\text{-}E=\emptyset \\ F'(\text{ծ})\cup S'(\text{ծ}), & \text{ծ}\in K\cap E=K \end{cases}$$

Hence,for all $\delta \in K$; $H(\delta)=F'(\delta)\cup S'(\delta)=F'(\delta)\cup U=U$, so $(H,K)=U_K$.

8) $(F,K)\tilde{*}\emptyset_\emptyset=(F,K)$.

Proof:Let $\emptyset_\emptyset=(S,\emptyset)$ and $(F,K)\tilde{*}(S,\emptyset)=(H,K)$. Hence, $\delta\in K$,

$$H(\delta)=\begin{cases} F(\delta), & \delta\in K-\emptyset=K \\ F'(\delta)\cup S'(\delta), & \delta\in K\cap\emptyset=\emptyset \end{cases}$$

Thus, for all $\delta\in K$; $H(\delta)=F(\delta)$, $(H,K)=(F,K)$. That is, $\emptyset_\emptyset$ is the right identity element for the operation $\tilde{*}$ in $S_E(U)$.

9) $\emptyset_\emptyset\tilde{*}(F,K)=\emptyset_\emptyset$.

Proof: Let $\emptyset_\emptyset=(S,\emptyset)$ and $(S,\emptyset)\tilde{*}(F,K)=(H,\emptyset)$. ,$\emptyset$). Since $\emptyset_\emptyset$ is the only soft set whose parameter set is the empty set, $(H,\emptyset)=\emptyset_\emptyset$.

That is, in $S_E(U)$, for the operation $\tilde{*}$, the left inverse of each element with respect to the right identity element $\emptyset_\emptyset$ is the soft set $\emptyset_\emptyset$. Moreover, in $S_E(U)$, the left absorbing element of the $\tilde{*}$ operation is the soft set $\emptyset_\emptyset$.

Theorem 3.3.2. From the properties of (1), (2), (8) and (9), the algebraic structure $(S_E(U),\tilde{*})$ is a right-left system with the right identity $\emptyset_\emptyset$, and the left inverses of each element is $\emptyset_\emptyset$ under the condition $K\cap Y\cap D=K\cap Y'\cap D=\emptyset$, where (F,K), (G,Y) and (H, D) are the elements of $S_E(U)$.

10) $(F,K)\tilde{*}U_K=U_K\tilde{*}(F,K)=(F,K)^r$.

Proof: Let $U_K=(T,K)$, where for all $\delta\in K$; $T(\delta)=U$. Let $(F,K)\tilde{*}(T,K)=(H,K)$, where for all $\delta\in K$;

$$H(\delta)=\begin{cases} F(\delta), & \delta\in K-K=\emptyset \\ F'(\delta)\cup T'(\delta), & \delta\in K\cap K=K \end{cases}$$

Thus, for all $\delta\in K$;$H(\delta)=F'(\delta)\cup T'(\delta)=F'(\delta)\cup\emptyset=F'(\delta)$, hence $(H,K)=(F,K)^r$.

11) $(F,K)\tilde{*}U_E=(F,K)^r$.

Proof: Let $U_E=(T,E)$, where for all $\delta\in E$; $T(\delta)=U$. Let $(F,K)\tilde{*}(T,E)=(H,K)$, where for all $\delta\in K$ için ;

$$H(\delta)=\begin{cases} F(\delta), & \delta\in K-E=\emptyset \\ F'(\delta)\cup T'(\delta), & \delta\in K\cap E=K \end{cases}$$

Thus,for all $\delta\in K$; $H(\delta)=F'(\delta)\cup T'(\delta)=F'(\delta)\cup\emptyset=F'(\delta)$. Thus, $(H,K)=(F,K)^r$.

12) $(F,K)\tilde{*}(F,K)^r=(F,K)^r\tilde{*}(F,K)=U_K$.

Proof: Let $(F,K)^r=(H,K)$, where for all $\delta\in K$; $H(\delta)=F'(\delta)$. Let $(F,K)\tilde{*}(H,K)=(T,K)$, where for all $\delta\in K$;

$$T(\delta)=\begin{cases} F(\delta), & \delta\in K-K=\emptyset \\ F'(\delta)\cup H'(\delta), & \delta\in K\cap K=K \end{cases}$$

Thus,for all $\delta\in K$; $T(\delta)=F'(\delta)\cup H'(\delta)=F'(\delta)\cup F(\delta)=U$, hence $(T,K)=U_K$.

13) $[(F,K)\tilde{*}(G,Y)]^r=(F,K)\underset{\cap}{\overset{*}{\sim}}(G,Y)$

Proof: Let $(F,K)\tilde{*}(G,Y)=(H,K)$, where for all $\delta\in K$,

$$H(\delta)=\begin{cases} F(\delta), & \delta\in K-Y \\ F'(\delta)\cup G'(\delta), & \delta\in K\cap Y \end{cases}$$

Let $(H,K)^r=(T,K)$, where for all $\delta\in K$,

$$T(\delta)=\begin{cases} F'(\delta), & \delta\in K-Y \\ F(\delta)\cap G(\delta), & \delta\in K\cap Y \end{cases}$$

Thus, $(T,K)=(F,K)\underset{\cap}{\overset{*}{\sim}}(G,Y)$.

14) $(F,K)\tilde{*}(G,K)=\emptyset_K \Leftrightarrow (F,K)=(G,K)=U_K$.

Proof: Let $(F,K)\tilde{*}(G,K)=(T,K)$, where for all $\delta\in K$,

$$T(\delta)=\begin{cases} F(\delta), & \delta\in K-K=\emptyset \\ F'(\delta)\cup G'(\delta), & \delta\in K\cap K=K \end{cases}$$

Since $(T,K)=\emptyset_K$, for all $\delta\in K$, $T(\delta)=\emptyset$. Thus, for all $\delta\in K$, $T(\delta)=F'(\delta)\cup G'(\delta)=\emptyset \Leftrightarrow$ for all $\delta\in K$, $F'(\delta)=\emptyset$ and $G'(\delta)=\emptyset \Leftrightarrow$ For all $\delta\in K$, $F(\delta)=U$ and $G(\delta)=U \Leftrightarrow (F,K)=(G,K)=U_K$.

15) $\emptyset_K\tilde{\subseteq}(F,K)\tilde{*}(G,Y)$ and $\emptyset_Y\tilde{\subseteq}(G,Y)\tilde{*}(F,K)$.

16) $(F,K)\tilde{*}(G,Y)\tilde{\subseteq}U_K$ and $(G,Y)\tilde{*}(F,K)\tilde{\subseteq}U_Y$.

17) $(F,K)^r\tilde{\subseteq}(F,K)\tilde{*}(G,K)$ and $(G,K)^r\tilde{\subseteq}(F,K)\tilde{*}(G,K)$.

Proof: Let $(F,K)\tilde{*}(G,K)=(H,K)$, where for all $\delta\in K$,

$$H(\delta)=\begin{cases} F(\delta), & \delta\in K-K=\emptyset \\ F'(\delta)\cup G'(\delta), & \delta\in K\cap K=K \end{cases}$$

Since for all $\delta\in K$, $H(\delta)=F'(\delta)\subseteq F'(\omega)\cup G'(\omega)$, $(F,K)^r\tilde{\subseteq}(F,K)\tilde{*}(G,K)$. $(G,K)^r\tilde{\subseteq}(F,K)\tilde{*}(G,K)$ can be shown similarly.

18) If $(F,K)\tilde{\subseteq}(G,Y)$, then $(F,K)\tilde{*}(G,Y)=(F,K)^r$.

Proof: Let $(F,K)\tilde{\subseteq}(G,Y)$. Then, $K\subseteq Y$ and for all $\delta\in K$, $F(\delta)\subseteq G(\omega)$. Let $(F,K)\tilde{*}(G,Y)=(H,K)$, where for all $\delta\in K$,

$$H(\delta)=\begin{cases} F(\delta), & \delta\in K-Y=\emptyset \\ F'(\delta)\cup G'(\delta), & \delta\in K\cap Y=K \end{cases}$$

Since for all $\delta\in K$,$F(\delta)\subseteq G(\omega)$,

then $G'(\delta)\subseteq F'(\delta)$. Thus, $H(\delta)=F'(\omega)\cup G'(\omega)=F'(\omega)$. Hence, $(F,K)\tilde{*}(G,Y)=(F,K)^r$.

19) If $(F,K)\tilde{\subseteq}(G,K)$, then $(H,Z)\tilde{*}(G,K)\tilde{\subseteq}(H,Z)\tilde{*}(F,K)$ and $(G,K)\tilde{*}(H,K)\tilde{\subseteq}(F,K)\tilde{*}(H,K)$.

Proof: Let $(F,K)\tilde{\subseteq}(G,K)$. Then, for all $\delta\in K$, $F(\delta)\subseteq G(\delta)$, so for all $\delta\in K$, $G'(\delta)\subseteq F'(\delta)$. Let $(H,Z)\tilde{*}(G,K)=(W,Z)$. Thus, for all $\delta\in Z$,

$$W(\delta)=\begin{cases} H(\delta), & \delta\in Z-K \\ H'(\delta)\cup G'(\delta), & \delta\in Z\cap K \end{cases}$$

Let $(H,Z)\tilde{*}(F,K)=(L,Z)$, where for all $\delta\in Z$,

$$L(\delta)=\begin{cases} H(\delta), & \delta\in Z-K \\ H'(\delta)\cup F'(\delta), & \delta\in Z\cap K \end{cases}$$

If for all $\eth \in Z\text{-}K$, then $W(\eth)=H(\eth)\subseteq H(\eth)=L(\eth)$, if for all $\eth \in Z\cap K$, then $W(\eth)=H'(\eth)\cup G'(\eth)\subseteq H'(\eth)\cup F'(\eth)=L(\eth)$. Thus, $(H,Z)\tilde{*}(G,K) \cong (H,Z)\tilde{*}(F,K)$. Moreover, since for all $\eth\in K$, $G'(\eth)\cup H'(\eth) \subseteq F'(\eth)\cup H'(\eth)$, $(G,K)\tilde{*}(H,K) \cong (F,K)\tilde{*}(H,K)$.

20) If $(H,Z)\tilde{*}(G,K) \cong (H,Z)\tilde{*}(F,K)$, then $(F,K) \cong (G,K)$ needs not to be true. Similarly, if $(G,K)\tilde{*}(H,K) \cong (F,K)\tilde{*}(H,K)$, then $(F,K) \cong (G,K)$ needs not to be true. That is, the converse of Theorem 3.3. (19) is not true.

Proof: To demonstrate that the converse of Theorem 3.3. (19) is not true, let's provide an example. Let $E=\{e_1,e_2,e_3,e_4,e_5\}$ be the parameter set, $K=\{e_1,e_3\}$ and $Z=\{e_1,e_3,e_5\}$ be two subsets of E, $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the universal set. Let (F,K), (G,K) and (H,Z) be soft sets over U as follows:

$(F,K)=\{(e_1,\{h_2,h_5\}),(e_3,\{h_1,h_2,h_5\})\}$, $(G,K)=\{(e_1,\{h_2\}),(e_3,\{h_1,h_2\})\}$,

$(H,Z)=\{(e_1,\emptyset),(e_3,\emptyset),(e_5,\{h_2,h_5\})\}$.

Let $(H,Z)\tilde{*}(G,K) = (L,Z)$, where *for all* $\eth \in Z\text{-}K=\{e_5\}$, $L(e_5)=H(e_5)=\{h_2,h_5\}$, *for all* $\eth \in Z\cap K=\{e_1,e_3\}$, $L(e_1)=H'(e_1)\cup G'(e_1)=U$, and $L(e_3)=H'(e_3)\cup G'(e_3)=U$. Thus, $(H,Z)\tilde{*}(G,K)=\{(e_1,U),(e_3,U),(e_5,\{h_2,h_5\})\}$.

Now let $(H,Z)\tilde{*}(F,K) = (W,Z)$, where for all $\eth \in Z\text{-}K=\{e_5\}$, $W(e_5)=H(e_5)=\{h_2,h_5\}$, *for all* $\eth \in Z\cap K=\{e_1,e_3\}$, $W(e_1)=H'(e_1)\cup F'(e_1)=U$, and $W(e_3)=H'(e_3)\cup F'(e_3)=U$. Thus, $(H,Z)\tilde{*}(F,K)=\{(e_1,U),(e_3,U),(e_5,\{h_2,h_5\})\}$.

Hence, $(H,Z)\tilde{*}(G,K) \cong (H,Z)\tilde{*}(F,K)$, but $(F,K) \cong (G,K)$ is not true. Similarly, if $(G,K)\tilde{*}(H,K) \cong (F,K)\tilde{*}(H,K)$, then $(F,K) \cong (G,K)$ needs not to be true can be shown by taking as $(H,K)=\{(e_1,\emptyset),(e_3,\emptyset)\}$.

21) Let $(F,T) \cong (G,T)$ and $(K,T) \cong (L,T)$, then $(G,T)\tilde{*}(L,T) \cong (F,T)\tilde{*}(K,T)$.

Proof: Let $(F,T)\cong (G,T)$ and $(K,T)\cong (L,T)$. Thus, for all $\eth \in T$, $F(\eth)\subseteq G(\omega)$ and $K(\eth)\subseteq L(\omega)$. Hence, for all $\eth \in T$, $G'(\eth)\subseteq F'(\eth)$ and $L'(\eth)\subseteq K'(\eth)$. Let $(G,T)\tilde{*}(L,T)=(M,T)$. Thus, for all $\eth \in T$, $M(\eth)=G'(\eth)\cup L'(\eth)$. Let $(F,T)\tilde{*}(K,T)=(N,T)$. Thus, for all $\eth \in T$, $N(\eth)=F'(\eth)\cup K'(\eth)$. Since for all $\eth \in T$, $G'(\eth)\subseteq F'(\eth)$ ve $L'(\eth)\subseteq K'(\omega)$, $M(\eth)=G'(\eth)\cup L'(\eth)\subseteq F'(\eth)\cup K'(\eth)=N(\eth)$. Thus, $(G,T)\tilde{*}(L,T) \cong (F,T)\tilde{*}(K,T)$

22) $(F,K)\tilde{\theta}(G,K) \cong (F,K)\tilde{*}(G,K)$.

Proof: Let $(F,K)\tilde{\theta}(G,K) = (T,K)$. Thus, for all $\eth\in K$,

$$T(\eth)=\begin{cases} F(\eth), & \eth\in K\text{-}K=\emptyset \\ F'(\eth)\cap G'(\eth), & \eth\in K\cap K=K \end{cases}$$

Let $(F,K)\tilde{*}(G,K) = (W,K)$. Thus, for all $\eth\in K$,

$$W(\eth)=\begin{cases} F(\eth), & \eth\in K\text{-}K=\emptyset \\ F'(\eth)\cup G'(\eth), & \eth\in K\cap K=K \end{cases}$$

Since for all $\eth\in K$, $T(\eth)=F'(\eth)\cap G'(\eth)\subseteq F'(\eth)\cup G'(\eth)=W(\eth)$. Hence, $(F,K)\tilde{\theta}(G,K) \cong (F,K)\tilde{*}(G,K)$.

4. Distribution Rules

This section provides a detailed examination of the distribution of the soft binary piecewise star operation over various soft set operations, leading to the discovery of several intriguing algebraic structures formed in the collection of soft sets together with the soft binary piecewise star operation and other various types of soft set operations.

Proposition 4.1. Let (F,K), (G,Y), and (H,D) be soft sets over U. Then, the soft binary piecewise star operation distributes over restricted operations as follows, under $K\cap Y\cap D=\emptyset$.

1) $[(F,K)\cup_R (G,Y)]\tilde{*}(H,D)=[(F,K)\tilde{*}(H,D)]\cup_R[(G,Y)\tilde{*}(H,D)]$.

Proof: First consider the LHS. Let $(F,K)\cup_R(G,Y)=(M,K\cap Y)$, where for all $\eth\in K\cap Y$, $M(\eth)=F(\eth)\cup G(\eth)$. Let $(M,K\cap Y)\tilde{*}(H,D)=(N,K\cap Y)$, where for all $\eth\in K\cap Y$,

$$N(\eth)=\begin{cases} M(\eth), & \eth\in(K\cap Y)\text{-}D \\ M'(\eth)\cup H'(\eth), & \eth\in(K\cap Y)\cap D \end{cases}$$

Thus,

$$N(\eth)=\begin{cases} F(\eth)\cup G(\eth), & \eth\in(K\cap Y)\text{-}D=K\cap Y\cap D' \\ [F'(\eth)\cap G'(\eth)]\cup H'(\eth), & \eth\in(K\cap Y)\cap D \end{cases}$$

Now consider the RHS, i.e., $[(F,K)\tilde{*}(H,D)]\cup_R[(G,Y)\tilde{*}(H,D)]$. Let $(F,K)\tilde{*}(H,D)=(V,K)$, where for all $\eth\in K$,

$$V(\eth)=\begin{cases} F(\eth), & \eth\in K\text{-}D \\ F'(\eth)\cup H'(\eth), & \eth\in K\cap D \end{cases}$$

Let $(G,Y)\tilde{*}(H,D)=(W,Y)$, where for all $\eth\in Y$,

$$W(\eth)=\begin{cases} G(\eth), & \eth\in Y\text{-}D \\ G'(\eth)\cup H'(\eth), & \eth\in Y\cap D \end{cases}$$

Let $(V,K)\cup_R (W,Y)=(T,K\cap Y)$, where for all $\eth\in K\cap Y$, $T(\eth)=V(\eth)\cup W(\eth)$,

$$T(\eth)=\begin{cases} F(\eth)\cup G(\eth), & \eth\in(K\text{-}D)\cap(Y\text{-}D)=K\cap Y\cap D' \\ F(\eth)\cup[G'(\eth)\cup H'(\eth)], & \eth\in(K\text{-}D)\cap(Y\cap D)=\emptyset \\ [F'(\eth)\cup H'(\eth)]\cup G(\eth), & \eth\in(K\cap D)\cap(Y\text{-}D)=\emptyset \\ [F'(\eth)\cup H'(\eth)]\cup[G'(\eth)\cup H'(\eth)], & \eth\in(K\cap D)\cap(Y\cap D)=K\cap Y\cap D \end{cases}$$

Thus,

$$T(\eth)=\begin{cases} F(\eth)\cup G(\eth), & \eth\in K\cap Y\cap D' \\ [F'(\eth)\cup H'(\eth)]\cup[G'(\eth)\cup H'(\eth)], & \eth\in K\cap Y\cap D \end{cases}$$

Hence, N=T, where $K\cap Y\cap D=\emptyset$.

2) $[(F,K)\cap_R (G,Y)]\tilde{*}(H,D)=[(F,K)\tilde{*}(H,D)]\cap_R[(G,Y)\tilde{*}(H,D)]$.

Corollary 4.1.1. $(S_E(U),\cup_R,\tilde{*})$ is an additive commutative and additive idempotent (right) near-semiring without zero and unity under certain conditions.

Proof: $(S_E(U),\cup_R)$ is a commutative, idempotent monoid with identity element $\emptyset_E$, that is, a bounded semilattice (hence a semigroup) (Ali et al., 2011). By Theorem 3.3.1, $(S_E(U),\tilde{*})$ is a non-commutative and not idempotent semigroup under the condition $T\cap Z'\cap M = T\cap Z\cap M=\emptyset$, where (F,T), (G,Z) and (H,M) are soft sets over U. Besides, by Proposition 4.1. (1), $\tilde{*}$ distributes over $\cap_R$ from RHS under the condition $T\cap Z\cap M=\emptyset$. Thus, $(S_E(U),\cup_R,\tilde{*})$ is an additive commutative and additive idempotent (right) near-semiring without zero and unity under certain conditions.

Corollary 4.1.2. $(S_E(U),\cap_R,\tilde{*})$ is an additive commutative and additive idempotent (right) near-semiring without zero and unity under certain conditions.

Proof: $(S_E(U),\cap_R)$ is a commutative, idempotent monoid with identity element U_E, that is, a bounded semilattice (hence a semigroup) (Ali et al., 2011). By Theorem 3.3.1, $(S_E(U),\tilde{*})$ is a non-commutative and not idempotent semigroup under the condition $T\cap Z'\cap M = T\cap Z\cap M = \emptyset$, where (F,T), (G,Z) and (H,M) are soft sets over U. Besides, by Proposition 4.1. (2), $\tilde{*}$ distributes over $\cap_R$ from RHS under the condition $T\cap Z\cap M = \emptyset$. Thus, $(S_E(U),\cap_R,\tilde{*})$ is an additive commutative and additive idempotent (right) near-semiring without zero and unity under certain conditions.

Proposition 4.2. Let (F,K), (G,Y), and (H,D) be soft sets over U. Then, the distributions of the soft binary piecewise star operation over extended soft set operations are as follows:

LHS Distributions: The followings hold, where $K\cap(Y\Delta D)= K\cap Y\cap D=\emptyset$.

1) $(F,K)\tilde{*}[(G,Y)\cup_\varepsilon(H,D)]=[(F,K)\tilde{*}(G,Y)]\cup_\varepsilon[(F,K)\tilde{*}(H,D)]$.

Proof: First, consider the LHS. Let $(G,Y)\cup_\varepsilon(H,D)=(M,Y\cup D)$, where for all ƍ∈Y∪D,

$$M(ƍ)=\begin{cases} G(ƍ), & ƍ\in Y-D \\ H(ƍ), & ƍ\in D-Y \\ G(ƍ)\cup H(ƍ), & ƍ\in Y\cap D \end{cases}$$

Let $(F,K)\tilde{*}(M,Y\cup D)=(N,K)$, where for all ƍ∈K,

$$N(ƍ)=\begin{cases} F(ƍ), & ƍ\in K-(Y\cup D) \\ F'(ƍ)\cup M'(ƍ), & ƍ\in K\cap(Y\cup D) \end{cases}$$

Thus,

$$N(ƍ)=\begin{cases} F(ƍ), & ƍ\in K-(Y\cup D)=K\cap Y'\cap D' \\ F'(ƍ)\cup G'(ƍ), & ƍ\in K\cap(Y-D)=K\cap Y\cap D' \\ F'(ƍ)\cup H'(ƍ), & ƍ\in K\cap(D-Y)=K\cap Y'\cap D \\ F'(ƍ)\cup[(G'(ƍ)\cap H'(ƍ)], & ƍ\in K\cap Y\cap D=K\cap Y\cap D \end{cases}$$

Now consider the RHS. Let $(F,K)\tilde{*}(G,Y)=(V,K)$, where for all ƍ∈K,

$$V(ƍ)=\begin{cases} F(ƍ), & ƍ\in K-Y \\ F'(ƍ)\cup G'(ƍ), & ƍ\in K\cap Y \end{cases}$$

Let $(F,K)\tilde{*}(H,D)=(W,K)$, where for all ƍ∈K,

$$W(ƍ)=\begin{cases} F(ƍ), & ƍ\in K-D \\ F'(ƍ)\cup H'(ƍ), & ƍ\in K\cap D \end{cases}$$

Let $(V,K)\cup_\varepsilon(W,K)=(T,K)$, where for all ƍ∈K,

$$T(ƍ)=\begin{cases} V(ƍ), & ƍ\in K-K=\emptyset \\ W(ƍ), & ƍ\in K-K=\emptyset \\ V(ƍ)\cap W(ƍ), & ƍ\in K\cap K=K \end{cases}$$

Thus,

$$T(ƍ)=\begin{cases} F(ƍ)\cup F(ƍ), & ƍ\in(K-Y)\cap(K-D)= K\cap Y'\cap D' \\ F(ƍ)\cup[F'(ƍ)\cup H'(ƍ)], & ƍ\in(K-Y)\cap(K\cap D)=K\cap Y'\cap D \\ [F'(ƍ)\cup G'(ƍ)]\cup F(ƍ), & ƍ\in(K\cap Y)\cap(K-D)= K\cap Y\cap D' \\ [F'(ƍ)\cup G'(ƍ)]\cup[F'(ƍ)\cup H'(ƍ)], & ƍ\in(K\cap Y)\cap(K\cap D)=K\cap Y\cap D \end{cases}$$

Thus,

$$T(ƍ)=\begin{cases} F(ƍ), & ƍ\in(K-Y)\cap(K-D)= K\cap Y'\cap D' \\ U & ƍ\in(K-Y)\cap(K\cap D)=K\cap Y'\cap D \\ U & ƍ\in(K\cap Y)\cap(K-D)= K\cap Y\cap D' \\ F'(ƍ)\cup G'(ƍ)\cup H'(ƍ), & ƍ\in(K\cap Y)\cap(K\cap D)=K\cap Y\cap D \end{cases}$$

N=T, where $K\cap Y'\cap D=K\cap Y\cap D'=K\cap Y\cap D=\emptyset$. It is obvious that the condition $K\cap Y'\cap D=K\cap Y\cap D'=\emptyset$ is equal to the condition $K\cap(Y\Delta D)=\emptyset$.

2) $(F,K)\tilde{*}[(G,Y)\cap_\varepsilon(H,D)]=[(F,K)\tilde{*}(G,Y)]\cap_\varepsilon[(F,K)\tilde{*}(H,D)]$.

RHS Distributions: The followings hold where $K\cap Y\cap D=\emptyset$.

1) $[(F,K)\cap_\varepsilon(G,Y)]\tilde{*}(H,D)=[(F,K)\tilde{*}(H,D)]\cap_\varepsilon[(G,Y)\tilde{*}(H,D)]$.

Proof: First consider the LHS. Let $(F,K)\cap_\varepsilon(G,Y)=(M,K\cup Y)$, where for all ƍ∈K∪Y

$$M(ƍ)=\begin{cases} F(ƍ), & ƍ\in K-Y \\ G(ƍ), & ƍ\in Y-K \\ F(ƍ)\cap G(ƍ), & ƍ\in K\cap Y \end{cases}$$

Let $(M,K\cup Y)\tilde{*}(H,D)=(N,K\cup Y)$, where for all ƍ∈K∪Y,

$$N(ƍ)=\begin{cases} M(ƍ), & ƍ\in(K\cup Y)-D \\ M'(ƍ)\cup H'(ƍ), & ƍ\in(K\cup Y)\cap D \end{cases}$$

Thus,

$$N(ƍ)=\begin{cases} F(ƍ), & ƍ\in(K-Y)-D=K\cap Y'\cap D' \\ G(ƍ), & ƍ\in(Y-K)-D=K'\cap Y\cap D' \\ F(ƍ)\cap G(ƍ), & ƍ\in(K\cap Y)-D=K\cap Y\cap D' \\ F'(ƍ)\cup H'(ƍ), & ƍ\in(K-Y)\cap D=K\cap Y'\cap D \\ G'(ƍ)\cup H'(ƍ), & ƍ\in(Y-K)\cap D=K'\cap Y\cap D \\ [F'(ƍ)\cup G'(ƍ)]\cup H'(ƍ), & ƍ\in(K\cap Y)\cap D=K\cap Y\cap D \end{cases}$$

Now consider the RHS, that is, $[(F,K)\tilde{*}(H,D)]\cap_\varepsilon[(G,Y)\tilde{*}(H,D)]$. Let $(F,K)\tilde{*}(H,D)=(V,K)$, where for all ƍ∈K,

$$V(ƍ)=\begin{cases} F(ƍ), & ƍ\in K-D \\ F'(ƍ)\cup H'(ƍ), & ƍ\in K\cap D \end{cases}$$

Let $(G,Y)\tilde{*}(H,D)=(W,Y)$, where for all ƍ∈Y,

$$W(ƍ)=\begin{cases} G(ƍ), & ƍ\in Y-D \\ G'(ƍ)\cup H'(ƍ), & ƍ\in Y\cap D \end{cases}$$

Let $(V,K)\cap_\varepsilon(W,Y)=(T,K\cup Y)$, where for all ƍ∈K∪Y,

$$T(ƍ)=\begin{cases} V(ƍ), & ƍ\in K-Y \\ W(ƍ), & ƍ\in Y-K \\ V(ƍ)\cap W(ƍ), & ƍ\in K\cap Y \end{cases}$$

Thus,

$$T(ƍ)=\begin{cases} F(ƍ), & ƍ\in(K-D)-Y=K\cap Y'\cap D' \\ F'(ƍ)\cup H'(ƍ), & ƍ\in(K\cap D)-Y=K\cap Y'\cap D \\ G(ƍ), & ƍ\in(Y-D)-K=K'\cap Y\cap D' \\ G'(ƍ)\cup H'(ƍ), & ƍ\in(Y\cap D)-K=K'\cap Y\cap D \\ F(ƍ)\cap G(ƍ), & ƍ\in(K-D)\cap(Y-D)=K\cap Y\cap D' \\ F(ƍ)\cap[G'(ƍ)\cup H'(ƍ)], & ƍ\in(K-D)\cap(Y\cap D)=\emptyset \\ [F'(ƍ)\cup H'(ƍ)]\cap G(ƍ), & ƍ\in(K\cap D)\cap(Y-D)=\emptyset \\ [F'(ƍ)\cup H'(ƍ)]\cap[G'(ƍ)\cup H'(ƍ)], & ƍ\in(K\cap D)\cap(Y\cap D)=K\cap Y\cap D \end{cases}$$

Thus,

$$T(ẟ)=\begin{cases} F(ẟ), & ẟ\in K\cap Y'\cap D' \\ F'(ẟ)\cup H'(ẟ), & ẟ\in K\cap Y'\cap D \\ G(ẟ), & ẟ\in K'\cap Y\cap D' \\ G'(ẟ)\cup H'(ẟ), & ẟ\in K'\cap Y\cap D \\ F(ẟ)\cap G(ẟ), & ẟ\in K\cap Y\cap D' \\ [F'(ẟ)\cup H'(ẟ)]\cap[G'(ẟ)\cup H'(ẟ)], & ẟ\in K\cap Y\cap D \end{cases}$$

Thus, N=Y, where $K\cap Y\cap D=\emptyset$.

2) $[(F,K)\cup_{\varepsilon}(G,Y)]\tilde{*}(H,D)=[(F,K)\tilde{*}(H,D)]\cup_{\varepsilon}[(G,Y)\tilde{*}(H,D)]$.

3) $[(F,K)\setminus_{\varepsilon}(G,Y)]\tilde{*}(H,D)=[(F,K)\tilde{*}(H,D)]\setminus_{\varepsilon}[(G,Y)\tilde{*}(H,D)]$.

4) $[(F,K)\Delta_{\varepsilon}(G,Y)]\tilde{*}(H,D)=[(F,K)\tilde{*}(H,D)]\Delta_{\varepsilon}[(G,Y)\tilde{*}(H,D)]$.

5) $[(F,K)+_{\varepsilon}(G,Y)]\tilde{*}(H,D)=[(F,K)\tilde{*}(H,D)]+_{\varepsilon}[(G,Y)\tilde{*}(H,D)]$.

6) $[(F,K)\gamma_{\varepsilon}(G,Y)]\tilde{*}(H,D)=[(F,K)\tilde{*}(H,D)]\gamma_{\varepsilon}[(G,Y)\tilde{*}(H,D)]$.

7) $[(F,K)*_{\varepsilon}(G,Y)]\tilde{*}(H,D)=[(F,K)\tilde{*}(H,D)]*_{\varepsilon}[(G,Y)\tilde{*}(H,D)]$.

8) $[(F,K)\theta_{\varepsilon}(G,Y)]\tilde{*}(H,D)=[(F,K)\tilde{*}(H,D)]\theta_{\varepsilon}[(G,Y)\tilde{*}(H,D)]$.

Corollary 4.2.1. $(S_E(U),\cup_{\varepsilon},\tilde{*})$ and $(S_E(U),\cap_{\varepsilon},\tilde{*})$ are additive commutative and additive idempotent (right) near-semirings with zero but without unity and without zero symmetric property under certain conditions. Furthermore, $(S_E(U),\setminus_{\varepsilon},\tilde{*})$, $(S_E(U),\Delta_{\varepsilon},\tilde{*})$, $(S_E(U),+_{\varepsilon},\tilde{*})$, $(S_E(U),\gamma_{\varepsilon},\tilde{*})$, $(S_E(U),\lambda_{\varepsilon},\tilde{*})$, $(S_E(U),*_{\varepsilon},\tilde{*})$, $(S_E(U),\theta_{\varepsilon},\tilde{*})$ are additive commutative, not idempotent (right) near-semirings with zero but without unity and zero symmetric property under certain conditions.

Proof: Ali et al. [6] showed that $(S_E(U),\cup_{\varepsilon})$ is a commutative, idempotent monoid with identity element $\emptyset_{\emptyset}$, that is, a bounded semilattice (hence a semigroup). By Theorem 3.3.1, $(S_E(U),\tilde{*})$ is a non-commutative and not idempotent semigroup under the condition $T\cap Z'\cap M = T\cap Z\cap M=\emptyset$, where (F,T), (G,Z) and (H,M) are soft sets over U. Besides, by Theorem 3.3 (9), $\emptyset_{\emptyset}\tilde{*}(F,T)=\emptyset_{\emptyset}$, that is $\emptyset_{\emptyset}$ is the left absorbing element for $\tilde{*}$ in $S_E(U)$, furthermore by Proposition 4.2, $\tilde{*}$ distributes over $\cup_{\varepsilon}$ from RHS under the condition $T\cap Z\cap M=\emptyset$. Thus, $(S_E(U),\cup_{\varepsilon},\tilde{\theta})$ is an additive commutative and additive idempotent (right) near-semiring with zero but without unity under certain conditions. Moreover, since $(F,K)\tilde{*}\emptyset_{\emptyset}\neq\emptyset_{\emptyset}$, $(S_E(U),\cup_{\varepsilon},\tilde{*})$ is a (right) near-semiring without zero symettric property. Similarly, $(S_E(U),\cap_{\varepsilon},\tilde{\theta})$ is an additive commutative and additive idempotent (right) near-semiring with zero, but without unity under certain conditions. Furthermore, $(S_E(U),\setminus_{\varepsilon},\tilde{*})$, $(S_E(U),\Delta_{\varepsilon},\tilde{*})$, $(S_E(U),+_{\varepsilon},\tilde{*})$, $(S_E(U),\gamma_{\varepsilon},\tilde{*})$, $(S_E(U),\lambda_{\varepsilon},\tilde{*})$, $(S_E(U),*_{\varepsilon},\tilde{*})$, $(S_E(U),\theta_{\varepsilon},\tilde{*})$ are all additive commutative not idempotent (right) near-semirings with zero, but without unity, and zero symmetric property under certain conditions. Here, note that Aybek (2024) showed that the first operation is associative in $S_E(U)$ under the condition $T\cap Z\cap M=\emptyset$ (for Δ_{ε}, without any conditions).

Corollary 4.2.2. $(S_E(U),\cup_{\varepsilon},\tilde{*})$ and $(S_E(U),\cap_{\varepsilon},\tilde{*})$ are additive commutative and additive idempotent semirings without zero and without unity under certain conditions.

Proof: Ali et al. [6] showed that $(S_E(U),\cup_{\varepsilon})$ is a commutative, idempotent monoid with identity element $\emptyset_{\emptyset}$, that is, a bounded semilattice (hence a semigroup). By Theorem 3.3.1, $(S_E(U),\tilde{*})$ is a non-commutative and not idempotent semigroup under the condition $T\cap Z'\cap M = T\cap Z\cap M=\emptyset$, where (F,T), (G,Z) and (H,M) are soft sets over U. Besides, by Proposition 4.2, $\tilde{*}$ distributes over $\cup_{\varepsilon}$ from LHS under the condition $T\cap(Z\Delta M)= T\cap Z\cap M=\emptyset$, and $\tilde{*}$ distributes over $\cup_{\varepsilon}$ from RHS under the condition $T\cap Z\cap M=\emptyset$. Thus, $(S_E(U),\cup_{\varepsilon},\tilde{\theta})$ is an additive commutative and additive idempotent semiring without zero and without unity under certain conditions. One can similarly show that $(S_E(U),\cap_{\varepsilon},\tilde{\theta})$ is an additive commutative and additive idempotent semiring without zero an unity under certain conditions.

Proposition 4.3. Let (F,K), (G,Y), (H,D) be soft sets on U. Then, the distributions of the soft binary piecewise star operation over soft binary piecewise operations are as follows: The followings hold where $K\cap Y\cap D=\emptyset$.

1) $[(F,K)\tilde{\cap}(G,Y)]\tilde{*}(H,D)=[(F,K)\tilde{*}(H,D)]\tilde{\cap}[(G,Y)\tilde{*}(H,D)]$.

Proof: First, consider the LHS. Let $(F,K)\tilde{\cap}(G,Y)=(M,K)$, where for all ẟ∈K,

$$M(ẟ)=\begin{cases} F(ẟ), & ẟ\in K-Y \\ F(ẟ)\cap G(ẟ), & ẟ\in K\cap Y \end{cases}$$

Let $(M,K)\tilde{*}(H,D)=(N,K)$, where for all ẟ∈K,

$$N(ẟ)=\begin{cases} M(ẟ), & ẟ\in K-D \\ M'(ẟ)\cup H'(ẟ), & ẟ\in K\cap D \end{cases}$$

Thus,

$$N(ẟ)=\begin{cases} F(ẟ), & ẟ\in(K-Y)-D=K\cap Y'\cap D' \\ F(ẟ)\cap G(ẟ), & ẟ\in(K\cap Y)-D=K\cap Y\cap D' \\ F'(ẟ)\cup H'(ẟ), & ẟ\in(K-Y)\cap D=K\cap Y'\cap D \\ [F'(ẟ)\cup G'(ẟ)]\cup H'(ẟ), & ẟ\in(K\cap Y)\cap D=K\cap Y\cap D \end{cases}$$

Now consider the RHS, i.e. $[(F,K)\tilde{*}(H,D)]\tilde{\cap}[(G,Y)\tilde{*}(H,D)]$. Let $(F,K)\tilde{*}(H,D)=(V,K)$, where for all ẟ∈K,

$$V(ẟ)=\begin{cases} F(ẟ), & ẟ\in K-D \\ F'(ẟ)\cup H'(ẟ), & ẟ\in K\cap D \end{cases}$$

Let $(G,Y)\tilde{*}(H,D)=(W,Y)$, where for all ẟ∈Y,

$$W(ẟ)=\begin{cases} G(ẟ), & ẟ\in Y-D \\ G'(ẟ)\cup H'(ẟ), & ẟ\in Y\cap D \end{cases}$$

Let $(V,K)\tilde{\cap}(W,Y)=(T,K)$, where for all ẟ∈K,

$$T(ẟ)=\begin{cases} V(ẟ), & ẟ\in K-Y \\ V(ẟ)\cap W(ẟ), & ẟ\in K\cap Y \end{cases}$$

Thus,

$$T(ẟ)=\begin{cases} F(ẟ), & ẟ\in(K-D)-Y=K\cap Y'\cap D' \\ F'(ẟ)\cup H'(ẟ), & ẟ\in(K\cap D)-Y=K\cap Y'\cap D \\ F(ẟ)\cap G(ẟ), & ẟ\in(K-D)\cap(Y-D)=K\cap Y\cap D' \\ F(ẟ)\cap[G'(ẟ)\cup H'(ẟ)], & ẟ\in(K-D)\cap(Y\cap D)=\emptyset \\ [F'(ẟ)\cup H'(ẟ)]\cap G(ẟ), & ẟ\in(K\cap D)\cap(Y-D)=\emptyset \\ [F'(ẟ)\cup H'(ẟ)]\cap[G'(ẟ)\cup H'(ẟ)], & ẟ\in(K\cap D)\cap(Y\cap D)=K\cap Y\cap D \end{cases}$$

Thus,

$$T(ẟ)=\begin{cases} F(ẟ), & ẟ\in K\cap Y'\cap D' \\ F'(ẟ)\cup H'(ẟ), & ẟ\in K\cap Y'\cap D \\ F(ẟ)\cap G(ẟ), & ẟ\in K\cap Y\cap D' \\ [F'(ẟ)\cap G'(ẟ)]\cup H'(ẟ), & ẟ\in K\cap Y\cap D \end{cases}$$

Thus, N=T, where K∩Y∩D=∅.

2) $[(F,K)\,\tilde{\cup}\,(G,Y)]\,\tilde{*}\,(H,D)=[(F,K)\,\tilde{*}\,(H,D)]\,\tilde{\cup}\,[(G,Y)\,\tilde{*}\,(H,D)]$.

3) $[(F,K)\,\tilde{\setminus}\,(G,Y)]\,\tilde{*}\,(H,D)=[(F,K)\,\tilde{*}\,(H,D)]\,\tilde{\setminus}\,[(G,Y)\,\tilde{*}\,(H,D)]$.

4) $[(F,K)\,\tilde{\Delta}\,(G,Y)]\,\tilde{*}\,(H,D)=[(F,K)\,\tilde{*}\,(H,D)]\,\tilde{\Delta}\,[(G,Y)\,\tilde{*}\,(H,D)]$.

5) $[(F,K)\,\tilde{+}\,(G,Y)]\,\tilde{*}\,(H,D)=[(F,K)\,\tilde{*}\,(H,D)]\,\tilde{+}\,[(G,Y)\,\tilde{*}\,(H,D)]$.

6) $[(F,K)\,\tilde{\gamma}\,(G,Y)]\,\tilde{*}\,(H,D)=[(F,K)\,\tilde{*}\,(H,D)]\,\tilde{\gamma}\,[(G,Y)\,\tilde{*}\,(H,D)]$.

7) $[(F,K)\,\tilde{*}\,(G,Y)]\,\tilde{*}\,(H,D)=[(F,K)\,\tilde{*}\,(H,D)]\,\tilde{*}\,[(G,Y)\,\tilde{*}\,(H,D)]$.

8) $[(F,K)\,\tilde{\theta}\,(G,Y)]\,\tilde{*}\,(H,D)=[(F,K)\,\tilde{*}\,(H,D)]\,\tilde{\theta}\,[(G,Y)\,\tilde{*}\,(H,D)]$.

9) $[(F,K)\,\tilde{\lambda}\,(G,Y)]\,\tilde{*}\,(H,D)=[(F,K)\,\tilde{*}\,(H,D)]\,\tilde{\lambda}\,[(G,Y)\,\tilde{*}\,(H,D)]$.

Corollary 4.3.1. $(S_E(U),\tilde{\cap},\tilde{*})$ and $(S_E(U),\tilde{\cup},\tilde{*})$ are additive idempotent, non-commutative (right) near-semirings without zero and unity under certain conditions.

Proof: Yavuz (2024) showed that $(S_E(U),\tilde{\cap})$ and $(S_E(U),\tilde{\cup})$ are idempotent, non-commutative semigroups (that is a band) under the condition T∩Z'∩M =∅, where (F,T), (G,Z) and (H,M) are soft sets over U. By Theorem 3.3.1, $(S_E(U),\tilde{*})$ is a non-commutative and not idempotent semigroup under the condition T∩Z'∩M = T∩Z∩M =∅, where (F,T), (G,Z) and (H,M) are soft sets over U. Besides, by Proposition 4.3, $\tilde{*}$ distributes over $\tilde{\cap}$ and $\tilde{\cup}$ from RHS under the condition T∩Z∩M=∅. Consequently, $(S_E(U),\tilde{\cap},\tilde{*})$ and $(S_E(U),\tilde{\cup},\tilde{*})$ are additive idempotent non-commutative (right) near-semiring without zero and unity under certain conditions.

Corollary 4.3.2. $(S_E(U),\tilde{\setminus},\tilde{*})$, $(S_E(U),\tilde{\Delta},\tilde{*})$, $(S_E(U),\tilde{+},\tilde{*})$, $(S_E(U),\tilde{\gamma},\tilde{*})$, $(S_E(U),\tilde{*},\tilde{*})$, $(S_E(U),\tilde{\theta},\tilde{*})$ are all not idempotent and non-commutative (right) near-semirings without zero and without unity under the condition T∩Z'∩M = T∩Z∩M =∅, where (F,T), (G,Z) and (H,M) are soft sets over U. Here, note that Yavuz (2024) showed that the first operation is associative in $S_E(U)$ under the condition T∩Z'∩M=T∩Z∩M=∅ (for $\tilde{\Delta}$, under the condition T∩Z'∩M=∅).

5. Conclusion

Parametric techniques like soft sets and soft operations are useful when dealing with uncertain data. Introducing new soft operations and figuring out their algebraic properties and uses provides new insights into handling parametric data problems. This work presents a unique kind of soft set operation in this respect. By putting forward a new soft set operation that we call the "soft binary piecewise star operation" and closely examining the algebraic structures that underlie it as well as other new soft set operations in the class of soft sets, we hope to make a significant contribution to the area of soft set theory. Specifically, the distributions of the soft binary piecewise star operation over different kinds of soft set operations are analyzed, and the whole algebraic properties of this novel soft set operation are investigated in detail. A thorough examination of the algebraic structures produced by the set of soft sets with these operations is given, taking into account the distribution laws and the algebraic characteristics of the soft set operations. We show that the collection of soft sets over the universe with the soft binary piecewise star operation, and other forms of soft sets, form different significant algebraic structures, such as semirings and near-semirings.

- $(S_E(U),\tilde{*})$ is a noncommutative, and not idempotent semigroup under certain conditions, moreover $(S_E(U),\tilde{*})$ is a right-left system under certain conditions.
- $(S_E(U),\cup_R,\tilde{*})$, $(S_E(U),\cap_R,\tilde{*})$ are additive commutative and additive idempotent (right) near-semirings without zero and unity under certain conditions.
- $(S_E(U),\cup_\varepsilon,\tilde{*})$ and $(S_E(U),\cap_\varepsilon,\tilde{*})$ are additive commutative and additive idempotent (right) near-semirings with zero, but without unity and zero symmetric property under certain conditions.
- $(S_E(U),\setminus_\varepsilon,\tilde{*})$, $(S_E(U),\Delta_\varepsilon,\tilde{*})$, $(S_E(U),+_\varepsilon,\tilde{*})$, $(S_E(U),\gamma_\varepsilon,\tilde{*})$, $(S_E(U),\lambda_\varepsilon,\tilde{*})$, $(S_E(U),*_\varepsilon,\tilde{*})$, $(S_E(U),\theta_\varepsilon,\tilde{*})$ are additive commutative not idempotent (right) near-semirings with zero but without unity and zero symmetric property under certain conditiosn.
- $(S_E(U),\cup_\varepsilon,\tilde{*})$ and $(S_E(U),\cap_\varepsilon,\tilde{*})$ are additive commutative and additive idempotent semirings without zero and unity under certain conditions.
- $(S_E(U),\tilde{\cap},\tilde{*})$ and $(S_E(U),\tilde{\cup},\tilde{*})$ are additive idempotent, non-commutative (right) near-semirings without zero and unity under certain conditions.
- $(S_E(U),\tilde{\setminus},\tilde{*})$, $(S_E(U),\tilde{\Delta},\tilde{*})$, $(S_E(U),\tilde{+},\tilde{*})$, $(S_E(U),\tilde{\gamma},\tilde{*})$, $(S_E(U),\tilde{*},\tilde{*})$, $(S_E(U),\tilde{\theta},\tilde{*})$ are all noncommutative, and not idempotent (right) near-semirings without zero and unity under certain conditions.

We obtain a complete understanding of their use by studying new soft set operations and the algebraic structures of soft sets. In addition to offering novel examples of algebraic structures, this might further the fields of soft set theory and classical algebraic literature. The goal of this study is to get the particular algebraic structures that the soft binary piecewise star operation forms in combination with other kinds of soft set operations in the collection of soft sets defined over a universal set. This kind of thorough investigation should improve our understanding of how soft sets are used. Subsequent investigations might explore in detail more variations of soft binary piecewise operations and their corresponding properties and distributions.

Author Contributions

All authors contributed to the study's conception and design. Material preparation, data collection, and analysis were performed by AS. The first draft of the manuscript was written by EY. All authors read and approved the final manuscript.

Acknowledgement

This paper is derived from the the second author's MSc. Thesis, supervised by the first author, at Amasya University, Amasya, Türkiye.

Funding

No funding was received for conducting this study.

Data Availability

The datasets generated during and/or analyzed during the current study are available from the corresponding author (Aslıhan Sezgin, aslihan.sezgin@amasya.edu.tr) on reasonable request.

Conflict of Interest

The authors stated that there are no conflicts of interest regarding the publication of this article.

References

Abbas, M., Ali, B. and Romaguera, S., 2014. On Generalized Soft Equality and Soft Lattice Structure. Filomat, 28 (6), Pp. 1191-1203.

Abbas, M., Ali, M.I., and Romaguera, S., 2017. Generalized operations in soft set theory via relaxed conditions on parameters, Filomat 31 (19), Pp. 5955–5964.

Akbulut E. 2024. New type of extended operations of soft set: Complementary extended difference and lambda operation. MSc Thesis, Amasya University, Amasya, Türkiye.

Ali MI, Feng F, Liu X, Min WK and Shabir M., 2009. On some new operations in soft set theory. Computational and Applied Mathematics, 57 (9), Pp. 1547-1553.

Ali MI, Shabir M, Naz M., 2011. Algebraic structures of soft sets associated with new operations, Computational and Applied Mathematics; 61, Pp. 2647–2654.

Ali, B., Saleem, N., Sundus, N, Khaleeq, S, Saeed, M., George, R. A., 2022. Contribution to the theory of soft sets via generalized relaxed operations, Mathematics, 10, Pp. 26-36.

Al-shami, T.M., 2019. Investigation and corrigendum to some results related to g-soft equality and g f -soft equality relations. Filomat, 33, Pp. 3375–3383.

Alshasi, T., El-Shafei., T., 2020. T-soft equality relation, Turkish Journal of Mathematics, Pp. 44 (4), Article 25.

Atagün AO and Sezgin A., 2018. Soft subnear-rings, soft ideals and soft n-subgroups of near-rings. Math Sci Letters, 7 (1), Pp. 37-42.

Aybek FN. 2024. New restricted and extended soft set operations, MSc Thesis, Amasya University, Amasya, Türkiye.

Beasley, L.B., and Pullman, N.G., 1988. Operators that preserves semiring matrix functions, Linear Algebra Application. 99 199–216.

Beasley, L.B., Pullman, N.G., 1992 Linear operators strongly preserving idempotent matrices over semirings, Linear Algebra Application. 160, Pp. 217–229.

Çağman N, Çitak F and Aktaş H., 2012. Soft int-group and its applications to group theory. Neural Computer Application, 2, Pp. 151–158.

Çağman N., 2021. Conditional complements of sets and their application to group theory. J New Results Sci. 10 (3), Pp. 67-74.

Clifford, A.H., 1954. Bands of Semigroups, Proceedings of the American Mathematical Society, 5 (3), Pp. 499–504.

Demirci, AM., 2024. New type of extended operations of soft set: Complementary extended union, plus and theta operation. MSc Thesis, Amasya University, Amasya, Türkiye.

Eren ÖF and Çalışıcı H. 2019. On some operations of soft sets, The Fourth International Conference on Computational Mathematics and Engineering Sciences; Antalya 2019.

Feng, F. and Li, Y., 2013. Soft Subsets and Soft Product Operations. Information Sciences, Pp. 44-57.

Fu L., 2011. Notes on soft set operations, ARPN Journal of Systems and Softwares, 1, Pp. 205-208.

Ge, X., Yang, S., 2011. Investigations on some operations of soft sets", World Academy of Science, Engineering and Technology, 75, Pp. 1113-1116.

Ghosh, S., 1996. Matrices over semirings, Inform. Sci. 90, Pp. 221–230.

Glazek, K., 2002. A Guide to Literature on Semirings and their Applications in Mathematics and Information Sciences: With Complete Bibliography, Kluwer Academy Publication., Nederland.

Golan, J.S., 1999. Semirings and their Applications, Kluwer Academy Publication.

Goodearl, K.R., 1979. Von Neumann Regular Rings, Pitman, London.

Hebisch, U., and Weinert, H.J., 1998. Semirings: Algebraic Theory and Applications in the Computer Science, World Scientific.

Hoorn, W.G.V. and Rootselaar, V.B., 1967. Fundamental Notions in the Theory of Seminearrings. Compositio Mathematica, 18 (1-2), Pp. 65-78.

Hopcroft, J.E, and Ullman, J.D., 1979. Introduction to Automata Theory, Languages and Computation, Addison Wesley, Reading, MA.

Husain, S., and. Shamsham K.M., 2018, A study of properties of soft set and its applications, International Research Journal of Engineering and Technology, 5 (1), Pp. 363-372.

Iftikhar M and Mahmood T., 2018. Some results on lattice ordered double framed soft semirings, International Journal of Algebra and Statistics, 7, Pp. 123-140

Jana C, Pal M, Karaaslan F and Sezgin A., 2019. (α, β)-soft intersectional rings and ideals with their applications. New Math Nat Computer. 15 (2), Pp. 333–350.

Jayanta S., 2014, On algebraic structure of soft sets, Annals of Fuzzy Mathematics and Informatics, 7 (6), Pp. 1013-1020.

Jun, Y. B. and Yang, X., 2011. A Note on The Paper Combination of Interval-Valued Fuzzy Set and Soft Set. Computers and Mathematics with Applications, 61 (5), Pp. 1468-1470.

Kilp, M., Knauer, U and Mikhalev, A., 2001. Monoids, Acts and Categories. De Gruyter Expositions in Mathematics, (29). https://doi.org/10.1515/9783110812909

Kolokoltsov, V.N., and Maslov, V.P., 1997. Idempotent Analysis and its Applications, in: Mathematics and its Applications, Pp. 401, Kluwer.

Liu, X., Feng, F. and Jun, Y. B., 2012. A Note on Generalized Soft Equal Relations. Computers and Mathematics with Applications, 64 (4), Pp. 572-578.

Maan H.B., 1944. On certain systems which are almost groups. Bulletin of the American Mathematical Society, 50 (12), Pp. 879-881.

Mahmood T, Rehman ZU and Sezgin A., 2018. Lattice ordered soft near rings. Korean Journal Math. 26 (3), Pp. 503-517.

Mahmood T, Waqas A and Rana M.A., 2015. Soft Intersectional Ideals in Ternary Semirings, Science International, 27 (5), Pp. 3929-3934.

Maji PK, Biswas R and Roy AR., 2003. Soft set theory. Comput Math Appl. 45 (1), Pp 555-562.

Molodtsov D., 1999. Soft set theory-first results. Comput Math Appl. 37 (1), Pp. 19-31.

Mordeson, J.N., and Malik, D.S., 2002. Fuzzy Automata and Languages, Theory and Applications, in: Computational Mathematics Series, Chapman and Hall, CRC, Boca Raton.

Muştuoğlu E, Sezgin A and Türk Z.K., 2016. Some characterizations on soft uni-groups and normal soft uni-groups. Int J Comput Appl. 155 (10), Pp. 1-8.

Neog, I.J and Sut D.K., 2011. A New Approach to the Theory of Soft Set, International Journal of Computer Applications, 32 (2), Pp. 1-6.

Onyeozili, I.A., and Gwary, T.M., 2014, A study of the fundamentals of soft set theory, International Journal of Scientific and Technology Research, 3 (4), Pp. 132-143.

Özlü Ş and Sezgin A., 2020. Soft covered ideals in semigroups. Acta Univ Sapientiae Math, 12 (2), Pp. 317-346.

Pant S, Dagtoros K, Kholil MI, Vivas A. 2024. Matrices: Peculiar determinant property, Optimum Science Journal, 1, pp. 1–7.

Pei D and Miao D., 2005. From soft sets to information systems. In: Proceedings of Granular Computing IEEE, 2, Pp. 617-621.

Petrich, M., 1973. Introduction to Semiring, Charles E Merrill Publishing Company, Ohio.

Ping, Z., and Qiaoyan, W., 2013, Operations on soft sets revisited, Journal of Applied Mathematics, 2013 (105752), Pp. 1-7.

Reutenauer C. and Straubing H., 1984. Inversion of matrices over a commutative semiring, Journal of Algebra, 88 (2), Pp. 350–360.

Sarıalioğlu M. New type of extended operations of soft set: Complementary extended intersection, gamma and star operation. MSc Thesis, Amasya University, Amasya, Türkiye.

Sezer AS, Çağman N and Atagün A.O., 2015. Uni-soft substructures of groups. Ann Fuzzy Math Inform, 9 (2): 235–246.

Sezer AS. 2014. Certain Characterizations of LA-semigroups by soft sets. J Intell Fuzzy System, 27 (2), Pp. 1035-1046.

Sezgin A. 2016. A new approach to semigroup theory I: Soft union semigroups, ideals and bi-ideals. Algebra Letters, 2016:3.

Sezgin A and Atagün AO., 2011. On operations of soft sets. Computational and Applied Mathematics, 61 (5), Pp. 1457-1467.

Sezgin A and Atagün AO., 2023. New soft set operation: Complementary soft binary piecewise plus operation. Matrix Science Mathematic, 7 (2), Pp. 125-142.

Sezgin A and Aybek FN., 2023. New soft set operation: Complementary soft binary piecewise gamma operation. Matrix Science Mathematic, (7) 1, Pp. 27-45.

Sezgin A and Çağman N., 2024. New soft set operation: Complementary soft binary piecewise difference operation. Osmaniye Korkut Ata Üniv Fen Biliml Derg, 7 (1), Pp. 58-94.

Sezgin A and Çalışıcı H., 2024. A comprehensive study on soft binary piecewise difference operation. Eskişehir Teknik Üniversitesi Bilim ve Teknoloji Dergisi B - Teorik Bilimler, 12 (1), Pp. 32-54.

Sezgin A and Dagtoros K., 2023. Complementary soft binary piecewise symmetric difference operation: a novel soft set operation. Scientific Journal of Mehmet Akif Ersoy University, 6 (2), Pp. 31-45.

Sezgin A and Demirci AM., 2023. New soft set operation: Complementary soft binary piecewise star operation. Ikonion Journal of Mathematics, 5 (2), Pp. 24-52.

Sezgin A and Orbay M., 2022. Analysis of semigroups with soft intersection ideals. Acta Univ Sapientiae Math. 14 (1), Pp. 166-210.

Sezgin A and Sarıalioğlu M., 2024. New soft set operation: Complementary soft binary piecewise theta operation. Journal of Kadirli Faculty of Applied Sciences, 4 (2), Pp. 325-357.

Sezgin A and Yavuz E., 2023a. A new soft set operation: Soft binary piecewise symmetric difference operation. Necmettin Erbakan University Journal of Science and Engineering, 5 (2): 189-208.

Sezgin A and Yavuz E., 2023b. New soft set operation: Complementary Soft Binary Piecewise Lambda Operation. Sinop University Journal of Natural Sciences, 8 (2), Pp. 101-133.

Sezgin A, Atagün AO and Çağman N and Demir H., 2022. On near-rings with soft union ideals and applications. New Math Nat Comput. 18 (2), Pp. 495-511.

Sezgin A, Aybek FN and Atagün AO., 2023c. New soft set operation: Complementary soft binary piecewise intersection operation. BSJ Eng Sci 6 (4), Pp. 330-346.

Sezgin A, Aybek FN and Güngör N.B., 2023b. New soft set operation: Complementary soft binary piecewise union operation. Acta Informatica Malaysia, 7 (1), Pp. 38-53.

Sezgin A, Çağman N and Atagün AO., 2017. A completely new view to soft intersection rings via soft uni-int product. Appl Soft Computer, 54, Pp. 366-392.

Sezgin A, Shahzad A and Mehmood A., 2019. New operation on soft sets: Extended difference of soft sets. J New Theory, (27), Pp. 33-42.

Sezgin A., 2018. A new view on AG-groupoid theory via soft sets for uncertainty modeling. Filomat, 32 (8), Pp. 2995–3030.

Sezgin A., Çağman N., Atagün AO and Aybek FN., 2023a. Complemental binary operations of sets and their application to group theory. Matrix Science Mathematic, 7 (2), Pp. 114-121.

Singh, D and Onyeozili, I.A., 2012a. Notes on Soft Matrices Operations. ARPN Journal of Science and Technology, 2 (9), Pp. 861-869.

Singh, D. and Onyeozili, I.A., 2012b. On Some New Properties on Soft Set Operations. International Journal of Computer Applications, 59 (4), Pp. 39-44.

Singh, D. and Onyeozili, I.A., 2012c. Some Results on Distributive and Absorption Properties on Soft Operations. IOSR Journal of Mathematics, 4 (2), Pp. 18-30.

Singh, D. and Onyeozili, I.A., 2012d. Some Conceptual Misunderstanding of The Fundamentals of Soft Set Theory. ARPN Journal of Systems and Softwares, 2 (9), Pp. 251-254.

Stojanovic N.S., 2021. A new operation on soft sets: Extended symmetric difference of soft sets. Military Technical Courier, 69 (4), Pp. 779-791.

Tunçay M., Sezgin A. 2016. Soft union ring and its applications to ring theory, International Journal of Computer Applications, 151, 9, 7-13.

Vandiver, H.S., 1934. Note on a simple type of algebra in which the cancellation law of addition does not hold, Bulletin of the American Mathematical Society, 40, (12), Pp. 914-920.

Wechler, W., 1978. The Concept of Fuzziness in Automata and Language Theory, Akademic Verlag, Berlin.

Yang, C.F., 2008. A note on: Soft set theory, Computers and Mathematics with Applications 45 (2003), no. 4-5, 555–562, Computers and Mathematics with Applications, 56 (7), Pp. 1899–1900.

Yavuz E. 2024. Soft binary piecewise operations and their properties. MSc Thesis, Amasya University, Amasya, Türkiye.