ZIBELINE INTERNATIONAL PUBLISHING

Matrix Science Mathematic (MSMK)

DOI: http://doi.org/10.26480/msmk.02.2020.51.54

ISSN: 2521-0831 (Print)
ISSN: 2521-084X (Online)
CODEN: MSMADH

RESEARCH ARTICLE

GLOBAL ANALYSIS OF AN SIR MODEL WITH VERTICAL TRANSMISSION AND VACCINATION

Tang Feng*

School of Chemical Engineering, Zhengzhou University, Zhengzhou 450000, China
****Corresponding Author E-mail: 3390722889@qq.com***

ARTICLE DETAILS

Article History:

Received 05 February 2021
Accepted 07 March 2021
Available online 09 March 2021

ABSTRACT

In this paper we study an SIR epidemic model with vertical transmission and vaccination. By carrying out a global analysis of the model and studying the stability of the disease-free equilibrium and the endemic equilibrium, we show that the vertical transmission and vaccination did not.

KEYWORDS

Vertical transmission, Vaccination, Global stability.

1. INTRODUCTION

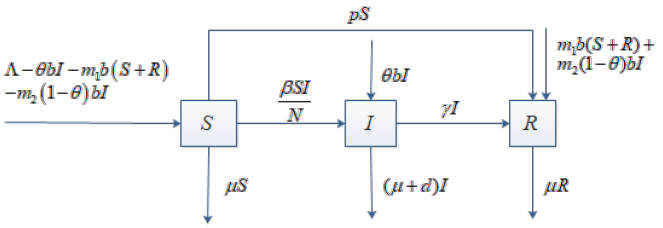
Mathematical models have become important tools in analyzing the spread and control of infectious diseases (Hethcote, 1994). Understanding the transmission characteristics of infectious diseases in com- munities, regions and countries may lead to implementation of better approaches of mitigating against infections or epidemics. In particular, mathematical models are useful in building and testing theories, and in comparing, planning, implementing and evaluating various detection, prevention, therapeutic and con- trol programs. The results of such studies may contribute to formulation of appropriate public health policies and guide the design of other relevant studies and development of methods for data collection.

Vertical transmission can be accomplished through transplacental transfer of disease agents (bacteria, viruses, parasites) from the mother to an embryo, fetus, or baby during pregnancy or childbirth (Li et al., 2001). A number of studies incorporating vertical transmission investigated the effects of various epidemiological and demographical factors on the disease transmission (Brauer, 2011; Long and Xiang, 2011; Gao and Hethcote, 1992; Busenberg and Cooke 1993; Kgosimore and Lungu, 2006). However, the literature on pulse vaccination dealing with the analysis of disease that are vertically and horizontally transmitted is not extensive. Lu et al and Alberto d'Onofrio considered that the vaccine treatment is taken to all of the susceptible, they did some original work on pulse vaccination under vertical transmission (Lu et al., 2002; d'Onofrio, 2005). In fact, on the one hand, under the situation of disease with vertical transmission, the vaccine treatment should be taken to the newborns who have not been infected by their infections mothers at birth and transfer to the susceptible . On the other hand, the vaccine treatment also should be considered to the newborns of the susceptible and the removed under the situation of disease with horizontal transmission (Meng et al., 2007).

In this paper, we study an SIR epidemic model with vertical transmission and continuous vaccination. This paper is organized as follows. Section 2 provides model formulation and analysis. In Section 3, we investigate the existence and the stability analysis of equilibria of the model and close with a discussion in Section 4.

2. MODEL CONSTRUCTION

In this section, we formulate an SIR model with vertical transmission and vaccination. Figure.1 shows the model diagram. The total population at time t denoted by N , is divided into three classes: susceptible (S), infectious (I) and recovered (R).



Recruitment occurs at a constant rate Λ . We assume that an individual may be infected only through contacts with infectious individuals. The natural death rate is μ and the natural birth rate is b . The infectious class has an additional death rate due to the disease with rate constant α . Infectious individuals are treated with rate constant γ , entering the treatment class. The incidence rate is the standard incidence rate $\frac{\beta SI}{N}$.

The vaccination proportion is $m_1(0 < m_1 < 1)$ to the newborn from S and R and $m_2(0 < m_2 < 1)$ to the newborn from I . A proportion $\theta(0 \le \theta \le 1)$ of new births are born infected through mother-to-child transmission (MTCT). Combining all the aforementioned assumptions, the model for the transmission dynamics is given by the following system of differential equations:

Quick Response Code	Access this article online	
	Website: www.matrixsmathematic.com	**DOI:** 10.26480/msmk.02.2020.51.54

Cite the Article: Tang Feng (2021). Global Analysis of An Sir Model with Vertical Transmission and Vaccination. *Matrix Science Mathematic, 4(2): 51-54.*

$$\begin{cases} \frac{dS}{dt} = \Lambda - \theta bI - m_1 b(S+R) - m_2(1-\theta)bI - pS - \frac{\beta SI}{N} - \mu S, \\ \frac{dI}{dt} = \frac{\beta SI}{N} + \theta bI - \gamma I - (\mu + d)I, \\ \frac{dR}{dt} = \gamma I + pS + m_1 b(S+R) + m_2(1-\theta)bI - \mu R, \\ S + I + R = N. \end{cases} \quad (1)$$

By adding all equation of system (1), the dynamics of the total population $N(t)$ is given by:

$$\frac{dN}{dt} = \Lambda - \mu N - dI. \quad (2)$$

Since $dN/dt < 0$ for $N > \Lambda/\mu$, then, without loss of generality, we can consider only solutions of system (1) in the following positively subset of R^3:

$$\Omega_\varepsilon = \{(S,I,R) \mid S,I,R \geq 0, S+I+R \leq \frac{\Lambda}{\mu}\}.$$

With respect to model system (1), we have the following result:

Proposition 2.1: The compact set Ω_ε is a positively invariant and absorbing set that attracts all solutions of system (1) in R^3.

Proof: Define a Lyapunov function as $W(t) = S(t) + I(t) + R(t)$, then we have:

$$\frac{dW(t)}{dt} = \Lambda - \mu W - \alpha I \leq \Lambda - \mu W. \quad (3)$$

Hence, that $\frac{dW}{dt} \leq 0$ for $W > \frac{\Lambda}{\mu}$, Ω_ε is a positively invariant set. On the other hand, solving the differential inequality Eq. (3) yields:

$$0 < W(t) < \frac{\Lambda}{\mu} + W(0)e^{-\mu t}.$$

$W(0)$ is the initial condition of $W(t)$. Thus, as $t \to \infty$, one has that $0 \leq W(t) \leq \frac{\Lambda}{\mu}$.

To analysis the system (1), we can consider the following system:

$$\begin{cases} \frac{dS}{dt} = \Lambda - \frac{\beta SI}{N} - \xi I - (p+\mu)S - m_1 bN, \\ \frac{dI}{dt} = \frac{\beta SI}{N} - \eta I, \\ \frac{dN}{dt} = \Lambda - \mu N - \alpha I. \end{cases} \quad (4)$$

where $\xi = m_1 b - m_2 b + m_2 \theta b - \theta b, \eta = r + \mu + \alpha - \theta b.$

3. Mathematical Analysis

In this section, the model is analyzed in order to obtain the basic reproduction number, conditions for the existence and uniqueness of non-trivial equilibria and asymptotic stability of equilibria.

3.1 Basic reproductive number

The disease-free equilibrium of system (2.4) is $X_0 = (S_0, 0, N_0)$ with

$$S_0 = \frac{\Lambda(\mu - m_1 b)}{\mu(p+\mu)}, N_0 = \frac{\Lambda}{\mu}.$$

Following Van den Driessche and Watmough (Van den Driessche, 2002), we can get the basic reproductive number:

$$R_0 = \frac{\beta(\mu - m_1 b)}{\eta(p+\mu)}.$$

3.2 Stability of the disease-free equilibrium

We have the following result about the global stability of the disease free equilibrium:

Theorem 3.1: When $R_0 > 1$, the disease free equilibrium X_0 is unstable. When $R_0 \leq 1$, the disease free equilibrium X_0 is globally asymptotically stable in Ω_ε; this implies the global asymptotic stability of the disease free equilibrium on the nonnegative orthant R^3. This means that the disease naturally dies out.

Proof: The Jacobian of system Eq. (2.4) at X_0 is

$$J(X_0) = \begin{pmatrix} -(p+\mu) & \xi & -m_1 b \\ 0 & R_0\eta - \eta & 0 \\ 0 & -\alpha & -\mu \end{pmatrix}$$

and the characteristic equation is

$$(\lambda + \mu)(\lambda + p + \mu)(\lambda + \eta - R_0\eta) = 0. \quad (5)$$

Therefore, if $R_0 < 1$, Eq. (5) has three negative real roots and hence X_0 is locally stable. If $R_0 > 1$, Eq. (5) has one positive real root and hence X_0 is unstable.

Next, we will prove the global stability of X_0 when $R_0 \leq 1$. From the first equation of Eq. (4), we get that

$$S'(t) \leq A - m_1 bN - (p+d)S$$

By solving the above equation, we can get

$$S(t) \leq \frac{A - m_1 bN}{p+d} + \left(S(0) - \frac{A - m_1 bN}{p+d}\right)\exp\left[-(p+d)t\right]$$

and then we have $S(t) \leq \frac{A - m_1 bN}{p+d}$ when t is large enough.

We can define the following Lyapunov function

$$V(t) = I(t)$$

Its time derivative along the trajectories of Eq. (4) satisfies

$$\begin{aligned} V'(t) = I'(t) &= \left[\frac{\beta S}{N} - (d + \alpha + \gamma - \theta b)\right] I \\ &\leq \left[\frac{\beta(A - m_1 bN)}{N(p+d)} - (d + \alpha + \gamma - \theta b)\right] I = (d + \alpha + \gamma - \theta b)(R_0 - 1) I \end{aligned} \quad (6)$$

We can that $I'(t) \leq 0$ if $0 < R_0 < 1$.It can be seen that $\lim_{t\to\infty} I(t) = 0$. When $t \to \infty$, the limit equation of system (4) is

$$\begin{cases} \frac{dS}{dt} = A - m_1 bN - pS - dS, \\ \frac{dN}{dt} = A - dN. \end{cases}$$

Solving the above equation, we can get that

$$\begin{cases} S = \frac{A - m_1 bN}{p+d} + c_1 e^{-(p+d)t}, \\ N = \frac{A}{d} + c_2 e^{-dt}. \end{cases}$$

where C_1, C_2 are arbitrary constants.

It is obvious from the above formula

$$\lim_{t\to\infty} S = \frac{A - m_1 bN}{p+d} = S_0, \quad \lim_{t\to\infty} N = \frac{A}{d} = N_0$$

Let $E = \left\{ (S, I, N) \middle| \frac{dV}{dt} = 0 \right\}$, it is easy to know that the largest invariant set in Ω_ε contained in $\{(S, I, N) \in \Omega_\varepsilon, \dot{V} = 0\}$ is reduced to the disease-free equilibrium X_0. This proves the global asymptotic stability of the disease-free equilibrium on Ω_ε (Bhatia and Szeg, 1970). Since Ω_ε is absorbing, this proves the global asymptotic stability on the nonnegative octant for $R_0 \le 1$. Generally, the LaSalle's invariance principle only proves the attractivity of the equilibrium. Considering Ω_ε permits to conclude for the stability (Bhatia and Szeg, 1970; LaSalle, 1968; LaSalle, 1976). This achieves the proof.

3.3 Existence and uniqueness of endemic equilibrium

To find the positive equilibrium, let

$$\begin{aligned} &\Lambda - \frac{\beta SI}{N} - (p+\mu)S - \xi I - m_1 bN = 0, \\ &\frac{\beta SI}{N} - \eta I = 0, \\ &\Lambda - \mu N - \alpha I = 0, \end{aligned}$$

which yields

$$\begin{aligned} S^* &= \frac{\Lambda\eta(\xi - \eta + \alpha)}{\alpha\beta m_1 b + \alpha\eta(p+\mu) + \beta\mu(\xi - \eta)}, \\ I^* &= \frac{\Lambda(m_1 b\beta + (p+\mu)\eta - \mu\beta)}{\alpha\beta m_1 b + \alpha\eta(p+\mu) + \beta\mu(\xi - \eta)}, \\ N^* &= \frac{\Lambda\beta(\xi - \eta + \alpha)}{\alpha\beta m_1 b + \alpha\eta(p+\mu) + \beta\mu(\xi - \eta)}. \end{aligned} \tag{7}$$

We can prove that when $R_0 > 1$,

$$\begin{aligned} &\xi - \eta + \alpha = m_1 b - m_2 b + m_2 \theta b - \mu - r < 0, \\ &\alpha\beta m_1 b + \alpha\eta(p+\mu) + \beta\mu(\xi - \eta) = \alpha\beta m_1 b + \alpha\beta\frac{\mu - m_1 b}{R_0} + \beta\mu(m_1 b - m_2 b + m_2\theta b - \mu - r - \alpha) \\ &\qquad = \alpha\beta(\mu - m_1 b)(\frac{1}{R_0} - 1) + \beta\mu(m_1 b - m_2 b + m_2\theta b - \mu - r) < 0. \end{aligned}$$

Then we can get that $S^* > 0, I^* > 0, N^* > 0$ and hence we have the following result:

Theorem 3.2: When $R_0 > 1$, there exist a unique endemic equilibrium $X^* = (S^*, I^*, N^*)$ for the system (4) where S^*, I^* and N^* are defined as in Eq. (3.3) which is in the nonnegative octant R_+^3.

3.4 Stability of endemic equilibrium

3.4.1 Locally stability of endemic equilibrium

Theorem 3.3 If $R_0 > 1$, the unique endemic equilibrium X^* of the system (4) is locally asymptotically stable.

Proof: The Jacobian matrix of system (2.4) at endemic equilibrium X^* is

$$J(X^*) = \begin{pmatrix} -(p+\mu) - \frac{\beta I^*}{N^*} & \xi - \eta & \frac{\beta S^* I^*}{N^*} - m_1 b \\ \frac{\beta I^*}{N^*} & 0 & -\frac{\beta S^* I^*}{N^*} \\ 0 & -\alpha & -\mu \end{pmatrix}.$$

The characteristic equation of $J(X^*)$ is

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0.$$

Where

$$\begin{aligned} &a = p + 2\mu + \beta\frac{I^*}{N^*} > 0, \\ &b = p\mu + + \mu^2 + \frac{\beta I^*(\eta - \xi + \mu)}{N^*} - \alpha\frac{\eta I^*}{N^*} > 0, \\ &c = -\frac{\alpha\beta I^*}{N^*} m_1 b + d(\eta - \xi)\frac{\beta I^*}{N^*} - (p+\mu)\eta\frac{\alpha I^*}{N^*} = \frac{I^*}{N^*}[-\alpha\beta m_1 b + \mu\beta(\eta - \xi) - \alpha\eta(p+\mu)] \\ &\quad = \frac{I^*}{N^*}[-\alpha\beta\mu + \alpha\eta R_0(p+\mu) + \mu\beta(\eta - \xi) - \alpha\eta(p+\mu)] \\ &\quad = \frac{I^*}{N^*}[\mu\beta(\eta - \xi - \alpha) + \alpha\eta(p+\mu)(R_0 - 1)] > 0. \end{aligned}$$

and we have

$$\begin{aligned} ab - c &= (p + 2\mu + \frac{\beta I^*}{N^*})(p\mu + \mu^2 + \frac{\beta I^*(\eta - \xi + \mu)}{N^*} - \frac{\alpha\eta I^*}{N^*}) + \frac{\alpha\beta I^*}{N^*} \cdot m_1 b - \mu(\eta - \xi)\frac{\beta I^*}{N^*} \\ &\quad + (p+\mu)\frac{\alpha\eta I^*}{N^*} \\ &= (p + \mu + \frac{\beta I^*}{N^*})(p\mu + + \mu^2 + \frac{\beta I^*(\eta - \xi + \mu)}{N^*} - \frac{\alpha\eta I^*}{N^*}) + \mu(p\mu + \mu^2 + \frac{\mu\beta I^*}{N^*}) \\ &\quad + \frac{\alpha\beta I^*}{N^*} \cdot m_1 b + \frac{p\alpha\eta I^*}{N^*} \\ &> 0. \end{aligned}$$

According to direct calculation we have $a, c > 0$ and $ab - c > 0$ when $R_0 > 1$. Therefore the endemic equilibrium X^* is locally asymptotically stable in Ω_ε by Routh-Hurwitz criterion.

3.4.2 Globally stability of endemic equilibrium

To prove the following theorem, we first give a lemma: Generalized Dulac-Bendixson criterion.

Lemma 3.4 (Busenberg and Van Driessche, 1990). Let $\mathbf{f}: R^3 \to R^3$ be a Lipschitz continuous vector field and let Γ be a closed, piecewise smooth, curve which is the boundary of an orientable smooth surface $\mathbf{S} \subset R^3$. Suppose that $\mathbf{g}: R^3 \to R^3$ is defined and piecewise smooth in a neighborhood of $\mathbf{S}$ and that is satisfies:

$$\mathbf{g}(\Gamma(t)) \cdot \mathbf{f}(\Gamma(t)) \le 0 \quad (\text{or} \ge 0) \ \text{ for all } t$$

And

$$(Curl\mathbf{g}) \cdot \mathbf{n} \ge 0 (\le 0) \ \text{ on } \mathbf{S}, \ \textit{and} \ (Curl\mathbf{g}) \cdot \mathbf{n} > 0 (< 0) \ \text{ for some point on } \mathbf{S}.$$

where $\mathbf{n}$ is the unit normal to $\mathbf{S}$. Then $\Gamma(t)$ is nor the finite union of solution trajectories of

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}).$$

which are traversed in the positive sense relation to the direction of $\mathbf{n}$.

Theorem 3.5 If $R_0 > 1$, the unique endemic equilibrium X^* of the system (4) is globally asymptotically stable.

Proof: We transform system (4) into proportions (using the change of variables $x = \frac{S}{N}, y = \frac{I}{N}, z = \frac{1}{N}$), performing the above manipulations gives

$$\begin{cases} \dfrac{dx}{dt} = \Lambda z + \xi y - (\beta - \alpha)xy - m_1 b - px - \Lambda xz, \\ \dfrac{dy}{dt} = \beta xy - (\eta - \mu)y - (\Lambda - \alpha)yz, \\ \dfrac{dz}{dt} = -\Lambda z^2 = \mu z + \alpha y. \end{cases} \quad (8)$$

We only need to prove the global stability of $P(x^*, y^*, z^*)$ of system (3.4) on the invariant set $\Omega^* = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1\}$. Obviously, the solutions of system (8) are bounded, and the endemic equilibrium $P(x^*, y^*, z^*)$ is locally asymptotically stable, it is only necessary to prove that system (8) has no periodic solution in the invariant domain Ω^*.

Obviously, the boundary curve of the domain Ω^* cannot form the periodic solution of system (8). We consider the following in the interior of Ω^*. Assuming that system (3.4) has a periodic solution $\varphi(t) = \{x(t), y(t), z(t)\}$, the image Γ of $\varphi(t)$ is the boundary of a plane domain Π which is in the interior of domain Ω^*.

Let $\boldsymbol{f} = (f_1, f_2, f_3)^T$ (T denotes transpose), $\boldsymbol{g}(x, y, z) = \dfrac{1}{xyz} \cdot \boldsymbol{r} \times \boldsymbol{f}$, ($\boldsymbol{r} = (x, y, z)^T$), then

$$\boldsymbol{g} \cdot \boldsymbol{f} = 0.$$

Let $\boldsymbol{g} = (g_1, g_2, g_3)$ and $\mathrm{Curl}\boldsymbol{g} = (\dfrac{\partial g_3}{\partial y} - \dfrac{\partial g_2}{\partial z}, \dfrac{\partial g_1}{\partial z} - \dfrac{\partial g_3}{\partial x}, \dfrac{\partial g_2}{\partial x} - \dfrac{\partial g_1}{\partial y})$.

By calculating straight forwardly, we get in the interior of domain Ω^*:

$$(\mathrm{Curl}\boldsymbol{g}) \cdot (1,1,1)^T = -\frac{q\delta(1-x-y-z)}{y^2}(\frac{1}{x} + \frac{1}{z}) - \frac{(1-q)\delta(1-x-y-z)}{x^2}(\frac{1}{y} + \frac{1}{z}) - k(\frac{1}{z^2} + \frac{1}{xz} + \frac{y}{xz^2}) - \frac{\mu}{x^2}(\frac{1}{y} + \frac{1}{z}) - c\beta(\frac{1}{y} + \frac{x}{y^2} + \frac{z}{y^2}) < 0.$$

If we choose the direction of plane domain Π upward, the direction of the image Ω^* conforms to the right-hand rule with the direction of plane domain Π. Vector $(1,1,1)$ is the normal vector of plane domain Π, then we get by Stoker's theorem:

$$\frac{1}{\sqrt{3}} \iint_{\Pi} \mathrm{Curl}\boldsymbol{g} \cdot (1,1,1)^T \, dS = \oint_{\Gamma} \frac{\boldsymbol{g} \cdot \boldsymbol{f}}{|\boldsymbol{f}|} dS.$$

This is in contradiction with the calculation above. The Theorem is proved.

4. Discussions

We have carried out a global qualitative analysis of an SIR model with vertical transmission and continuous vaccination and studied the existence and stability of the disease-free and endemic equilibria. Interestingly, this model does not exhibit complicated dynamics. In terms of the basic reproduction number $R_0 = \dfrac{\beta(\mu - m_1 b)}{\eta(p + \mu)}$, our main results indicate that when $R_0 \leq 1$, the disease-free equilibrium is globally attractive. When $R_0 > 1$, the endemic equilibrium exists and is globally stable.

References

Bhatia. N.P., Szeg, G.P. 1970. Stability theory of dynamical systems, Springer-Verlag.

Brauer, F. 2011. Backward bifurcations in simple vaccination/treatment models, Journal of Biological Dynamics, 5:410–418.

Busenberg, S., K. Cooke, 1993. Vertically transmitted diseases, Models and Dynamics: Biomathematics, 23, Springer-Verlag, Berlin.

Busenberg, S., Van Driessche, P. 1990. Analysis of a disease transmission model in a population with varying size, Journal of Mathematical Biology, 28: 257–270.

d'Onofrio, A. 2005. On pulse vaccination strategy in the SIR epidemic model with vertical transmission. Applied Mathematics Letters, 18(7): 29-732.

Gao, L.Q., H.W. Hethcote, 1992. Disease transmission models with desity-dependent demographics, Journal of Mathematical Biology, 30(7): 717-731.

Hethcote, H.W. 1994. A Thousand and One Epidemic Models. Frontiers in Mathematical Biology. Springer Berlin Heidelberg.

Kgosimore, M., E.M. Lungu. 2006. The effects of vertical transmission on the spread of HIV/AIDS in the presence of treatment, Mathematical Bioscience & Engineering, 23: 297–312.

LaSalle, J.P. 1968. Stability theory for ordinary differential equations, Journal of Differential Equations, 41:57-65.

LaSalle, J.P. 1976. The stability of dynamical systems, Society for Industrial and Applied Mathematics, Philadelphia, PA., 1976. With an appendix: "Limiting equations and stability of nonautonomous ordinary differential equations" by Z. Artstein, Regional Conference Series in Applied Mathematics.

Li, M.Y., H.L. Simth, L. Wang, 2001. Global dynamics of an SEIR epidemic model with vertical transmission, SIAM J. Appl. Math, 62:58–69.

Long, D., Z. Xiang, 2011. On the study of an SEIV epidemic model concerning vaccination and vertical transmission, Journal of Applied Mathematics and Bioinformatics, 1: 21–30.

Lu, Z.H., Chi, X.B., Chen, L.S. 2002. The effect of constant and pulse vaccination on SIR epidemic model with horizontal and vertical transmission. Mathematical and Computer Modelling, 36: 1039-1057.

Meng, X., L. Chen, et al. 2007. Global dynamics behaviors for new delay SEIR epidemic disease model with vertical transmission and pulse vaccination. Applied Mathematics & Mechanics, 28(9): 1259-1271.

Van den Driessche, P., Watmough, J. 2002. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. Mathematical Bioscience, 180: 28-29.