Matrix Science Mathematic (MSMK)

DOI: http://doi.org/10.26480/msmk.01.2020.06.09

ZIBELINE INTERNATIONAL PUBLISHING

ISSN: 2521-0831 (Print)
ISSN: 2521-084X (Online)
CODEN: MSMADH

CrossMark

REVIEW ARTICLE

ON NEW WAYS OF VARIOUS IDEALS IN TERNARY SEMIGROUPS

M. Palanikumar*, K. Arulmozhi

Department of Mathematics, Annamalai University, India.
**Corresponding Author Email: palanimaths86@gmail.com*

ARTICLE DETAILS

Article History:

Received 18 April 2020
Accepted 21 May 2020
Available online 15 June 2020

ABSTRACT

We discuss tri-quasi ideals and bi-quasi ideals in ternary semigroups and give some characterizations. The intersection of left, lateral and right ideals is a tri-ideal and product of left, lateral and right ideals is again a tri-ideal. We also discuss m-tri-ideals towards some characterizations in terms of tri-ideals. Some relevant counter examples are also indicated.

KEYWORDS

m-tri-ideals, m-bi-ideals, m-quasi ideals, tri-quasi ideal.

1. INTRODUCTION

A researcher initiated the concept of ternary algebraic systems in 1932 (Lehmer, 1932). In other study researchers was introduced by the notion of quasi-ideal in semigroup and ring. Quasi-ideals are generalization of right ideals, lateral ideals and left ideals whereas bi-ideals are generalization of quasi-ideals (Arulmozhi, 2020). The notion of bi-ideals in rings and semigroups were introduced (Lajos, 1969). Rao introduced bi-quasi ideals of semigroups. The notion of tri-ideals as a generalization of quasi ideal, bi-ideal, interior ideal, left (right) ideal and ideal of semirings and study the properties of tri-ideals of a semiring (Rao, 2020). In this paper, we introduce m-quasi ideal, m-bi-ideal, left bi-quasi ideal, lateral bi-quasi ideal, right bi-quasi ideal, left tri-quasi ideal, lateral tri-quasi ideal, right tri-quasi ideal of ternary semigroup (Lajos and Szasz, 1970; Lister, 1971; Dubey and Anuradha, 2013; Munir and Shafiq, 2018).

2. PRELIMINARIES

Here $\mathbf{S}$ will denotes a ternary semigroup unless otherwise stated:

Definition 2.1 A non-empty subset A of $\mathbf{S}$ is called a ternary subsemigroup (shortly TSS) if $[t_1t_2t_3]\in A$ for all $t_1, t_2, t_3 \in A.$

Definition 2.2 A non-empty subset A of $\mathbf{S}$ is called a left (right, lateral) ideal of $\mathbf{S}$ if $s_1s_2a\in A$ $(as_1s_2\in A,\ s_1as_2\in A)$ for all $s_1, s_2 \in \mathbf{S}$ and $a\in A.$ If A is a left, right and lateral ideal of $\mathbf{S},$ then A is called an ideal of $\mathbf{S}$.

Definition 2.3 (i) A TSS B of $\mathbf{S}$ is called a bi-ideal if $B\mathbf{S}B\mathbf{S}B\subseteq B.$

(ii) A TSS Q of $\mathbf{S}$ is called a quasi ideal if $Q\mathbf{S}\mathbf{S}\cap[\mathbf{S}Q\mathbf{S}\cup \mathbf{S}\mathbf{S}Q\mathbf{S}\mathbf{S}]\cap\mathbf{S}\mathbf{S}Q\subseteq Q.$

Definition 2.4 For any positive integers l, n, a subset L of $\mathbf{S}$ is called l -left (n -right) ideal if $\mathbf{S}^l L\subseteq L$ $(L\mathbf{S}^n\subseteq L).$

Remark 2.5 For a semigroup $\mathbf{S}$ and a positive integer m, $\mathbf{S}^m = \mathbf{SSS}.....\mathbf{S}(m-times).$ Now $\mathbf{S}^2 = \mathbf{SS}\subseteq \mathbf{S}$ and $\mathbf{S}^3 = \mathbf{SSS}\subseteq \mathbf{S}^2\subseteq \mathbf{S}.$ We conclude that $\mathbf{S}^l\subseteq\mathbf{S}^m$ for all positive integers l and m, such that $l\geq m.$ Consequently $\mathbf{S}^m\subseteq\mathbf{S},$ for all m.

Definition 2.6 A non-empty subset A of a semigroup $\mathbf{S}$ is called a:

(i) right tri-ideal of $\mathbf{S}$ if $AA\mathbf{S}A\subseteq A.$

(ii) left tri-ideal of $\mathbf{S}$ if $A\mathbf{S}AA\subseteq A.$

(iii) tri-ideal of $\mathbf{S}$ if A is a right tri-ideal and left tri-ideal of $\mathbf{S}$.

(iv) left bi-quasi ideal of $\mathbf{S}$ if A is a subsemigroup of $\mathbf{S}$ and $\mathbf{S}A\cap A\mathbf{S}A\subseteq A.$

(v) right bi-quasi ideal of $\mathbf{S}$ if A is a subsemigroup of $\mathbf{S}$ and $A\mathbf{S}\cap A\mathbf{S}A\subseteq A.$

(vi) bi-quasi ideal of $\mathbf{S}$ if A is a left bi-quasi ideal and right bi-quasi ideal of $\mathbf{S}$.

3. VARIOUS TRI IDEALS

Definition 3.1 A TSS A of $\mathbf{S}$ is called a:

(i) left tri-ideal of $\mathbf{S}$ if $AA\mathbf{S}\mathbf{S}AAA\subseteq A.$

(ii) lateral tri-ideal of $\mathbf{S}$ if $AA\mathbf{S}A\mathbf{S}AA\subseteq A.$

(iii) right tri-ideal of $\mathbf{S}$ if $AAA\mathbf{S}\mathbf{S}AA\subseteq A.$

(iv) tri-ideal of $\mathbf{S}$ if A is a left tri-ideal, lateral tri-ideal and right tri-ideal.

Theorem 3.2 Every left (lateral, right) ideal is a left (lateral, right) tri-ideal.

Converse of the Theorem 3.2 may not be true as in the given Example.

Quick Response Code	Access this article online	
	Website: www.matrixsmathematic.com	**DOI:** 10.26480/msmk.01.2020.06.09

Cite the Article: M. Palanikumar, K. Arulmozhi (2020). On New Ways Of Various Ideals In Ternary Semigroups. *Matrix Science Mathematic, 4(1): 06-09.*

Example 3.3

Let
$$\mathbf{S}_1 = \left\{ \begin{pmatrix} 0 & u_1 & u_2 & u_3 \\ 0 & 0 & u_4 & u_5 \\ 0 & 0 & 0 & u_6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \ \mathrm{u_i}^{'s} \text{ are non positive intergers} \right\}$$

and
$$\mathbf{S}_2 = \left\{ \begin{pmatrix} 0 & v_1 & v_2 & v_3 & v_4 & v_5 \\ 0 & 0 & v_6 & v_7 & v_8 & v_9 \\ 0 & 0 & 0 & v_{10} & v_{11} & v_{12} \\ 0 & 0 & 0 & 0 & v_{13} & v_{14} \\ 0 & 0 & 0 & 0 & 0 & v_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \ \mathrm{v_i}^{'s} \text{ are non positive intergers} \right\}.$$

Let
$$A_1 = \left\{ \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \ \text{x, y are non positive intergers} \right\}$$

and
$$A_2 = \left\{ \begin{pmatrix} 0 & x_1 & 0 & x_2 & 0 & x_3 \\ 0 & 0 & x_4 & 0 & 0 & x_5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \ \mathrm{x_i}^{'s} \text{ are non positive intergers} \right\}.$$

Hence A_1 is a left tri-ideal and right tri-ideal of $\mathbf{S}_1$ but not left ideal and not right ideal of $\mathbf{S}_1$ and A_2 is a lateral tri-ideal of $\mathbf{S}_2$, but not lateral ideal of $\mathbf{S}_2$.

Theorem 3.4 If A_1 is a left ideal, A_2 is a lateral ideal and A_3 is a right ideal of $\mathbf{S}$, then $A_1 \cap A_2 \cap A_3$ and $A_1 \circ A_2 \circ A_3$ is a tri-ideal of $\mathbf{S}$.

Definition 3.5 A non-empty subset A of $\mathbf{S}$ is called a:

(i) left bi-quasi ideal of $\mathbf{S}$ if A is a TSS of $\mathbf{S}$ and $\mathbf{S}\mathbf{S}A \cap ASASA \subseteq A$.

(ii) lateral bi-quasi ideal of $\mathbf{S}$ if A is a TSS of $\mathbf{S}$ and $\mathbf{S}A\mathbf{S} \cap ASASA \subseteq A$.

(iii) right bi-quasi ideal of $\mathbf{S}$ if A is a TSS of $\mathbf{S}$ and $A\mathbf{S}\mathbf{S} \cap ASASA \subseteq A$.

(iv) bi-quasi ideal of $\mathbf{S}$ if A is a left bi-quasi ideal, lateral bi-quasi ideal and right bi-quasi ideal.

Definition 3.6 A non-empty subset A of $\mathbf{S}$ is called a:

(i) left tri-quasi ideal of $\mathbf{S}$ if A is a TSS of $\mathbf{S}$ and $\mathbf{S}\mathbf{S}A \cap AA\mathbf{S}\mathbf{S}AAA \subseteq A$.

(ii) lateral tri-quasi ideal of $\mathbf{S}$ if A is a TSS of $\mathbf{S}$ and $\mathbf{S}A\mathbf{S} \cap AASASAA \subseteq A$.

(iii) right tri-quasi ideal of $\mathbf{S}$ if A is a TSS of $\mathbf{S}$ and $A\mathbf{S}\mathbf{S} \cap AAA\mathbf{S}\mathbf{S}AA \subseteq A$.

(iv) tri-quasi ideal of $\mathbf{S}$ if A is a left tri-quasi ideal, lateral tri-quasi ideal and right tri-quasi ideal.

Theorem 3.7 Every bi-quasi ideal is a tri-quasi ideal.

Theorem 3.8 (i) If A is a left bi-quasi ideal of $\mathbf{S}$, then A is a tri-ideal of $\mathbf{S}$.

(ii) If A is a lateral bi-quasi ideal of $\mathbf{S}$, then A is a tri-ideal of $\mathbf{S}$.

(iii) If A is a right bi-quasi ideal of $\mathbf{S}$, then A is a tri-ideal of $\mathbf{S}$.

Proof. (iii) Suppose that A is a right bi-quasi ideal of $\mathbf{S}$, then $A\mathbf{S}\mathbf{S} \cap ASASA \subseteq A$. Now, $AAA\mathbf{S}\mathbf{S}AA \subseteq A\mathbf{S}\mathbf{S} \cap ASASA \subseteq A$ and $AA\mathbf{S}\mathbf{S}AAA \subseteq A\mathbf{S}\mathbf{S} \cap ASASA \subseteq A$ and $AASASAA \subseteq A\mathbf{S}\mathbf{S} \cap ASASA \subseteq A$. Thus, A is a tri-ideal of $\mathbf{S}$. Similarly to prove (i) and (ii).

Corollary 3.9 If A is a bi-quasi ideal of $\mathbf{S}$, then A is a tri-ideal of $\mathbf{S}$.

Theorem 3.10 Every bi-ideal is a left (lateral, right) tri-ideal of $\mathbf{S}$.

Converse of the Theorem 3.10 is need not true by the following Example.

Example 3.11

Let
$$\mathbf{S}_1 = \left\{ \begin{pmatrix} 0 & u_1 & u_2 & u_3 & u_4 & u_5 \\ 0 & 0 & u_6 & u_7 & u_8 & u_9 \\ 0 & 0 & 0 & u_{10} & u_{11} & u_{12} \\ 0 & 0 & 0 & 0 & u_{13} & u_{14} \\ 0 & 0 & 0 & 0 & 0 & u_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \ \mathrm{u_i}^{'s} \text{ are non positive intergers} \right\}.$$

$$\mathbf{S}_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ v_1 & 0 & 0 & 0 & 0 & 0 \\ v_2 & v_3 & 0 & 0 & 0 & 0 \\ v_4 & v_5 & v_6 & 0 & 0 & 0 \\ v_7 & v_8 & v_9 & v_{10} & 0 & 0 \\ v_{11} & v_{12} & v_{13} & v_{14} & v_{15} & 0 \end{pmatrix} \middle| \ \mathrm{v_i}^{'s} \text{ are non positive intergers} \right\}.$$

Let
$$A_1 = \left\{ \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \ \mathrm{x_i}^{'s} \text{ are non positive intergers} \right\}.$$

$$A_2 = \left\{ \begin{pmatrix} 0 & y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_2 & 0 & y_3 \\ 0 & 0 & 0 & 0 & y_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \ \mathrm{y_i}^{'s} \text{ are non positive intergers} \right\}.$$

And
$$A_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ z_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_3 & 0 \end{pmatrix} \middle| \ \mathrm{z_i}^{'s} \text{ are non positive intergers} \right\}.$$

Hence A_1 and A_2 are left tri-ideal and lateral tri-ideal of $\mathbf{S}_1$ respectively, but not bi-ideal of $\mathbf{S}_1$ and A_3 is a right tri-ideal of $\mathbf{S}_2$, but not bi-ideal of $\mathbf{S}_2$.

Definition 3.12 A non-empty subset A of $\mathbf{S}$ is called a interior ideal of $\mathbf{S}$ if $SISIS \subseteq I$.

Theorem 3.13 Every interior ideal is a left (lateral, right) tri-ideal of $\mathbf{S}$.

Proof. Suppose that I is a interior ideal of $\mathbf{S}$, then $SISIS \subseteq I$. Now, $II\mathbf{S}\mathbf{S}III \subseteq SISIS \subseteq I$. Thus, I is a left tri-ideal of $\mathbf{S}$.

Theorem 3.14 Let A be a TSS of $\mathbf{S}$. If U_1 is a right ideal, U_2 is a lateral ideal and U_3 is a left ideal of $\mathbf{S}$ such that $U_1U_2U_3 \subseteq A \subseteq U_1 \cap U_2 \cap U_3$, then A is a tri-ideal of $\mathbf{S}$.

Proof. Suppose that U_1 is a right ideal, U_2 is a lateral ideal and U_3 is a left ideal of $\mathbf{S}$ such that $U_1U_2U_3 \subseteq A \subseteq U_1 \cap U_2 \cap U_3$. Then $AAS\mathbf{S}AAA \subseteq (U_1 \cap U_2 \cap U_3)(U_1 \cap U_2 \cap U_3)$ $\mathbf{S}\mathbf{S}(U_1 \cap U_2 \cap U_3)(U_1 \cap U_2 \cap U_3)(U_1 \cap U_2 \cap U_3) \subseteq U_1U_2\mathbf{S}\mathbf{S}U_3U_3U_3 \subseteq U_1U_2\mathbf{S}\mathbf{S}U_3$ $\subseteq U_1U_2U_3 \subseteq A$. Thus A is a left tri-ideal of $\mathbf{S}$. Similarly, A is a right (lateral) tri-ideal of $\mathbf{S}$. Hence A is a tri-ideal of $\mathbf{S}$.

Theorem 3.15 The intersection of a left (lateral, right) tri-ideal A of $\mathbf{S}$ and an ideal I of $\mathbf{S}$ is a left (lateral, right) tri-ideal of $\mathbf{S}$.

Proof. Suppose $J = A \cap I$. Then $JJ\mathbf{S}\mathbf{S}JJJ \subseteq AA\mathbf{S}\mathbf{S}AAA \subseteq A$. Since I is an ideal, $JJ\mathbf{S}\mathbf{S}JJJ \subseteq II\mathbf{S}\mathbf{S}III \subseteq I\mathbf{S}\mathbf{S} \subseteq I$ and $JJ\mathbf{S}\mathbf{S}JJJ \subseteq II\mathbf{S}\mathbf{S}III \subseteq \mathbf{S}I\mathbf{S} \subseteq I$ and $JJ\mathbf{S}\mathbf{S}JJJ \subseteq II\mathbf{S}\mathbf{S}III \subseteq \mathbf{S}\mathbf{S}I \subseteq I$. Thus, $JJ\mathbf{S}\mathbf{S}JJJ \subseteq A \cap I = J$. Hence J is a left tri-ideal of $\mathbf{S}$.

Corollary 3.16 (i) The intersection of a tri-ideal and an ideal is a tri-ideal of $\mathbf{S}$. (ii) The intersection of tri-ideals is a tri-ideal of $\mathbf{S}$.

Theorem 3.17 The intersection of a tri-ideal and an interior (bi-quasi, tri-quasi) ideal of $\mathbf{S}$ is a tri-ideal of $\mathbf{S}$.

Proof. Suppose that I is a tri-ideal of $\mathbf{S}$ and A is a bi-quasi ideal of $\mathbf{S}$. To prove that $A \cap I$ is a tri-ideal of $\mathbf{S}$. Now, $(A\cap I)(A\cap I)\mathbf{S}\mathbf{S}(A\cap I)(A\cap I)(A\cap I) \subseteq A\mathbf{S}A\mathbf{S}A$ and $(A\cap I)(A\cap I)\mathbf{S}\mathbf{S}(A\cap I)(A\cap I)(A\cap I) \subseteq \mathbf{S}\mathbf{S}A$.

Thus, $(A\cap I)(A\cap I)\mathbf{S}\mathbf{S}(A\cap I)(A\cap I)(A\cap I) \subseteq \mathbf{S}\mathbf{S}A \cap A\mathbf{S}A\mathbf{S}A \subseteq A$.
Now, $(A\cap I)(A\cap I)\mathbf{S}\mathbf{S}(A\cap I)(A\cap I)(A\cap I) \subseteq I\mathbf{S}I\mathbf{S}I \subseteq I$.
Hence $(A\cap I)(A\cap I)\mathbf{S}\mathbf{S}(A\cap I)(A\cap I)(A\cap I) \subseteq A \cap I$.
Hence $A \cap I$ is a left tri-ideal of $\mathbf{S}$. Similarly, $A \cap I$ is a lateral and right tri-ideal of $\mathbf{S}$. Thus, $A \cap I$ is a tri-ideal of $\mathbf{S}$.

4 m-Tri ideals.

Definition 4.1 A non-empty subset A of $\mathbf{S}$ is called a:

(i) left m-tri-ideal if $AA\mathbf{S}^m AAA \subseteq A$

(ii) lateral m-tri-ideal if $AA\mathbf{S}^m A\mathbf{S}^m AA \subseteq A$

(iii) right m-tri-ideal if $AAA\mathbf{S}^m AA \subseteq A$

(iv) m-tri-ideal A of $\mathbf{S}$ if A is a left m-tri-ideal, lateral m-tri-ideal, right m-tri-ideal of $\mathbf{S}$, where m is a positive integer.

Remark 4.2 If m = 1, then A is called tri-ideal of $\mathbf{S}$.

Theorem 4.3 For $m \geq 1$, (i) Every left tri-ideal is an m-left tri-ideal of $\mathbf{S}$.

(ii) Every lateral tri-ideal is an m-lateral tri-ideal of $\mathbf{S}$.

(iii) Every right tri-ideal is an m-right tri-ideal of $\mathbf{S}$.

Converse of the Theorem 4.3 may not be true as by the following counter example.

Example 4.4

$$\mathbf{S}_1 = \left\{ \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0 \\ x_1&0&0&0&0&0&0&0&0&0 \\ x_2&x_3&0&0&0&0&0&0&0&0 \\ x_4&x_5&x_6&0&0&0&0&0&0&0 \\ x_7&x_8&x_9&x_{10}&0&0&0&0&0&0 \\ x_{11}&x_{12}&x_{13}&x_{14}&x_{15}&0&0&0&0&0 \\ x_{16}&x_{17}&x_{18}&x_{19}&x_{20}&x_{21}&0&0&0&0 \\ x_{22}&x_{23}&x_{24}&x_{25}&x_{26}&x_{27}&x_{28}&0&0&0 \\ x_{29}&x_{30}&x_{31}&x_{32}&x_{33}&x_{34}&x_{35}&x_{36}&0&0 \\ x_{37}&x_{38}&x_{39}&x_{40}&x_{41}&x_{42}&x_{43}&x_{44}&x_{45}&0 \end{pmatrix} \middle| \; x_i\text{'s are non positive intergers} \right\}$$

and

$$\mathbf{S}_2 = \left\{ \begin{pmatrix} 0&y_1&y_2&y_3&y_4&y_5&y_6&y_7&y_8&y_9 \\ 0&0&y_{10}&y_{11}&y_{12}&y_{13}&y_{14}&y_{15}&y_{16}&y_{17} \\ 0&0&0&y_{18}&y_{19}&y_{20}&y_{21}&y_{22}&y_{23}&y_{24} \\ 0&0&0&0&y_{25}&y_{26}&y_{27}&y_{28}&y_{29}&y_{30} \\ 0&0&0&0&0&y_{31}&y_{32}&y_{33}&y_{34}&y_{35} \\ 0&0&0&0&0&0&y_{36}&y_{37}&y_{38}&y_{39} \\ 0&0&0&0&0&0&0&y_{40}&y_{41}&y_{42} \\ 0&0&0&0&0&0&0&0&y_{43}&y_{44} \\ 0&0&0&0&0&0&0&0&0&y_{45} \\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \middle| \; y_i\text{'s are non positive intergers} \right\}$$

$$\mathbf{A}_1 = \left\{ \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0 \\ a_1&0&0&0&0&0&0&0&0&0 \\ 0&a_2&0&0&0&0&0&0&0&0 \\ a_3&0&a_4&0&0&0&0&0&0&0 \\ 0&a_5&0&a_6&0&0&0&0&0&0 \\ a_7&0&a_8&0&a_9&0&0&0&0&0 \\ 0&a_{10}&0&a_{11}&0&a_{12}&0&0&0&0 \\ a_{13}&0&a_{14}&0&a_{15}&0&a_{16}&0&0&0 \\ 0&a_{17}&0&a_{18}&0&a_{19}&0&a_{20}&0&0 \\ a_{21}&0&a_{22}&0&a_{23}&0&a_{24}&0&a_{25}&0 \end{pmatrix} \middle| \; a_i\text{'s are non positive intergers} \right\}$$

$$\mathbf{A}_2 = \left\{ \begin{pmatrix} 0&b_1&0&b_2&0&b_3&0&b_4&0&b_5 \\ 0&0&b_6&0&b_7&0&b_8&0&b_9&0 \\ 0&0&0&b_{10}&0&b_{11}&0&b_{12}&0&b_{13} \\ 0&0&0&0&b_{14}&0&b_{15}&0&b_{16}&0 \\ 0&0&0&0&0&b_{17}&0&b_{18}&0&b_{19} \\ 0&0&0&0&0&0&b_{20}&0&b_{21}&0 \\ 0&0&0&0&0&0&0&b_{22}&0&b_{23} \\ 0&0&0&0&0&0&0&0&b_{24}&0 \\ 0&0&0&0&0&0&0&0&0&b_{25} \\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \middle| \; b_i\text{'s are non positive intergers} \right\}$$

Hence A_1 is a left m-tri-ideal but not left tri-ideal of $\mathbf{S_1}$ and A_2 is a right m-tri-ideal but not right tri-ideal of $\mathbf{S_2}$.

Definition 4.5 An m-bi-ideal A of $\mathbf{S}$ is a TSS of $\mathbf{S}$ such that $A\mathbf{S}^m A\mathbf{S}^m A \subseteq A$, where m is a positive integer.

Theorem 4.6 The product of atleast three m-tri-ideals is also m-tri-ideal of $\mathbf{S}$.

Theorem 4.7 If A is a m-tri-ideal of $\mathbf{S}$ and T_1, T_2 are two TSS with identity element e, then AT_1T_2, T_1AT_2 *and* T_1T_2A are m-tri-ideals of $\mathbf{S}$.

Proof. Let A be a m-tri-ideal of $\mathbf{S}$, T_1 and T_2 are two TSS with identity element e. Now,

$$(AT_1T_2)(AT_1T_2)\,\mathbf{S}^{\mathbf{m}}\,(AT_1T_2)(AT_1T_2)(AT_1T_2) \subseteq AA\mathbf{S}^{\mathbf{m}}AAAT_1T_2 \subseteq AT_1T_2.$$

Thus, AT_1T_2 is a m-left tri-ideal of $\mathbf{S}$. Similarly AT_1T_2 are m-lateral tri-ideal and m-right tri-ideal of $\mathbf{S}$. Thus, AT_1T_2 is a m-tri-ideal of $\mathbf{S}$. Similarly, T_1AT_2 *and* T_1T_2A are m-tri-ideal of $\mathbf{S}$.

Theorem 4.8 If A is a m-tri-ideal of $\mathbf{S}$ and T is a TSS of $\mathbf{S}$, then $A \cap T$ is a m-tri-ideal of T.

Proof. Since $A \cap T \subseteq A$ and $A \cap T \subseteq T$, $(A\cap T)(A\cap T)\,(A\cap T) \subseteq AAA \subseteq A$ and $(A\cap T)(A\cap T)\,T^m\,(A\cap T)(A\cap T)(A\cap T) \subseteq (A\cap T)(A\cap T)\mathbf{S}^{\mathbf{m}}\,(A\cap T)(A\cap T)(A\cap T)$ $\subseteq AA\mathbf{S}^{\mathbf{m}}AAA \subseteq A$. Therefore $A \cap T$ is a left m-tri-ideal of T. Similarly, $A \cap T$ is a lateral m-tri-ideal and right m-tri-ideal of T.

Definition 4.9 A TSS Q of $\mathbf{S}$ is called a m-quasi ideal if $Q\mathbf{S}^{\mathbf{m}} \cap \mathbf{S}^{\mathbf{m}}Q\mathbf{S}^{\mathbf{m}} \cap \mathbf{S}^{\mathbf{m}}Q \subseteq Q$.

Theorem 4.10 Every m-quasi ideal is a m-tri-ideal of $\mathbf{S}$.

Proof. Suppose Q is a m-quasi ideal of $\mathbf{S}$. Then $Q\mathbf{S}^{\mathbf{m}} \cap \mathbf{S}^{\mathbf{m}}Q\mathbf{S}^{\mathbf{m}} \cap \mathbf{S}^{\mathbf{m}}Q \subseteq Q$. Now,

$QQ\mathbf{S^m}QQQ \subseteq Q\mathbf{S^m}$, $QQ\mathbf{S^m}QQQ \subseteq \mathbf{S^m}Q\mathbf{S^m}$ and $QQ\mathbf{S^m}QQQ \subseteq \mathbf{S^m}Q$. Hence $QQ\mathbf{S^m}QQQ \subseteq Q$. Thus, Q is m-left tri-ideal of $\mathbf{S}$. Similarly, Q is m-lateral (right) tri-ideal of $\mathbf{S}$.

Converse of the Theorem 4.10 need not true by the following Example.

Example 4.11

Let

$$\mathbf{S} = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & x_6 & x_7 & x_8 & x_9 \\ 0 & 0 & 0 & x_{10} & x_{11} & x_{12} \\ 0 & 0 & 0 & 0 & x_{13} & x_{14} \\ 0 & 0 & 0 & 0 & 0 & x_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \ x_i^{'s} \text{ are non positive intergers} \right\}.$$

$$Q = \left\{ \begin{pmatrix} 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \ x, y, z \text{ are non positive intergers} \right\}$$

Hence Q is a m-tri-ideal but not 2-quasi ideal of $\mathbf{S}$.

Definition 4.12 (i) A non-empty subset L of $\mathbf{S}$ is called l -left tri-ideal if $LL\mathbf{S}^l LLL \subseteq L$.

(ii) A non-empty subset X of $\mathbf{S}$ is called m-lateral tri-ideal if $XX\,\mathbf{S}^m X\,\mathbf{S}^m XX \subseteq X$.

(iii) A non-empty subset N of $\mathbf{S}$ is called n -right tri-ideal if $NNN\,\mathbf{S}^n\,NN \subseteq N$, where l, m, n are positive integers.

Theorem 4.13 Every l-left ideal, m-lateral ideal and n -right ideal of $\mathbf{S}$ with e is an l-tri-ideal, m-tri-ideal, n-tri-ideal of $\mathbf{S}$ respectively.

Proof. Let L be a m-lateral ideal of $\mathbf{S}$, then $\mathbf{S}^m L\mathbf{S}^m \subseteq L$. Now, $LL\mathbf{S}^m LLL \subseteq LL\mathbf{S}^m Lee....eLL \subseteq LL\mathbf{S}^m L\mathbf{S}^m LL \subseteq L$ and $LL\mathbf{S}^m L\mathbf{S}^m LL \subseteq LLLLL \subseteq L$ and $LLL\mathbf{S}^m LL \subseteq LLee.....eL\mathbf{S}^m LL \subseteq LL\mathbf{S}^m L\mathbf{S}^m LL \subseteq L$.

Therefore L is a m-tri-ideal of $\mathbf{S}$. Similarly, L is a l-tri-ideal and n-tri-ideal of $\mathbf{S}$.

Theorem 4.14 The intersection of l-left ideal, m-lateral ideal and n-right ideal is an l-left ideal, m-lateral ideal and n -right ideal of $\mathbf{S}$ respectively.

Theorem 4.15 Let Y_1,Y_2, Y_3 be an l -left tri-ideal, lateral m-tri-ideal and n -right tri-ideal of $\mathbf{S}$ respectively. Then $Y_1 \cap Y_2 \cap Y_3$ is an t -tri-ideal, where t = max (l, m, n).

Proof. Clearly $Y = Y_1 \cap Y_2 \cap Y_3$ is a TSS of $\mathbf{S}$. By Theorem 4.13, Y_1, Y_2 and Y_3 are l-tri-ideal, m-tri-ideal and n -tri-ideal respectively. The intersection of Y_1,Y_2 and Y_3 becomes max (l, m, n) tri-ideal.

$$\begin{aligned} YY\mathbf{S^t}YYY &\subseteq Y_1Y_1\mathbf{S^t}Y_1Y_1Y_1 \\ &\subseteq \mathbf{SSS^tSS}Y_1 \\ &\subseteq \mathbf{S^t}Y_1 \\ &\subseteq Y_1 \end{aligned}$$

Similarly $YY\mathbf{S^t}YYY \subseteq Y_2$, $YY\mathbf{S^t}YYY \subseteq Y_3$.

Hence $YY\mathbf{S^t}YYY \subseteq Y$.

Similarly, $YYY\mathbf{S^t}YY \subseteq Y$ and $YY\,\mathbf{S^t}Y\mathbf{S^t}YY \subseteq Y$

Theorem 4.16 Let Y_1 be a m-left (lateral, right) tri-ideal of $\mathbf{S}$ and Y_2 be a m-left (lateral, right) tri- ideal of Y_1 such that $Y_2^3 = Y_2$. Then Y_2 is a m-left (lateral, right) tri-ideal of $\mathbf{S}$.

Proof. Since Y_1 is a left m-tri-ideal of $\mathbf{S}$, $Y_1Y_1\mathbf{S}^m Y_1Y_1Y_1 \subseteq Y_1$ and Y_2 is a m-left tri-ideal of Y_1, $Y_2Y_2Y_1^m Y_2Y_2Y_2 \subseteq Y_2$.

Now,

$$\begin{aligned} Y_2Y_2\mathbf{S}^m Y_2Y_2Y_2 &= (Y_2Y_2Y_2)(Y_2Y_2Y_2)\mathbf{S}^m Y_2(Y_2Y_2Y_2)(Y_2Y_2Y_2) \\ &= Y_2Y_2Y_2Y_2(Y_2Y_2\mathbf{S}^m Y_2Y_2Y_2)Y_2Y_2Y_2Y_2 \\ &\subseteq Y_2Y_2Y_2Y_2(Y_1Y_1\mathbf{S}^m Y_1Y_1Y_1)Y_2Y_2Y_2Y_2 \\ &\subseteq Y_2Y_2Y_2Y_2Y_1Y_2Y_2Y_2Y_2 \\ &= Y_2Y_2Y_2Y_2Y_1Y_2Y_2Y_2(Y_2Y_2Y_2) \\ &= Y_2Y_2Y_1^3Y_2Y_2Y_2 \\ &\circ \\ &\circ \\ &\circ \\ &\circ \\ &= Y_2Y_2Y_1^m Y_2Y_2Y_2 \\ &\subseteq Y_2 \end{aligned}$$

Thus, Y_2 is a left m-tri-ideal of $\mathbf{S}$.

Corollary 4.17 Let Y_1 be a m-tri-ideal of $\mathbf{S}$ and Y_2 be a m-tri-ideal of Y_1 such that $Y_2^3 = Y_2$. Then Y_2 is a m-tri-ideal of $\mathbf{S}$.

Theorem 4.18 (i) Let Y_1, Y_2 *and* Y_3 be three TSS of $\mathbf{S}$ and $A = Y_1Y_2Y_3$. Then A is a m tri-ideal if at least one of Y_1, Y_2 *and* Y_3 is a m-right or m-left ideal of $\mathbf{S}$.

(ii) Let Y_1, Y_2 *and* Y_3 be three TSS of $\mathbf{S}$ and $A = Y_1Y_2Y_3$. Then A is a m- lateral tri-ideal if at least one of Y_1, Y_2 *and* Y_3 is a lateral m-tri-ideal of $\mathbf{S}$.

ACKNOWLEDGEMENT

The research of the first author is partially supported by UGC-BSR grant : F.25-1/2014-15(BSR)/7-254/2009(BSR) dated 20-01-2015 in India.

REFERENCES

Arulmozhi, K., 2020. New approach of various ideals in Ternary Semirings. Journal of Xidian University, 14(3), Pp. 22-29.

Dubey, M.K., Anuradha., 2013. A note on prime quasi-ideals in ternary semirings. Kragujevac Journal of Mathematics, 37(2), Pp. 361- 367.

Lajos, S., 1969. On the bi-ideals in semigroups. Proc. Japan Acad., 45, Pp. 710 -712.

Lajos, S., Szasz, F.A., 1970. On the bi-ideals in associative rings. Proc. Japan Acad., 49(6), Pp. 505- 507 - 317-321.

Lehmer, D.H., 1932. A ternary analogue of abelian groups. American Journal of Mathematics, Pp. 329-338.

Lister, W.G., 1971. Ternary Rings, Trans. Amer. Math. Soc., 154, Pp. 37-55.

Munir, M., Shafiq, A., 2018. A generalization of bi-ideals in semirings. Bull. Int. Math. Virtual Inst., 8, Pp. 123-133.

Rao, M.K, 2020. Tri-ideals of semirings. Bull. Int. Math. Virtual Inst., 10(1), Pp. 145-155.