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RESEARCH ARTICLE

BESSEL MATRIX POLYNOMIALS AND FIFTH-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT

There are numerous methods for numerically solving a fifth-order ordinary differential equation. We show in this paper that the Bessel matrix polynomials method is a powerful tool for numerically solving this equation. Our method is tested on examples, and it is compared to other methods. It is demonstrated that the Bessel matrix polynomials method produces good results.

KEYWORDS

Bessel Functions, Matrix Polynomials, fifth Problems, Numerical Solutions.

1. INTRODUCTION

In this paper, we attempt to introduce a collocation method, based on the Bessel functions of the first kind for solving:

$$u^{(5)}(t) + \sum_{k=1}^{4} p_k(t)\, u^{(k)}(t) + \sum_{s=1}^{r} q_s(t)\, u^s(t) = f(t), \quad t \in [a,b], \quad r \geq 1, \qquad (1)$$

with initial and boundary conditions

$$\begin{cases} u^{(j)}(a) = \mu_j, & j = 0,1,2. \\ u^{(m)}(b) = \alpha_m, & m = 0,1. \end{cases} \qquad (2)$$

Where in equation (1.1) the term $u^{(k)}(t)$ refers to the k-th differentiation of the unknown function u(t), whereas the power term of the unknown function is defined as us(t). The known functions $p_k(t), q_s(t)$, and $f(t)$ are well-defined and bounded on the interval $[a,b]$. In equation (2), μ_j and α_m are real constants.

In the sciences, higher-order boundary value issues might be found, for instance when modelling induction motors using current electricity. The mathematical modelling of viscoelastic flows and other areas of the mathematical, physical, and engineering sciences are other areas where this kind of boundary value problem is applied (Caglar et al., 1999; Davies et al., 1988; Fyfe, 1971; Noor and Mohyud-Din, 2007; Noor and Mohyud-Din, 2007; Wazwaz, 2007; Davies et al., 1988). Additionally, a fifth-order or seventh-order differential equation model is used to represent the behaviour of induction motors (Richards and Sarma, 1994). The mathematical modelling of AFTI-F16 fighters uses ninth-order differential equations (Lyshevski and Dunipace, 1997).

Higher-order boundary value problems were solved by many methods. Numerical and analytical methods have included the sextic spline method [10], nonpolynomial spline technique, decomposition method, collocation method with sixth-degree B-splines, reproducing kernel space method, finite element method, finite-difference, variational iteration method, homotopy perturbation method, Sinc-collocation, Chebyshev method, and Euler matrix method (Siddiqi and Akram, 2007; Siddiqi and Akram, 2006; Noor and Mohyud-DinT, 2009; Viswanadham et al., 2010; Akram and Rehman, 2011; Viswanadham and Raju, 2012; Khan, 1994; El-Gamel, 2007; El-Gamel, 2006; El-Gamel, 2019; El-Gamel, 2015; El-Gamel and Adel, 2018). Recently, a group researchers applied the Genocchi collocation technique to solve nonlinear high-order boundary value problems (El-Gamel et al., 2022).

Agarwal's book contains an extensive discussion of theorems that list the conditions for the existence and uniqueness of solutions to such problems (Agarwal, 1986). In the last two decades, numerical methods have been used widely to find the approximate solution to applied and engineering problems. One of these methods is the collocation method which is used to approximate the solution of both ordinary and partial differential equations at some points in the solution interval called collocation points. In the collocation methods, the unknown function is written in the form of truncated series of basis functions with unknown coefficients which are calculated by solving linear or nonlinear systems of equations depending on the type of the original problem. Researchers use different types of basis functions to solve boundary value problems such as Chebyshev functions, Bessel functions, Legendre functions, B-spline functions, Sinc functions, etc. Further information about using collocation methods can be found in (El-Gamel et al., 2022; El-Gamel and El-Hady, 2020; El-Gamel and El-Hady, 2021; El-Baghdady Galal et al., 2017; El-Baghdady Galal and El-Azab, 2016; El-Baghdady Galal and El-Azab, 2015).

Bessel collocation method has been known for almost thirty years. Since that time, Bessel collocation scheme has been developed. For instance, Bessel collocation scheme on several problems has been applied to dealing with linear neutral delay differential equations with variable coefficients, a class of Lane-Emden type equations, pantograph equations, fractional optimal control problems, high-order linear Fredholm-Volterra integro-differential, high-order linear Volterra integro-differential equations, differential and integro-differential equations of the fractional order, Multi-Order Fractional Differential Equations, Fredholm-Volterra integro-differential equations of multi fractional order within the Caputo sense and variable-order fractional problems (Yuzbasi et al., 2011; Yuzbasi and Sezer, 2011; Yuzbasi et al., 2010; Tohidi and Saberi Nik, 2015; Yuzbasi et al., 2012; Yuzbasi et al., 2011; Parand and Nikarya, 2014; Izadi and Cattani, 2020; Ahmed and Mohammed Faeq, 2012; Dehestani and Ordokhani, 2018). The following is how the paper is structured. Bessel polynomials and some of their properties are introduced in the next section. Section 3 introduces the fundamental concept for solving linear and nonlinear fifth-order differential equations. Method convergence and error estimation Section four. Section 5 includes numerical experiments to demonstrate the applicability of the presented technique. Conclusions are found in Section 6.

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2. Bessel Polynomials and Some of Their Properties

In this work, we aim to find an approximation solution of (1) in terms of truncated Bessel functions taking the form:

$$u_N(t) \cong u_N(t) = \sum_{n=0}^{N} a_N J_N(t), \quad a \le t \le b, \quad n = 0,1,2,\cdots,N \tag{3}$$

Here, N is taken as a positive integer such that $N \ge 5$, a_N, and $J_N(t)$are the unknown Bessel coefficients and the Bessel functions of the first kind respectively.

The Bessel functions can be represented as:

$$J_N(t) = \sum_{k=0}^{\left\lfloor\frac{N-n}{2}\right\rfloor} \frac{(-1)^k}{k!(k+n)!}\left(\frac{t}{2}\right)^{2k+n}, \quad n \in N, \quad 0 \le t \le \infty \tag{4}$$

Here, $\lfloor x \rfloor$ is the floor function that rounds the element (x) to the nearest integer less than or equal to that element.

In this section, to consider the approximation solution of (1.1) we will create the matrix forms of each term in the equation. For this purpose, _rstly we transform the truncated Bessel series defined by (2.1) into the next matrix form:

$$u_N(t) = \boldsymbol{J}(t) = \boldsymbol{T}(t)\boldsymbol{H}^T\boldsymbol{A} \tag{5}$$

where;

$$\boldsymbol{J}(t) = [J_{N,0}(t) \quad J_{N,1}(t) \quad \cdots \quad J_{N,N}(t)], \qquad \boldsymbol{A} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}$$

for N is odd and even, respectively, the following $(N+1)\times(N+1)$ matrices form are true:

$$\boldsymbol{H} = \begin{bmatrix} \frac{1}{0!\,0!\,2^0} & 0 & \frac{-1}{1!\,1!\,2^2} & \cdots & \frac{(-1)^{\frac{N-1}{2}}}{\left(\frac{N-1}{2}\right)!\left(\frac{N-1}{2}\right)!\,2^{N-1}} & 0 \\ 0 & \frac{1}{0!\,1!\,2^1} & 0 & \cdots & 0 & \frac{(-1)^{\frac{N-1}{2}}}{\left(\frac{N-1}{2}\right)!\left(\frac{N+1}{2}\right)!\,2^{N}} \\ 0 & 0 & \frac{1}{0!\,2!\,2^2} & \cdots & \frac{(-1)^{\frac{N-3}{2}}}{\left(\frac{N-3}{2}\right)!\left(\frac{N+1}{2}\right)!\,2^{N-1}} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!\,(N-1)!\,2^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!\,(N)!\,2^{N}} \end{bmatrix},$$

$$\boldsymbol{H} = \begin{bmatrix} \frac{1}{0!\,0!\,2^0} & 0 & \frac{-1}{1!\,1!\,2^2} & \cdots & 0 & \frac{(-1)^{\frac{N}{2}}}{\left(\frac{N}{2}\right)!\left(\frac{N}{2}\right)!\,2^{N}} \\ 0 & \frac{1}{0!\,1!\,2^1} & 0 & \cdots & \frac{(-1)^{\frac{N-2}{2}}}{\left(\frac{N-2}{2}\right)!\left(\frac{N}{2}\right)!\,2^{N-1}} & 0 \\ 0 & 0 & \frac{1}{0!\,2!\,2^2} & \cdots & 0 & \frac{(-1)^{\frac{N-2}{2}}}{\left(\frac{N-2}{2}\right)!\left(\frac{N+2}{2}\right)!\,2^{N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!\,(N-1)!\,2^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!\,(N)!\,2^{N}} \end{bmatrix},$$

nd $\boldsymbol{T}(t) = [1 \quad t \quad \cdots \quad t^N]$.

Lemma 2.1. Let $\boldsymbol{J}(t)$ be the Bessel polynomials with the form

$\boldsymbol{J}(t) = \boldsymbol{T}(t)\boldsymbol{H}^T$;

then;

$$\boldsymbol{J}^{(k)}(t) = \boldsymbol{T}(t)(\boldsymbol{D})^{(k)}\boldsymbol{H}^T \tag{6}$$

where $\boldsymbol{D}$ is the first differentiation matrix defined as:

$$\boldsymbol{D} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(N+1)\times(N+1)}$$

From the previous relation (6), so the derivatives of the approximation solution in (5) can be formulated as;

$$u_N^{(k)}(t) = \boldsymbol{J}^{(k)}(t)\boldsymbol{A} = \boldsymbol{T}(t)(\boldsymbol{D})^{(k)}\boldsymbol{H}^T\boldsymbol{A} \tag{7}$$

where;

$$\boldsymbol{D}^{(k)} = (\boldsymbol{D})^{(k-1)}.\boldsymbol{D}, \qquad k = 2, 3, \cdots$$

By using the shifted Chebyshev-Gauss-Lobatto (**SCGL**) collocation nodes (t_i)

calculated by

$$t_i = a + \left(\frac{b-a}{2}\right)\left(1 - \cos\left(\frac{\pi i}{N}\right)\right) \tag{8}$$

the approximation solutions in (3) and (7) will be in matrix form respectively as;

$$u_N(t_i) = \boldsymbol{T}(t_i)\boldsymbol{H}^T\boldsymbol{A}$$

$$u_N^{(k)}(t_i) = \boldsymbol{T}(t_i)(\boldsymbol{D})^{(k)}\boldsymbol{H}^T\boldsymbol{A} \tag{9}$$

Equation (9) can be written with the next compact form

$$\boldsymbol{U} = \boldsymbol{T}\boldsymbol{H}^T\boldsymbol{A}$$

$$\boldsymbol{U}^{(k)} = \boldsymbol{T}(\boldsymbol{D})^{(k)}\boldsymbol{H}^T\boldsymbol{A} \tag{10}$$

where;

$$\boldsymbol{T} = \begin{bmatrix} T(t_0) \\ T(t_1) \\ \vdots \\ T(t_N) \end{bmatrix}, \qquad \boldsymbol{U}^{(k)} = \begin{bmatrix} u^{(k)}(t_0) \\ u^{(k)}(t_1) \\ \vdots \\ u^{(k)}(t_N) \end{bmatrix}.$$

3. Bessel Matrix-Collocation Method

The following matrix method is used to obtain a Bessel polynomial approximation solution to Eq. (1) under the mixed conditions in Eq. (2). The Bessel coefficients are computed using collocation points in this method (6). First, in Eq. (1), the collocation points are substituted as:

$$u^{(5)}(t_i) + \sum_{k=1}^{4} p_k(t_i)\,u^{(k)}(t_i) + \sum_{s=1}^{r} q_s(t_i)\,u^{(s)}(t_i) = f(t_i); \quad t_i \in [a,b]$$

Then, the previous equation can be written in the following matrix form:

$$\boldsymbol{U}^{(5)} + \sum_{k=1}^{4} \boldsymbol{P}_k\,\boldsymbol{U}^{(k)} + \sum_{s=1}^{r} \boldsymbol{Q}_s\,\boldsymbol{U}^{s} = \boldsymbol{F}, \quad r \ge 1, \tag{11}$$

where;

$$\boldsymbol{F} = \begin{bmatrix} f(t_0) \\ f(t_1) \\ \vdots \\ f(t_N) \end{bmatrix}, \qquad \boldsymbol{P}_k = \begin{bmatrix} p^{(k)}(t_0) & 0 & \cdots & 0 \\ 0 & p^{(k)}(t_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p^{(k)}(t_N) \end{bmatrix},$$

$$\boldsymbol{Q}_s = \begin{bmatrix} q^{(k)}(t_0) & 0 & \cdots & 0 \\ 0 & q^{(k)}(t_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q^{(k)}(t_N) \end{bmatrix} \tag{12}$$

The final matrix forms for linear and non-linear approximation solutions will be introduced in the next subsections.

3.1 Linear Case Approximation Estimation

By setting $r = 1$ in Eq. (11) to

get the linear matrix form as:

$$\boldsymbol{U}^{(5)} + \sum_{k=1}^{4} \boldsymbol{P}_k\,\boldsymbol{U}^{(k)} + \boldsymbol{Q}_1\boldsymbol{U} = \boldsymbol{F}. \tag{13}$$

Now, by substituting the relations (9) and (12) in Eq. (13), the matrix equation can be written as:

$$\left\{\boldsymbol{T}(\boldsymbol{D})^{(5)}\boldsymbol{H}^T + \sum_{k=1}^{4} \boldsymbol{P}_k\,\boldsymbol{T}(\boldsymbol{D})^{(k)}\boldsymbol{H}^T + \boldsymbol{Q}_1\boldsymbol{T}\boldsymbol{H}^T\right\}\boldsymbol{A} = \boldsymbol{F}. \tag{14}$$

Eq. (14) also can be written in the canonical form as:

$$\boldsymbol{\Theta}\boldsymbol{A} = \boldsymbol{F} \quad \Longrightarrow \quad [\boldsymbol{\Theta}; \mathbf{F}] \tag{15}$$

where;

$$\boldsymbol{\Theta} = \boldsymbol{T}(\boldsymbol{D})^{(5)}\boldsymbol{H}^T + \sum_{k=1}^{4} \boldsymbol{P}_k\,\boldsymbol{T}(\boldsymbol{D})^{(k)}\boldsymbol{H}^T + \boldsymbol{Q}_1\boldsymbol{T}\boldsymbol{H}^T$$

Eq.(15) refers to a system of the $(N + 1)$ linear algebraic equations with unknown Bessel coefficients a_N. To complete the solution, the boundary conditions in Eq. (2) must be presented in a matrix form and then placed in (3.5) to find the unknown Bessel coefficients a_N. By using relations in (2.7) at $t_i = t_0$ and $t_i = t_N$, we get the following matrix form:

$$\begin{aligned} &\{\boldsymbol{T}(t_0)(\boldsymbol{D})^{(j)}\boldsymbol{H}^T\}\boldsymbol{A} = \mu_j, \qquad j = 0,1,2,\\ &\{\boldsymbol{T}(t_N)(\boldsymbol{D})^{(m)}\boldsymbol{H}^T\}\boldsymbol{A} = \alpha_m, \qquad m = 0,1. \end{aligned} \tag{16}$$

Or

$$\begin{cases} \boldsymbol{\Phi}_j\ \boldsymbol{A} = [\mu_j] \quad \Longrightarrow \quad [\boldsymbol{\Phi}_j; \mu_j], \qquad j = 0,1,2, \\ \boldsymbol{\Psi}_m \boldsymbol{A} = [\alpha_m] \quad \Longrightarrow \quad [\boldsymbol{\Psi}_m; \alpha_m], \qquad m = 0,1 \end{cases} \tag{17}$$

where;

$$\boldsymbol{\Phi}_j = \boldsymbol{T}(t_0)(\boldsymbol{D})^{(j)}\boldsymbol{H}^T = [\emptyset_{j,0} \quad \emptyset_{j,1} \quad \emptyset_{j,2} \quad \cdots \quad \emptyset_{j,N}]$$

$$\boldsymbol{\Psi}_m = \boldsymbol{T}(t_N)(\boldsymbol{D})^{(m)}\boldsymbol{H}^T = [\psi_{m,0} \quad \psi_{m,1} \quad \psi_{m,2} \quad \cdots \quad \psi_{m,N}]$$

By replacing the row matrices in (3.7) with the (j + m = 5) rows in the augmented matrix (3.5), the new solution augmented matrix becomes

$$\bar{\boldsymbol{\Theta}}\boldsymbol{A} = \mathbb{F} \quad \Longrightarrow \quad [\bar{\boldsymbol{\Theta}}; \mathbb{F}], \tag{18}$$

$$[\bar{\boldsymbol{\Theta}}; \mathbb{F}] = \begin{bmatrix} \theta_{0,0} & \theta_{0,1} & \theta_{0,2} & \cdots & \theta_{0,N} & ; & f(t_0) \\ \theta_{1,0} & \theta_{1,1} & \theta_{1,2} & \cdots & \theta_{1,N} & ; & f(t_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & ; & \vdots \\ \theta_{N-5,0} & \theta_{N-5,1} & \theta_{N-5,2} & \cdots & \theta_{N-5,N} & ; & f(t_{N-5}) \\ \emptyset_{0,0} & \emptyset_{0,1} & \emptyset_{0,2} & \cdots & \emptyset_{0,N} & ; & \mu_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \emptyset_{2,0} & \emptyset_{2,1} & \phi_{2,2} & \cdots & \emptyset_{2,N} & ; & \mu_2 \\ \psi_{0,0} & \psi_{0,1} & \psi_{0,2} & \cdots & \psi_{0,N} & ; & \alpha_0 \\ \psi_{1,0} & \psi_{1,1} & \psi_{1,2} & \cdots & \psi_{1,N} & ; & \alpha_1 \end{bmatrix} \tag{19}$$

The previous augmented matrix in (3.9) describes a linear algebraic system consisting of $(N + 1)$ equations, by solving it, the unknown Bessel coefficients matrix **A** is calculated and then substituted into Eq. (2.3), finally the Bessel polynomial approximation solution is found.

3.2 Non-Linear Case Approximation Estimation

In this subsection, we will introduce the Bessel polynomial approximation solution to the nonlinear form in Eq. (3.1) when $s = 2, r \geq 2$, which takes the following matrix form:

$$\boldsymbol{U}^{(5)} + \sum_{k=1}^{4} \boldsymbol{P}_k\, \boldsymbol{U}^{(k)} + \sum_{s=2}^{r} \boldsymbol{Q}_s\, \boldsymbol{U}^s = \boldsymbol{F}, \quad r \geq 2, \tag{20}$$

To illustrate how we deal with the nonlinear situation in Eq. (20), it can be formulated as the following form:

$$\boldsymbol{U}^{(5)} + \sum_{k=1}^{4} \boldsymbol{P}_k\, \boldsymbol{U}^{(k)} + \left(\sum_{s=2}^{r} \boldsymbol{Q}_s\, \mathbb{U}^{s-1}\right)\boldsymbol{U} = \boldsymbol{F} \tag{21}$$

where;

$$\mathbb{U}^{v} = \begin{bmatrix} u^{v}(t_0) & 0 & \cdots & 0 \\ 0 & u^{v}(t_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u^{v}(t_N) \end{bmatrix}; \quad v \geq 1,$$

$$\mathbb{U} = \mathbb{T}\mathbb{H}^T\mathbb{A}, \quad T = \begin{bmatrix} \boldsymbol{T}(t_0) & 0 & \cdots & 0 \\ 0 & \boldsymbol{T}(t_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{T}(t_N) \end{bmatrix},$$

$$\mathbb{H}^T = \begin{bmatrix} \boldsymbol{H}^T & 0 & \cdots & 0 \\ 0 & \boldsymbol{H}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{H}^T \end{bmatrix}, \quad \mathbb{A} = \begin{bmatrix} \boldsymbol{A} & 0 & \cdots & 0 \\ 0 & \boldsymbol{A} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{A} \end{bmatrix}. \tag{22}$$

Now, by using the relations (2.8) and (22) in Eq. (21), the matrix equation can be written as:

$$\{\boldsymbol{T}(\boldsymbol{D})^{(5)}\boldsymbol{H}^T + \textstyle\sum_{k=1}^{4} \boldsymbol{P}_k\, \boldsymbol{T}(\boldsymbol{D})^{(k)}\boldsymbol{H}^T + \sum_{s=2}^{r} \boldsymbol{Q}_s\, (\mathbb{T}\mathbb{H}^T\mathbb{A})^{S-1}\boldsymbol{T}\boldsymbol{H}^T\}\boldsymbol{A} = \boldsymbol{F}. \tag{23}$$

Eq. (23) can be written in the canonical form as:

$$\boldsymbol{V}\boldsymbol{A} = \boldsymbol{F} \quad \Longrightarrow \quad [\boldsymbol{V}\boldsymbol{A}; \boldsymbol{F}], \tag{24}$$

where,

$$\boldsymbol{V} = \boldsymbol{T}(\boldsymbol{D})^{(5)}\boldsymbol{H}^T + \sum_{k=1}^{4} \boldsymbol{P}_k\, \boldsymbol{T}(\boldsymbol{D})^{(k)}\boldsymbol{H}^T + \sum_{s=2}^{r} \boldsymbol{Q}_s\, (\mathbb{T}\mathbb{H}^T\mathbb{A})^{S-1}\boldsymbol{T}\boldsymbol{H}^T.$$

Then after replacing the row matrices of boundary conditions in relation (3.7) by the (j + m = 5) rows in the augmented matrix (24), the new resulting solution augmented matrix will be:

$$\mathbb{V}\boldsymbol{A} = \mathbb{F} \quad \Longrightarrow \quad [\mathbb{V}\boldsymbol{A}; \mathbb{F}], \tag{25}$$

$$[\mathbb{V}\boldsymbol{A}; \mathbb{F}] = \begin{bmatrix} v_{0,0} & v_{0,1} & v_{0,2} & \cdots & v_{0,N} & ; & f(t_0) \\ v_{1,0} & v_{1,1} & v_{1,2} & \cdots & v_{1,N} & ; & f(t_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & ; & \vdots \\ v_{N-5,0} & v_{N-5,1} & v_{N-5,2} & \cdots & v_{N-5,N} & ; & f(t_{N-5}) \\ \phi_{0,0}a_0 & \phi_{0,1}a_1 & \phi_{0,2}a_2 & \cdots & \phi_{0,N}a_N & ; & \mu_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \phi_{2,0}a_0 & \phi_{2,1}a_1 & \phi_{2,2}a_2 & \cdots & \phi_{2,N}a_N & ; & \mu_2 \\ \psi_{0,0}a_0 & \psi_{0,1}a_1 & \psi_{0,2}a_2 & \cdots & \psi_{0,N}a_N & ; & \alpha_0 \\ \psi_{1,0}a_0 & \psi_{1,1}a_1 & \psi_{1,2}a_2 & \cdots & \psi_{1,N}a_N & ; & \alpha_1 \end{bmatrix} \tag{26}$$

The result in Eq. (26) shows a nonlinear system with (N+1) equations and (N+1) unknowns that can be solved by using the Newton technique.

4. Convergence and Error Estimation of The Method

4.1 Convergence of Bessel Polynomial

Now, we will introduce the convergence and error bound to Bessel polynomials. Let u be a function and t_i be any point from the interior of its domain. Suppose that u has a real continuously differentiable function at t_i. Then, we define its Taylor series at t_i by the formula

$$u(t) = \sum_{k=0}^{\infty} \frac{u^{(i)}(t)}{k!}(t - t_i)^k$$

Therefore, we can write it as;

$$u(t) = \sum_{k=0}^{N} \frac{u^{(i)}(t)}{k!}(t - t_i)^k \zeta_N(t),$$

such that

$$|\zeta_N(t)| = \frac{M_N(t)}{(N+1)!}(t - t_i)^{N+1}, \quad M_N = \max\left|u^{(N+1)}(t)\right|, \quad 0 \leq t \leq 1.$$

Theorem 4.1. (Lamnii et al., 2008)

Suppose that u(t) are real (N+1)-times continuously differentiable function on the bounded interval [0, l] and $u_N(t) = \boldsymbol{J}(t)\boldsymbol{A}$ be the approximate solution obtained by the present method in section 3. If $\tilde{u}_N(t) = \tilde{\boldsymbol{J}}(t)\tilde{\boldsymbol{A}}$ be the Bessel polynomials expansion of the exact solution u(t), where

$$\tilde{\boldsymbol{A}} = [\tilde{a}_0, \tilde{a}_1, \cdots, \tilde{a}_N], \ \ and\ \tilde{\boldsymbol{J}}(t) = [\tilde{J}_0(t) \quad \tilde{J}_1(t) \quad \cdots \quad \tilde{J}_N(t)]$$

and $J_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(K+n)!}\left(\frac{t}{2}\right)^{2k+n}, \quad n \in N, \quad 0 \leq t \leq \infty$

where the set of Bessel polynomials of the first kind $\tilde{J}_N(t)$ is orthogonal in $L_2[0,1]$ with respect to the weight function $w(t) = t$, and

$$\int_0^1 w(t)[\bar{J}_n(t)]^2\, dt = \frac{1}{2}[\bar{J}_{n+1}(t)]^2.$$

Then the upper bound of the error is

$$\|u(t) - u_N(t)\|_2 \leq \frac{M_N(t)}{(N+1)!\sqrt{2N+4}} + \|\bar{\boldsymbol{A}} - \boldsymbol{A}\|_2 \alpha_N + \|\boldsymbol{A}\|_2 \beta_N$$

where

$$\alpha_N = \left[\sum_{k=0}^{N} \frac{1}{2}[\bar{J}_{N+1}(1)]^2\right]^{\frac{1}{2}},$$

$$\beta_N = \left[\sum_{n=0}^{N} \sum_{k=\frac{N-n}{2}}^{\infty} \frac{1}{(k!\,(k+n)!\,2k+n)^2(4k+2n+2)}\right]^{\frac{1}{2}}$$

Proof. in [40]

4.2 Error Estimation of Bessel-Collocation Method

In this part, we can confirm the accuracy of the developed technique by using residual error analysis. The residual function can be defined in the form

$$R_N(t) = u_N^{(5)}(t) + \sum_{k=1}^{4} p_k(t)\, u_N^{(k)}(t) + \sum_{s=1}^{r} q_s\, u_N^{s}(t) - f(t) \tag{27}$$

where the $u_N(t) = \sum_{i=0}^{N} a_i J_i(t)$ is the approximate solution of Eq. (1), By subtracting Eq. (27) from Eq. (1.1), we will get

$$u^{(5)}(t) - u_N^{(5)}(t) + \sum_{k=1}^{4} p_k(t)\left[u^{(k)}(t) - u_N^{(k)}(t)\right] + \sum_{s=1}^{r} q_s(t)\left[u^{s}(t) - u_N^{s}(t)\right] =..-R_n(t) \tag{28}$$

For the linear part of Eq.(28), we obtain

$$L[\varepsilon_N(t)] = L[u(t) - u_N(t)] = \varepsilon_N^{(5)} + \sum_{k=1}^{4} p_k(t)\, \varepsilon_N^{(k)}(t) \tag{29}$$

where $u(\mathrm{t})$ is the exact solution of Eq. (1) and $\varepsilon_N(t)$ is the error function. Similarly

for the nonlinear part in Eq. (29), we have

$$\aleph[u(t)] = \sum_{s=1}^{r} q_s(t) u^s(t) \tag{30}$$

$$\aleph[u_N(t) + \varepsilon_N(t)] = \sum_{s=1}^{r} q_s(t)[u_N(t) + \varepsilon_N(t)]^s \tag{31}$$

Eq. (31) can be written in operator form as follows

$$L[\varepsilon_N(t)] + \aleph[u(t)] - \aleph[u_N(t)] = -R_n(t) \tag{32}$$

So Eqs. (29) and (31) are substituted into Eq. (32) we obtain

$$\varepsilon_N^{(5)} + \sum_{k=1}^{4} p_k(t)\varepsilon_N^{(k)}(t) + \sum_{s=1}^{r} q_s(t)\left[\sum_{k=0}^{s}\binom{s}{k} u_N^{s-k}(t)\, \varepsilon_N^{k}(t) - u_N^{s}(t)\right] = -R_n(t) \tag{33}$$

Then, by solving the system of equations in Eq.(33) using the Bessel collocation method with the homogenous initial and boundary conditions

$$\begin{cases} \varepsilon^{(j)}(a) = 0, & j = 0,1,2, \\ \varepsilon^{(m)}(b) = 0, & m = 0,1. \end{cases} \tag{34}$$

The solution of equations (33)and (34) is;

$$\varepsilon_{j,N}(t) = \sum_{i=0}^{N} \tilde{a}_i J_i(t)$$

In the same manner as Section 3, we will obtain unknown coefficients $\tilde{a}_i, i = 0, 1, 2, \dots, N$. So, the maximum residual error can be evaluated by

$$\|\varepsilon_j\| = \max\{|\varepsilon_N(t)|, \quad 0 \le t \le 1\}$$

Using maximum residual error, we can check the accuracy of the results especially if the exact solution is unknown.

5. Numerical Simulation

In this section, we test the utility of the proposed method by presenting several examples whose exact solutions are provided in each case. To show the efficiency of the proposed technique, we compare the numerical results with the exact solution and some methods used to solve the same problem (Lang and Xu, 2011; Khan, 1993; Siddiqi et al., 2008). The formulas for the absolute error and the maximum absolute error will be:

- The absolute error E_N of the solution is defined by

$$|E_N| = |u(t) - u_N(t)|,$$

- The max. error $E_{Max.}$ of the solution is defined by

$$E_{Max.} = \max_{1\le i\le N-1} |u(t_i) - u_N(t_i)|.$$

Example 5.1: Consider $p_k(t) = 0$, $q_s(t) = 0$, and $q_1(t) = -1$ in equation (1) to give the linear fifth order differential equation of the form;

$$u^{(5)}(t) - u(t) = f(t), \qquad t \in (0,1), \tag{35}$$

with the boundary conditions:

$$\begin{cases} u(0) = 0, \quad u^{(1)}(0) = 1, \quad u^{(2)}(0) = 0, \\ u(1) = 0 \quad u^{(1)}(1) = -\exp(1). \end{cases}$$

Here, $f(t) = -(15 + 10t)\exp(t)$, so that the exact solution is $u(t) = t(1 - t)\exp(t)$. For $N = 6$, and by using equation (2.6) the set of collocation points is

$$\{t_0 = 0, t_1 = 0.067, t_2 = 0.25, t_3 = 0.5, t_4 = 0.75, t_5 = 0.933, t_6 = 1.0\}.$$

From equation (35) we can get the fundamental matrix equation of the problem:

$$\{\mathbf{T}(\mathbf{D})^{(5)}\mathbf{H}^{\mathbf{T}} + \mathbf{Q_1}\mathbf{T}\mathbf{H}^{\mathbf{T}}\}\mathbf{A} = \mathbf{F}$$

where,

$$\boldsymbol{D} = \begin{bmatrix} 0&1&0&0&0&0&0\\ 0&0&2&0&0&0&0\\ 0&0&0&3&0&0&0\\ 0&0&0&0&4&0&0\\ 0&0&0&0&0&5&0\\ 0&0&0&0&0&0&6\\ 0&0&0&0&0&0&0 \end{bmatrix}, \quad \boldsymbol{Q_1} = \begin{bmatrix} -1&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0\\ 0&0&-1&0&0&0&0\\ 0&0&0&-1&0&0&0\\ 0&0&0&0&-1&0&0\\ 0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&-1 \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} -15.000000\\ -16.755511\\ -22.470445\\ -32.974425\\ -47.632500\\ -61.850988\\ -67.957045 \end{bmatrix}, \quad \mathrm{F} = \begin{bmatrix} -15.000000\\ -16.755511\\ 0.00000\\ 1.00000\\ 0.00000\\ 0.00000\\ -2.7182818 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} \mathbf{T}(0.000)\\ \mathbf{T}(0.067)\\ \mathbf{T}(0.250)\\ \mathbf{T}(0.500)\\ \mathbf{T}(0.750)\\ \mathbf{T}(0.933)\\ \mathbf{T}(1.000) \end{bmatrix},$$

$$\mathbf{T} = \begin{bmatrix} 1&0&0&0&0&0&0\\ 1&0.06699&0.004&3.01E-4&2.013E-5&1.34E-6&9.03E-8\\ 1&0.25000&0.06250&0.01563&3.906E-3&9.765E-4&2.441E-4\\ 1&0.50000&0.25000&0.12500&0.062500&0.03125&0.01563\\ 1&0.75000&0.56250&0.42188&0.316406&0.23731&0.17798\\ 1&0.93301&0.87051&0.81220&0.757792&0.70703&0.658967\\ 1&1&1&1&1&1&1 \end{bmatrix}$$

$$\mathbf{H}^T = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1/2&0&0&0&0&0\\ -1/4&0&1/8&0&0&0&0\\ 0&-1/16&0&1/48&0&0&0\\ 1/64&0&-1/96&0&1/384&0&0\\ 0&1/384&0&-1/768&0&1/3840&0\\ -1/2304&0&1/3072&0&-1/7680&0&1/46080 \end{bmatrix}$$

$$\boldsymbol{A} = [1.1975E-15\ 2.00\ 3.2996E-15-16.8836-139.486-584.418-2450.285]^T$$

then, the approximate solution can be found as

$$u_6(t) = (1.1975E-15)J_0(t) + 2J_1(t) + (3.2996E-15)J_2(t) - 16.8836J_3(t) - 139.486J_4(t) - 584.418J_5(t) - 2450.285J_6(t),$$

or, equivalent to

$$u_6(t) = (1.1975200E-15) + t + (1.13074986E-16)t^2 - 0.47674284765t^3 - 0.3632448143(t^4) - 0.1250(t)^5 - 0.0350123380(t)^6.$$

Table 1: Numerical results of t_i for Example 5.1.

t_i	$u(t_i)$	$u_8(t_i)$	$E_8(t_i)$	$E_{12}(t_i)$
0.1	0.0994653826	0.0994657181	3.35492296E-07	3.47553930E-12
0.2	0.1954244413	0.1954267801	2.33875124E-06	2.45707066E-11
0.3	0.2834703496	0.2834770781	6.72853214E-06	7.18516913E-11
0.4	0.3580379274	0.3580511137	1.31862599E-05	1.44034062E-10
0.5	0.4121803177	0.4122006742	2.03565178E-05	2.29970098E-10
0.6	0.4373085121	0.4373344303	2.59181814E-05	3.08663039E-10
0.7	0.4228880686	0.4229150113	2.69427082E-05	3.49249629E-10
0.8	0.3560865486	0.3561075516	2.10030255E-05	3.11271897E-10
0.9	0.2213642800	0.2213731352	8.85514822E-06	1.60047697E-10

Table 2: Maximum error for different N	
N	Max. error
8	9.94997016E-04
10	7.76765086E-05
12	5.29499652E-06
14	3.31794318E-07
16	1.96712734E-08
18	1.13287021E-09

Table 3: Maximum absolute error in some methods for Example 5.1.	
The method	Max. error
Present method	1.133E-09
Sinc-Galerkin method (El Gamel, 2007)	0.951E-05
Sextic spline method (Lamnii et al., 2008)	4.844E-07
Cubic B-spline method (Lanf and Xu, 2011)	1.41E-05
Finite difference method (Khan 1993)	1.15E-02
Quartic spline method (Siddiqi et al., 2008)	7.66E-05

In "Table 1", we introduce some results in (0,1) of the exact solution, the approximate solution, and the absolute error values obtained by the present method.

whereas "Table 2", introduces the maximum error obtained for the different values of N in Example 5.1. "Table 3" shows the comparison of the proposed method with some other methods for the same problem.

Example 5.2: Consider the non-linear fifth order differential equation by setting $p_k(t) = 0, p_4(t) = 1$, and $q_s(t) = 0$ except $q_2(t) = \exp(-2t)$ in equation (1.1) to get the equation

$$u^{(5)}(t) + u^{(4)}(t) + \exp(-2t)\, u^{(2)}(t) = f(t), \qquad t \in (0,1), \qquad (36)$$

with the boundary conditions;

$$\begin{cases} u(0) = u^{(1)}(0) = u^{(2)}(0) = 1, \\ u(1) = u^{(1)}(1) = \exp(1). \end{cases}$$

Here, $f(t) = 2\exp(t) + 1$ so that the exact solution will be $u(t) = \exp(t)$. To introduce the analysis of the method, let N = 6, and using (2.6) the set of collocation points will be

$$\{t_0 = 0, t_1 = 0.067, t_2 = 0.25, t_3 = 0.5, t_4 = 0.75, t_5 = 0.933, t_6 = 1.0\}$$

From equations (3.13) and (3.14), we get the fundamental matrix equation for this problem:

$$\{\mathbf{T}(\mathbf{D})^{(5)}\mathbf{H}^T + \mathbf{P_4}\mathbf{T}(\mathbf{D})^{(4)}\mathbf{H}^T + \mathbf{Q_2}(\mathbb{T}\mathbb{H}^T\mathbb{A})\mathbf{T}\mathbf{H}^\mathbf{T}\}\mathbf{A} = \mathbf{F}$$

where $\mathbf{T}$, $\mathbf{D}$; and $\mathbf{H}^T$ were described in the previous example, whereas

$$\mathbf{Q_2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.874612 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.606531 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.367879 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.223130 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.154737 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.135335 \end{bmatrix},$$

$$\mathbf{P_4} = \boldsymbol{I}, \quad \mathbf{F} = \begin{bmatrix} 3.0000000000 \\ 3.1385637924 \\ 3.5680508334 \\ 4.2974425414 \\ 5.2340000332 \\ 6.0843128237 \\ 6.4365636569 \end{bmatrix}, \quad \mathbb{F} = \begin{bmatrix} 3.0000000000 \\ 3.1385637924 \\ 1.0000000000 \\ 1.0000000000 \\ 1.0000000000 \\ 2.7182818285 \\ 2.7182818285 \end{bmatrix},$$

$$\boldsymbol{A} = [1.00 \quad 2.00 \quad 6.00 \quad 13.9630 \quad 34.4382 \quad 80.9386 \quad 204.4941]^T$$

then, the approximate solution will be found as

$$u_6(t) = J_0(t) + 2J_1(t) + 6J_2(t) + 13.9630J_3(t) + 34.4382J_4(t) + 80.9386J_5(t) + 204.4941J_6(t),$$

or,

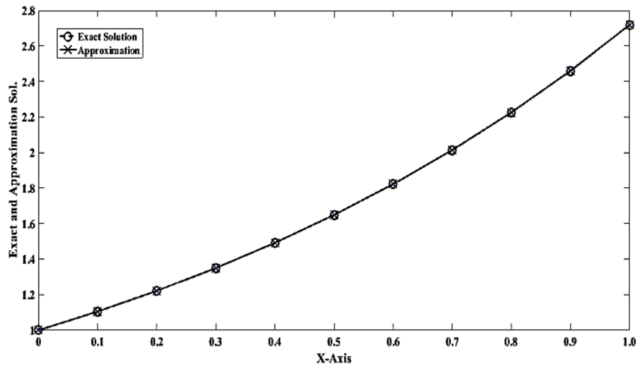
$$u_6(t) = 1 + t + 0.500(t^2) + 0.165896095(t^3) + 0.04280788(t^4) + 0.008105090(t)^5 + 0.001472760(t)^6.$$

Table 4: Numerical results for t_i for Example 5.2.				
t_i	$u(t_i)$	$u_6(t_i)$	$E_6(t_i)$	$E_{10}(t_i)$
0.1	1.1051709181	1.1051702594	6.58668117E-07	1.89184224E-11
0.2	1.2214027582	1.2213983493	4.40889759E-06	1.30335298E-10
0.3	1.3498588076	1.3498467074	1.21001340E-05	3.71688236E-10
0.4	1.4918246976	1.4918022605	2.24371781E-05	7.25544069E-10
0.5	1.6487212707	1.6486888006	3.24701401E-05	1.12488308E-09
0.6	1.8221188004	1.8220804231	3.83772440E-05	1.45753298E-09
0.7	2.0137527075	2.0137160247	3.66828027E-05	1.57144076E-09
0.8	2.2255409285	2.2255148608	2.60676550E-05	1.30099576E-09
0.9	2.4596031111	2.4595931653	9.94589378E-06	5.97708993E-10

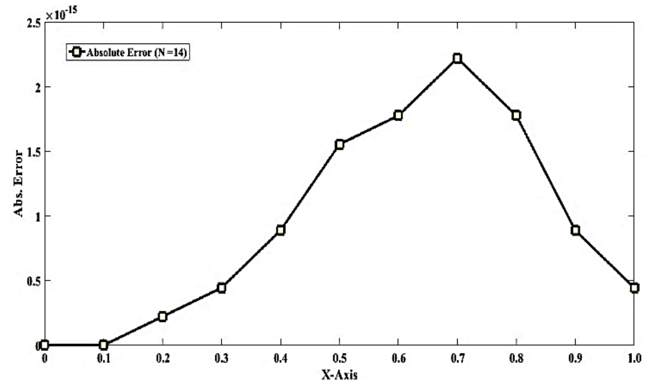
Table 5: Maximum error for different N.	
N	Max. error
6	3.83772440 E-05
8	4.41496335 E-07
10	1.57144076E-09
12	2.51043630 E-12
14	2.22044605 E-15

Table 6: Maximum absolute error in some methods for Example 5.2.	
The method	Max. error
Present method	2.22044605 E-15
Sinc-Galerkin method (El Gamel, 2007)	0.951E-05
Sextic spline method (Lamnii et al., 2008)	4.844E-07
Cubic B-spline method (Lang and Xu, 2011)	1.41E-05
Finite difference method (Khan, 1993)	1.15E-02
Quartic spline method (Siddiqi et al., 2008)	7.66E-05

In "Table 4", we introduce the results in (0, 1) of the exact solution, the approximate solution, and the absolute error values obtained by the present method. "Table 6" shows the highest absolute error to compare the current technique to several methods for Example 5.2. Also, "Figure 1" shows the exact solution, the approximation solution, and the absolute error when $t \in [0,1]$, and $N = 14$.



(a) Exact and Approximation solutions



(b) Absolute Error

Figure 1: Exact, Approximation solutions, and Absolute Error when $t \in [0,1]$, and $N = 14$

Example 5.3: In this example, we will consider another case of non-linear fifth order differential equation in exponential form as the following problem

$$u^{(5)}(t) + 24\, exp\big(-5u(t)\big) = f(t), \quad t \in (0,1), \tag{37}$$

with the boundary conditions;

$$\begin{cases} u(0) = 0, \quad u^{(1)}(0) = 1, \quad u^{(2)}(0) = -1 \\ u(1) = \ln(2), \quad u^{(1)}(1) = 0.5, \end{cases}$$

Here, $f(t) = 48/(1+t)^5$, so that the exact solution is $u(t) = \ln(1+t)$, and setting $q(t) = 24$. As before, for $N = 6$, and if we use (2.6) as the set of collocation points lead to:

$$t_i = \{t_0 = 0, t_1 = 0.067, t_2 = 0.25, t_3 = 0.5, t_4 = 0.75, t_5 = 0.933, t_6 = 1.0\}.$$

From equations (3.13) and (3.14), the fundamental matrix equation of the problem will be:

$$\{\mathbf{T}(\mathbf{D})^{(5)}\mathbf{H}^T\mathbf{A} + \mathbf{Q}\exp(-5\mathbf{T}\mathbf{H}^T\mathbf{A})\} = \mathbf{F}, \tag{38}$$

where

$$\boldsymbol{Q} = \begin{bmatrix} 24 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 24 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 24 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 24 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 24 \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} 48.00000000 \\ 34.70923088 \\ 15.72864000 \\ 6.320987654 \\ 2.924495745 \\ 1.778556632 \\ 1.500000000 \end{bmatrix}, \quad \mathbb{F} = \begin{bmatrix} 48.000000 \\ 34.709230 \\ 1.0000000 \\ 1.0000000 \\ -1.000000 \\ 0.6931472 \\ 0.5000000 \end{bmatrix},$$

$\boldsymbol{A} = [1.000 \quad 2.000 \quad -4.000 \quad 15.45066 \quad -41.30238 \quad 825.2533 \quad -6539.941]^T$, then, we get the approximate solution as

$$u_6(t) = J_0(t) + 2J_1(t) - 4J_2(t) + 15.45066J_3(t) - 41.30238J_4(t) + 825.2533J_5(t) - 6539.941J_6(t),$$

or,

$$u_6(t) = t - 0.50(t^2) + 0.1968888(t^3) - 0.0658916(t^4) + 0.2000(t)^5 - 0.13784997(t)^6.$$

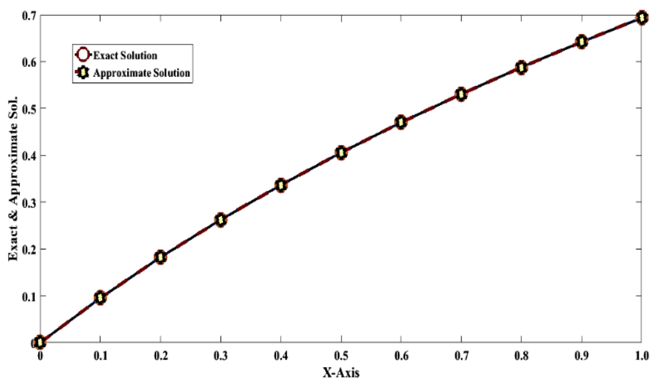
In “Table 7”, we introduce the results in (0, 1) of the exact solution, the approximate solution, and the absolute error values obtained by the present method. “Figure 2” also shows the exact solution, the approximation solution, and the absolute error when $t \in [0.1]$, and $N = 16$ for Example 5.3. The maximum error for different N will be demonstrated in “Table 8”.

Table 7: Numerical results for t_i for Example 5.3.

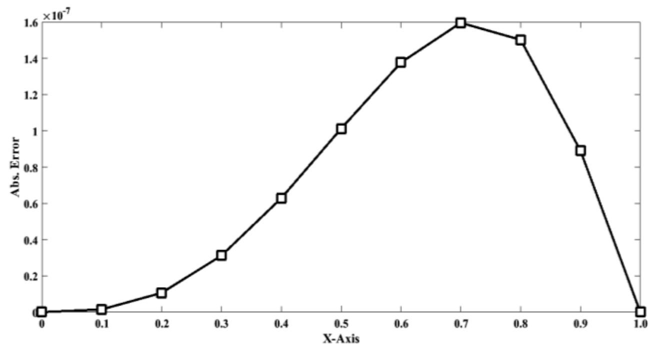
t_i	$u(t_i)$	$u_6(t_i)$	$E_6(t_i)$	$E_{10}(t_i)$
0.1	0.0953101798	0.0951921618	1.18018038E-04	4.00681679E-06
0.2	0.1823215568	0.1815248612	7.96695564E-04	2.81438767E-05
0.3	0.2623642645	0.2601677822	2.19648224E-03	8.17863452E-05
0.4	0.3364722366	0.3323974227	4.07481390E-03	1.62582423E-04
0.5	0.4054651081	0.3995889649	5.87614322E-03	2.56503956E-04
0.6	0.4700036292	0.4631088931	6.89473610E-03	3.37780702E-04
0.7	0.5306282510	0.5241083603	6.51989078E-03	3.69189927E-04
0.8	0.5877866649	0.5832173015	4.56936336E-03	3.08455253E-04
0.9	0.6418538862	0.6401392968	1.71458941E-03	1.42056029E-04

Table 8: Maximum error for different N.

N	Max. error
10	3.6918993E-04
12	3.4689325E-05
14	2.5430047E-06
16	1.5954943E-07
18	8.9867772E-08
20	4.67441308E-10



(a) Exact and Approximation solutions



(b) Absolute Error

Figure 2: Exact, Approximation solutions, and Absolute Error when $t \in [0,1]$, and $N = 16$

6. Conclusions

The target of the present work is to evolve an efficient and accurate method for solving fifth-order Boundary Value Problems in an induction motor. The problem has been reduced to solving a system of algebraic equations. Application of the Bessel collocation algorithm is more effective and faster algorithms than the ordinary ones. The numerical and graphical results elucidate that the suggested approach is very straightforward and accurate.

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