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RESEARCH ARTICLE

A CONDITIONAL MARGINAL APPROACH IN MULTIVARIATE POISSON REGRESSION MODEL

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ABSTRACT

In recent times, the conditional marginal modeling approach has emerged as a new area in longitudinal studies. A new multivariate Poisson regression model has been proposed for count data. The model's validity was assessed using simulation techniques, followed by fitting the model to real data from the Health and Retirement Study. The correlation coefficients of the response variable's impact among the first, second and the third follow-up were estimated.

KEYWORDS

Conditional Marginal model, Count Data, Follow-up

1. INTRODUCTION

Regression model determines the effect of dependent variable on changes in independent variable. There are different types of regression models to characterize the independent and dependent variables. If the independent and dependent variables are normally distributed, then we fit the model through classical linear regression model. In other cases, if the independent and dependent variables are not normally distributed, then the model is fitted through the generalized linear model. The field of generalized linear model is very wide, like if the dependent variable is binary then binary logistic regression is used and if the value of dependent variable is of multiple choice type or ordinal then multinomial logistic regression is used. Similarly, Poisson regression is used when the underlying variable values are count data. Data are basically of two types, such as cross sectional data and longitudinal data. The above-mentioned models are used in two types of data, if the independent and dependent variables are normally distributed in cross sectional data, then it is called classical linear regression model, in case of longitudinal data, it is called general linear model. Generalized linear model is used for both cross sectional and longitudinal studies for non-normal data. In this study, longitudinal data has been used, many researcher have done research on longitudinal data. The field of research on longitudinal data is expanding day by day. Longitudinal data were analyzed using generalized linear model (Liang and Zeger, 1986). McCullagh and Nelder (1989) the concept of Generalized Linear Models (GLMs) has been playing an increasingly important role in statistical theory and applications.

The specialty of this study is that a model has been considered from a follow-up. Usually, the model obtained from the first follow-up is independent, so it is called a marginal model. Again the second follow-up depends on the first follow-up, similarly the third follow-up depends on the second and first follow-up and so on. Therefore, except the first follow-up, the model obtained from all follow-ups is called conditional model. Analyzed conditional and marginal regression model for longitudinal data (Ritz and Spiegelman, 2004). Pinheiro (2006) worked on conditional versus marginal covariance representation for linear and nonlinear models. Chowdhury *et al.* (2020) analyzing the effect of duration on the daily new cases of COVID-19 infections and deaths using bivariate Poisson regression: a marginal conditional approach. A bivariate Poisson models with covariate dependence was employed by Islam and Chowdhury (2015). Karlis and Ntzoufras (2003) analyzed sports data by using bivariate Poisson models. Islam and Chowdhury (2017) used a generalized bivariate Poisson regression model with application to health data.

Currently, the conditional marginal modeling approach is a new area in multivariate models used in longitudinal data. In this study, the count response variable was taken at three follow-ups, allowing for the simultaneous estimation of conditional and marginal models for multivariate Poisson regression. The model was fitted using both simulation data and real data from the Health and Retirement Study (2010), specifically focusing on the number of days hospitalized in the previous two years.

Table 1: A Sample example of longitudinal data in wide format

Serial No.	***Time***	***First Follow-up***					***Second Follow-up***					***Third Follow-up***				
	ID	*Response*	*Covariates*				*Response*	*Covariates*				*Response*	*Covariates*			
		Y_1	X_{11}	X_{12}	..	X_{1p}	Y_2	X_{21}	X_{22}	..	X_{2p}	Y_3	X_{31}	X_{32}	..	X_{3p}
01	101	Y_{1101}	X_{11101}	X_{12101}	..	X_{1p101}	Y_{2101}	X_{21101}	X_{22101}	..	X_{2p101}	Y_{3101}	X_{31101}	X_{32101}	..	X_{3p101}
02	102	Y_{1102}	X_{11102}	X_{12102}	..	X_{1p102}	Y_{2102}	X_{21102}	X_{22102}	..	X_{2p102}	Y_{3102}	X_{31102}	X_{32102}	..	X_{3p102}
..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..

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Table 1 (Cont.): A Sample example of longitudinal data in wide format

..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
...	...	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
N	10N	Y_{110N}	X_{1110N}	X_{1210N}	..	X_{1p10N}	Y_{210N}	γ	X_{2210N}	..	X_{2p10N}	Y_{310N}	X_{3110N}	X_{3210N}	..	X_{3p10N}

Where, Y_1, Y_2 and Y_3 represent the outcome of first, second and third follow-up respectively. X_{jk} be the corresponding explanatory variable of j^{th} follow-up $j = 1,2,\ldots,T$ and k^{th} number of variable $k = 1,2,\ldots,p$. In the above table can get an independent model from first follow-up be $g(y_1) = \gamma_{10} + \gamma_{11}x_{11} + \gamma_{12}x_{12} + \ldots + \gamma_{1p}x_{1p} + e_1$. Also, get an independent model from second and third follow-up be $g(y_2) = \gamma_{20} + \gamma_{21}x_{21} + \gamma_{22}x_{22} + \ldots + \gamma_{2p}x_{2p} + e_2$ and $g(y_3) = \gamma_{30} + \gamma_{31}x_{31} + \gamma_{32}x_{32} + \ldots + \gamma_{3p}x_{3p} + e_3$respectively.

Actually, the first follow-up is independent but the subsequent follow-ups are not independent, there is a dependency between them. That's why the model get from the first follow-up is the marginal model $g(y_1) = \beta_{10} + \beta_{11}x_{11} + \beta_{12}x_{12} + \ldots + \beta_{1p}x_{1p} + e_1$ and the rest are the conditional model such as $g(y_2|y_1) = \beta_{20} + \beta_{21}x_{21} + \beta_{22}x_{22} + \ldots + \beta_{2p}x_{2p} + e_2$, $g(y_3|y_2, y_1) = \beta_{30} + \beta_{31}x_{31} + \beta_{32}x_{32} + \ldots + \beta_{3p}x_{3p} + e_3$ and so on.

2. Model and Estimation

Let Y_{ij}be a time dependent count outcome variable (Islam and Chowdhury, 2017) for subject $i$at time j, $i = 1,2,\ldots,N$and $j = 1,2,\ldots,T$. Then the outcome vector for subject $i$can be defined as $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{iT})'$with mean vector

$\mu_i = E(Y_i) = \left(E(Y_{i1}), E(Y_{i2}), \ldots, E(Y_{iT})\right)' = (\mu_{1i}, \mu_{2i}, \ldots, \mu_{Ti})' = (\lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{Ti})'$. Also let X_{ij}be the $p \times 1$ vector of covariates for subject i at j^{th} occasion.

Suppose that the number of occurrences Y_1 recorded in a given time interval has a Poisson distribution with density

$$g_1(y_1) = \frac{e^{-\lambda_1}\lambda_1^{y_1}}{y_1!}; y_1 = 0,1,\ldots.. \quad (1)$$

Let Y_2 be a random variable with the number of occurrences from the 2^{nd} time point, and suppose it has a Poisson distribution with parameter, λ_2; that is,

$$Pr[Y_2 = y_2] = \frac{e^{-\lambda_2}\lambda_2^{y_{2k}}}{y_2!}; y_2 = 0,1,\ldots.. \quad (2)$$

Now if the $Y_2's$ are assumed to be mutually independent, then the conditional distribution of $Y_2 = Y_{21} + Y_{22} + \ldots + Y_{2y_1}$ the total number of occurrences recorded among the Y_1 occurring in the j^{th} time interval is Poisson with parameter $\lambda_2 y_1$. Thus

$$g_2(y_2|y_1) = \frac{e^{-(\lambda_2 y_1)}(\lambda_2 y_1)^{y_2}}{y_2!}; y_2 = 0,1,\ldots.. \quad (3)$$

Again, let Y_3 be a random variable with the number of occurrences from the 3^{rd} time point, and suppose it has a Poisson distribution with parameter, λ_3; that is,

$$Pr[Y_3 = y_3] = \frac{e^{-\lambda_3}\lambda_3^{y_{3k}}}{y_3!}; y_3 = 0,1,\ldots.. \quad (4)$$

Now if the $Y_3's$ are assumed to be mutually independent, then the conditional distribution of $Y_3 = Y_{31} + Y_{32} + \ldots + Y_{3y_2}$ the total number of occurrences recorded among the Y_2 occurring in the j^{th} time interval is Poisson with parameter $\lambda_3 y_2 y_1$. Thus

$$g_3(y_3|y_2, y_1) = \frac{e^{-(\lambda_3 y_2 y_1)}(\lambda_3 y_2 y_1)^{y_3}}{y_3!}; y_3 = 0,1,\ldots.. \quad (5)$$

Therefore the joint distribution y_1, $y_2$and y_3 is

$$g(y_1, y_2, y_3) = \frac{e^{-(\lambda_3 y_2 y_1)}(\lambda_3 y_2 y_1)^{y_3} e^{-(\lambda_2 y_1)}(\lambda_2 y_1)^{y_2} e^{-\lambda_1}\lambda_1^{y_1}}{y_1! \times y_2! \times y_3!} \quad (6)$$

In this study, three time points or three follow-ups have been considered.

Therefore the log likelihood function is

$$log\, L = \sum [-\lambda_3 y_2 y_1 + y_3\, log(\lambda_3 y_2 y_1) - \lambda_2 y_1 + y_2\, log(\lambda_2 y_1) - \lambda_1 + y_1\, log(\lambda_1) - log(y_1! \times y_2! \times y_3!)] \quad (7)$$

Link functions of multivariate Poisson regression are

$\lambda_1 = e^{X_1\beta_1} or X_1\beta_1 = log(\lambda_1)$

$\lambda_2 = e^{X_2\beta_{2.1}} or X_2\beta_{2.1} = log(\lambda_2)$

$\lambda_3 = e^{X_3\beta_{3.21}} or X_3\beta_{3.21} = log(\lambda_3)$

Where, $\beta_1$is the vector of parameter of the marginal model of y_1, $\beta_{2.1}$ and $\beta_{3.21}$ are the vector of parameters of the conditional models of y_2 given y_1 and y_3 given y_2 and y_1 respectively.

The log likelihood function become

$$log\, L = \sum \left[-y_1 y_2 e^{X_3\beta_{3.21}} + y_3 X_3\beta_{3.21} - y_1 e^{X_2\beta_{2.1}} + y_2 X_2\beta_{2.1} - e^{X_1\beta_1} + y_1 X_1\beta_1\right] + \text{Constant} \quad (8)$$

Now differentiating partially with respect to $\beta_{3.21}$, $\beta_{2.1}$and $\beta_1$and equating to zero.

$$\frac{\partial\, log\, L}{\partial\beta_{3.21}} = \sum\left(y_3 - y_1 \times y_2 \times e^{X_3\beta_{3.21}}\right) \times X_3 = 0 \quad (9)$$

$$\frac{\partial\, log\, L}{\partial\beta_{2.1}} = \sum\left(y_2 - y_1 \times e^{X_2\beta_{2.1}}\right) \times X_2 = 0 \quad (10)$$

$$\frac{\partial\, log\, L}{\partial\beta_1} = \sum\left(y_1 - e^{X_1\beta_1}\right) \times X_1 = 0 \quad (11)$$

Where equation (9), (10) and (11) are the estimating equations or score vectors of conditional and marginal models respectively.

The score vectors are

$$U(\beta_{3.21}) = \left[\frac{\partial\, log\, L}{\partial\beta 0_{3.21}}, \frac{\partial\, log\, L}{\partial\beta 1_{3.21}}, \ldots, \frac{\partial\, log\, L}{\partial\beta p_{3.21}}\right]$$

$$U(\beta_{2.1}) = \left[\frac{\partial\, log\, L}{\partial\beta 0_{2.1}}, \frac{\partial\, log\, L}{\partial\beta 1_{2.1}}, \ldots, \frac{\partial\, log\, L}{\partial\beta p_{2.1}}\right]$$

$$U(\beta_1) = \left[\frac{\partial\, log\, L}{\partial\beta 0_1}, \frac{\partial\, log\, L}{\partial\beta 1_1}, \ldots, \frac{\partial\, log\, L}{\partial\beta p_1}\right]$$

Where, $U(\beta_{3.21})$, $U(\beta_{2.1})$ and $U(\beta_1)$ are the score functions of conditional-marginal models.

The second derivatives are

$$\frac{\partial^2\, log\, L}{\partial\beta_{3.21}^2} = -\sum\left(y_1 \times y_2 \times e^{X_3\beta_{3.21}}\right) \times X_3^2 \quad (12)$$

$$\frac{\partial^2\, log\, L}{\partial\beta_{2.1}^2} = -\sum\left(y_1 \times e^{X_2\beta_{2.1}}\right) \times X_2^2 \quad (13)$$

$$\frac{\partial^2\, log\, L}{\partial\beta_1^2} = -\sum\left(e^{X_1\beta_1}\right) \times X_1^2 \quad (14)$$

Using the second derivatives, define the observed information matrices are

$$I_0(\beta_{3.21}) = \begin{bmatrix} -\left(\frac{\partial^2\, log\, L}{\partial\beta_{3.21}^2}\right)_{1\times1} & O_{1\times1} \\ O_{1\times1} & -\left(\frac{\partial^2\, log\, L}{\partial\beta_{3.21}^2}\right)_{p\times p} \end{bmatrix}$$

Where, $I_0(\beta_{3.21})$ is the information matrix of the conditional model y_3 given y_2 and y_1.

Again,

$$I_0(\beta_{2.1}) = \begin{bmatrix} -\left(\frac{\partial^2 \log L}{\partial \beta_{2.1}^2}\right)_{1\times 1} & O_{1\times 1} \\ O_{1\times 1} & -\left(\frac{\partial^2 \log L}{\partial \beta_{2.1}^2}\right)_{p\times p} \end{bmatrix}$$

Where, $I_0(\beta_{2.1})$ is the information matrix of the conditional model y_2 given y_1.

And

$$I_0(\beta_1) = \begin{bmatrix} -\left(\frac{\partial^2 \log L}{\partial \beta_1^2}\right)_{1\times 1} & O_{1\times 1} \\ O_{1\times 1} & -\left(\frac{\partial^2 \log L}{\partial \beta_1^2}\right)_{p\times p} \end{bmatrix}$$

Where, $I_0(\beta_1)$ is the information matrix of the marginal model y_1.

Using Newton-Raphson method (Henrici, 1974), to estimate parameters of the model as shown below:

$$\hat{\beta}_{3.21j} = \hat{\beta}_{3.21j-1} + I_0^{-1}\left(\hat{\beta}_{3.21j-1}\right) U\left(\hat{\beta}_{3.21j-1}\right) \quad (15)$$

$$\hat{\beta}_{2.1j} = \hat{\beta}_{2.1j-1} + I_0^{-1}\left(\hat{\beta}_{2.1j-1}\right) U\left(\hat{\beta}_{2.1j-1}\right) \quad (16)$$

$$\hat{\beta}_{1j} = \hat{\beta}_{1j-1} + I_0^{-1}\left(\hat{\beta}_{1j-1}\right) U\left(\hat{\beta}_{1j-1}\right) \quad (17)$$

Where, $\hat{\beta}_{3.21j}$, $\hat{\beta}_{2.1j}$and $\hat{\beta}_{1j}$ are the estimated value of the parameters of conditional marginal models.

3. CORRELATION

The number of occurrences as a response variable for two time points y_1 and y_2, which is follows Poisson distribution The correlation between y_1 and $y_2$shown by is Leiter and Hamdan (1973) is

$$corr(Y_1, Y_2) = \frac{\lambda_1\lambda_2}{[\lambda_1\lambda_1\lambda_2(\lambda_2+1)]^{\frac{1}{2}}} = \left(\frac{\lambda_2}{1+\lambda_2}\right)^{\frac{1}{2}} \quad (18)$$

4. SIMULATION STUDY

Simulation techniques are very important for a new model. Because the justification of the model is verified using simulation techniques. Simulation techniques are used to verify the accuracy of a new model. First, we generate data for specific parameters of the new or proposed model using simulation techniques, and then estimate the parameters of the model using that data. If there is no significant difference found between the parametric values and the estimated values of the parameters, then the proposed model is considered valid.

In table 2 to table 4, different three sets of values of Poisson regression parameter β have been taken. Then based on those parametric value data have been simulated and then estimate the parametric value. It has been observed from table 2 to table 4 that, all the estimates are closed to parametric value and corresponding standard error are very small. Converge probability (CP) indicates the probability that the estimated value is equal to the parametric value. In table 2, for the parametric value $\beta_0 = 0.1$ and $\beta_1 = 0.6$, the estimated values are $\hat{\beta}_0 = 0.093$and $\hat{\beta}_1 = 0.605$ corresponding standard error $SE\left(\hat{\beta}_0\right) = 0.007$ and $SE\left(\hat{\beta}_1\right) = 0.008$. Converge probability of β_0 and β_1 are $CP\left(\hat{\beta}_0\right) = 0.978$ and $CP\left(\hat{\beta}_1\right) = 0.904$. Except the converge probability of β_1 for parametric value $\beta_0 = 0.1$ and $\beta_1 = 0.6$ in table 2, all the estimates are converge to more than 95% with its parametric value.

Result from table 2 to table 4 indicate that the proposed multivariate Poisson regression model is appropriate for analyzing the data.

Table 2: Simulation study of the marginal model for Y_1

$\beta_0 = 0.1$ and $\beta_1 = 0.6$					$\beta_0 = 0.5$ and $\beta_1 = 0.9$					$\beta_0 = -0.2$ and $\beta_1 = -0.3$			
Parameter	Estimate	SE	CP		Parameter	Estimate	SE	CP		Parameter	Estimate	SE	CP
β_0	0.093	0.007	0.978		β_0	0.507	0.006	0.991		β_0	-2.004	0.003	0.997
β_1	0.605	0.008	0.904		β_1	0.892	0.006	0.995		β_1	-2.996	0.003	0.997

Note: Standard Error (SE), Converge Probability (CP)

Table 3: Simulation study of the Conditional model for Y_2 given Y_1

$\beta_0 = 0.2$ and $\beta_1 = 0.5$					$\beta_0 = 0.6$ and $\beta_1 = 0.8$					$\beta_0 = -0.3$ and $\beta_1 = -0.5$			
Parameter	Estimate	SE	CP		Parameter	Estimate	SE	CP		Parameter	Estimate	SE	CP
β_0	0.198	0.004	0.970		β_0	0.597	0.003	0.998		β_0	-0.298	0.005	0.998
β_1	0.496	0.006	0.998		β_1	0.805	0.003	0.999		β_1	-0.504	0.004	0.997

Table 4: Simulation study of the Conditional model for Y_3 given Y_2 andY_1

$\beta_0 = 0.3$ and $\beta_1 = 0.7$					$\beta_0 = 0.4$ and $\beta_1 = 0.6$					$\beta_0 = -0.4$ and $\beta_1 = -0.6$			
Parameter	Estimate	SE	CP		Parameter	Estimate	SE	CP		Parameter	Estimate	SE	CP
β_0	0.304	0.002	0.999		β_0	0.399	0.001	0.982		β_0	-0.402	0.002	0.979
β_1	0.697	0.002	0.996		β_1	0.601	0.001	0.960		β_1	-0.603	0.001	0.964

5. APPLICATION TO REAL DATA

In this paper the proposed model are applied Health and Retirement Study (HRS) data (2010). This Health and Retirement Study (HRS) is a longitudinal household survey conducted by the Institute for Social Research at the University of Michigan. The multidisciplinary data provide researchers the opportunity to investigate many different aspects related to population aging in the United States.

The outcome variable Y_1, Y_2 and Y_3as are the number of days hospitalized in previous two years for the year 2016, 2018 and 2020 respectively. The highly significant explanatory variables age, high blood pressure (BP), diabetes, cancer and lung disease have been considered.

Table 5: Marginal model for Y_1

Covariate	Estimate	SE	Wald test	P-Value
Constant	0.264	0.031	8.646	<0.001
Age (60+)	-0.040	0.027	-1.478	0.070
High BP (Yes)	0.054	0.029	1.881	0.030
Diabetes (Yes)	0.154	0.026	6.013	<0.001
Cancer (Yes)	0.117	0.029	4.070	<0.001
Lung (Yes)	0.126	0.035	3.650	<0.001

Significantly positive impact for hospitalization have been observed who have been high BP, diabetes, cancer and lung disease and there are no significant impact found in age.

Table 6: Conditional model for Y_2 given Y_1

Covariate	Estimate	SE	Wald test	P-Value
Constant	-0.433	0.039	-11.010	<0.001
Age (60+)	-0.162	0.032	-5.056	<0.001
High BP (Yes)	0.257	0.036	7.214	<0.001
Diabetes (Yes)	0.170	0.028	6.037	<0.001
Cancer (Yes)	-0.022	0.031	-0.694	0.244
Lung (Yes)	0.343	0.032	10.633	<0.001
Correlation of y_2 given y_1 $\hat{\rho}_{y_2\|y_1} = 0.664,\ P\ Value = 0.002$				

Except covariate cancer diseases, remaining all are significantly associated with hospitalization. Also observed from table 6, impact of age is negative and others impact positive for hospitalization. The significantly positively highly correlation of y_2 given y_1 be 0.664, that means second survey was positively highly correlated.

Table 7: Conditional model for Y_3 given Y_2 andY_1

Covariate	Estimate	SE	Wald test	P-Value
Constant	-0.899	0.052	-17.135	<0.001
Age (60+)	0.199	0.044	4.574	<0.001
High BP (Yes)	0.234	0.042	5.600	<0.001
Diabetes (Yes)	-0.203	0.033	-6.221	<0.001
Cancer (Yes)	-0.156	0.035	-4.477	<0.001
Lung (Yes)	-0.066	0.036	-1.832	0.034
Correlation of y_3 given $y_2$and y_1 $\hat{\rho}_{y_3\|y_2,y_1} = 0.584,\ P\ Value = 0.002$				

Age and high BP are positively and diabetes, cancer and lung disease are negativity associated with hospitalization, diabetes and lung diseases are shown positive impact for hospitalization in above model in table 6. Significant positive correlation of y_3 given $y_2$and y_1. All the calculations have been performed by R-Language (Version-4.1.3).

6. Conclusions

The use of multivariate count regression models is highly beneficial for analyzing longitudinal data from various fields such as health, economics, and insurance. Poisson regression is typically used to analyze count data. When dealing with longitudinal data involving more than two count response variables, multivariate Poisson regression becomes applicable. In longitudinal studies, each follow-up (except the first) depends on the previous one, making the conditional marginal approach essential for accurate analysis.

This paper proposes a new model using the Poisson-Poisson-Poisson conditional marginal approach. The model is applied to both simulation data and real data on the number of days hospitalized over the past two years from the Health and Retirement Study.

In the marginal model, high blood pressure, diabetes, cancer, and lung diseases are positively associated. In the conditional model, high blood pressure, diabetes, and lung diseases are positively associated, while age is negatively associated. Additionally, age and high blood pressure are positively associated, whereas diabetes, cancer and lung disease are negatively associated in the conditional model. This pattern holds consistently across different follow-ups.

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