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RESEARCH ARTICLE

A UNIQUE METHOD FOR THE TRISECTION OF AN ARBITRARY ANGLE

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ABSTRACT

Trisection of an arbitrary angle is considered an impossible event in mathematical history. In this paper, we are trying to establish a general procedure for trisecting an arbitrary angle using only an unmarked straightedge and a compass. The basic geometrical compass constructions described in our main results section will ensure the possibilities of known and unknown angles trisection indeed. Before reaching the main results or general trisecting procedure, we will try to prove that Angles of sixty, thirty, seventy-five, forty-five, and ninety degrees can be trisected successfully by following the way of Archimedes and other discrete endeavors. Although based on algebra, Pierre Laurent Wantzel declared the verdict, “Arbitrary angle trisection is impossible.” we experienced that our achieved procedure was possibly not arrested by this verdict. After applying our general process to trisect an arbitrary angle, we hope readers can trust algebra cannot affect the successful geometrical constructions of angle trisection.

KEYWORDS

Archimedes, Wantzel, Circle, Arc-trisection, GeoGebra, Line-segment

1. INTRODUCTION

One of the three famous classical unsolved construction problems of antiquity left in Greek Philosophy is – whether it is possible to divide an arbitrary angle into three equal parts using a compass and a straightedge only. To find the answer mathematical world is being made relentless efforts for almost 2.6 millennia. Some particular angles (e.g., 90° and 180°) can be divided into three equal parts only using a compass and straightedge , but such partitions are not applicable for an arbitrary angle (Corte, 2013). Although angles have been trisected successfully with the help of other tools, together with the compass and straightedge, it is not accepted because there is a ban on using anything other than a compass and straightedge (Corte, 2013; Angle Trisection Wikipedia, 2023; Blonder, 2015; Alex and Mutembei, 2017). In 1837, France’s mathematician Pierre Laurent Wantzel proved at age 23 that this work trisection of an angle is impossible (Angle Trisection Wikipedia, 2023). Pierre Laurent Wantzel died prematurely in 1848. Since his verdict, the trisection of an arbitrary angle has been considered impossible. An article says that (Marianne, 2015). “*Wantzel showed that the problem of trisecting an angle is equivalent to solving a cubic equation using a straightedge-and-ruler construction. He also showed that only very few cubic equations can be solved in this way — most cannot. He thus deduced that most angles cannot be trisected.*” But the valuable mathematicians are not satisfied with this verdict, or it does not satisfy their thirst. That’s why they are still trying to make it possible that seems to be going continually (Robert and Yates., 2015; The Geometry Center, 2023).

The famous great scientist, mathematician, physicist, engineer, astronomer, and inventor Archimedes (287 BC – 212 BC) resolved this problem only one step away from the fulfillment of satisfaction of straightedge-compass embargo 2.05 millennia before providing this verdict (Angle Trisection Wikipedia, 2023; Archimedes Trisection of Angle, 2015).

Many math-related researchers admitted Laurent Wantzel's verdict on angle trisection. But they have shown the trisection of an angle is possible with a little bit of violating the rules, e.g., using extra dimension with a cylinder , Neusis method, Origami etc. Why do they do it? The answer is to try to catch the same allegiance (Blonder, 2015; Suzuki, 2008).

Also, many mathematicians do not want to admit Wantzel's verdict. They tried to trisect the angle and claimed they found approximately the correct result of the trisection (Corte, 2013). Again someone made a mistake (Marianne, 2015).

From these close shadows and Archimedean logical approaches, we are hopeful to discover a process for constructing the arbitrary angle trisection using the straightedge-compass rule. To reach our desired goal of angle trisection, we are also trying to trisect the special angles here.

2. GEOMETRICAL EXPERIMENTS ON ANGLE TRISECTION

From the ancient Greek period mathematical realm believed that angle trisection was not possible using classical tools, and it observed without expressing proof until 1837. After delivering the algebraic proof of the impossibility of the angle trisection based on algebra on behalf of Laurent Wantzel in 1837, most mathematicians considered this proof an absolute judgment like the belief of the non-existence of irrational numbers to Pythagoras. Pythagoras, the father of mathematics, believed all numbers can be expressed as the ratio of two integers. The followers of Pythagoras, i.e., Pythagoreans, did not also believe in irrational numbers. When Hippasus discovered irrational numbers and died in the ocean, Pythagoras thought this death was the justice of God. But in the existing math realm, the Real number system is divided into rational and irrational. This phenomenon teaches us no belief is absolute, which is analogous to the so-called death of the possibility of angle trisection. Article says that “*Algebraic irrationality is not a geometrical impossibility.*” (Rediske, 2018) We think that impossibility should not stop our gradual development, and we believe that if our world mathematics authority

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emphasizes their positive attention on establishing the beauty of imperfection, the approximation can achieve the possibility of the impossibility of angle trisection. However, with our little effort, we want to prove the geometrical construction of the angle trisection is approximately possible.

We do not want to miss the opportunity to express our idea for completing the angle trisection task because we can see a ray of light at the end of the tunnel.

From the algebraic perspective, Wantzel proved the trisection of a 60degree angle is impossible, so we all acknowledge that an arbitrary angle is impossible to trisect. He also said, *A few angles can be trisected, but most cannot*. There he did not specify the number and amount of such Angles. We will prove through this article that the trisection of a 60-degree Angle is possible by following prescribed rules. Therefore, it may make us hopeful that algebraic arguments do not affect the successful geometric drawing process. So we are trying to succeed in our mission with the following schedules:

i) Twist Archimedes' method in his final step in various ways for some particular angles, of course, including a 60-degree angle to solve the problem by following in the footsteps of Archimedes.

ii) To derive a unique and general trisection procedure for an arbitrary angle based on the traditional arc-trisection procedure of 180-degree and 90-degree angles.

2.1 Archimedes' proof

Archimedes proved that an arbitrary angle can be trisected using a marked straightedge and compass in the following simple way:

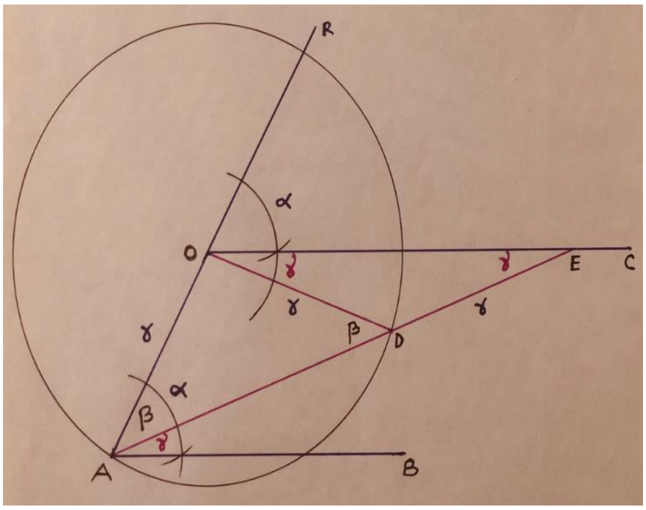

Figure 1: (Archimedes' Proof of angle trisection)

Suppose $< BAR = \alpha$ is an arbitrary angle that we want to trisect. We draw a line OC parallel to AB at any point O on the line segment AR using a straightedge and a compass. Now we draw a circle with a radius of $OA = r$ and center at point O. Finally, by using a marked straightedge and trial-error method, we draw a straight line passing through point A so that it meets the circle and the line OC at the points D and E respectively, in such a way that $OD = DE$. Since, $OA = OD = DE = r \therefore$ Let, $< OAD = < ODA = \beta$ and $< DEO =< DOE = \gamma$ for isosceles triangles OAD and ODE. Therefore, we get $\alpha = \beta + \gamma$ from $< BAR$ and alternate angles $< BAE = < OEA = \gamma$. Again, the extended angle $< ODA = \beta = \gamma + \gamma = 2\gamma$ for the triangle ODE. So, $2\gamma = \alpha - \gamma \therefore \gamma = \frac{1}{3}\alpha$. Hence, the angle α has been trisected with angle γ and the proof is completed.

Proof of the trisection of a sixty-degree angle using GeoGebra and Archimedes' procedure:

We consider $(x-1)^2 + (y-1.732)^2 = 4$ to be the equation of the respective circle and $y = 1.732$ to be the equation of the parallel line for trisecting a sixty-degree angle, followed by Archimedes method, where $C(1, 1.732)$ is the center and $OC = 4 =$ radius. Let $D(a, b)$ and $E(c, d)$ be respective points on the circle and line, which are shown in the figure below:

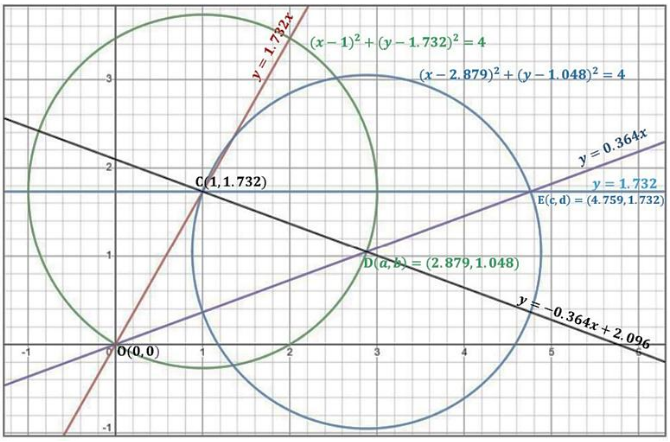

Figure 2: (Trisection of 60° angle through GeoGebra)

From figure 2, we have the following system of equations:

$$(a-1)^2 + (b-1.732)^2 = 4$$
$$(a-c)^2 + (b-1.732)^2 = 4$$

$$d = 1.732 \ and$$
$$\left(\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0 \ or \ ad - bc = 0 \right)$$

We can choose a solution of the above system as $(a, b) = (2.879, 1.048)$, $(c, d) = (4.759, 1.732)$. Now, if we draw a circle centering $D(2.879, 1.048)$ and a radius of $CO = 2$, it passes through points $C(1, 1.732)$ and $E(4.759, 1.732)$. From the equation $y = 0.364x$ of diameter ODE of the new circle, we see that OD makes an angle of 20° with the x-axis. So, it has been proved by GeoGebra that an angle of 60° can trisect if we can find out point D on the circle and point E on the straight line OC using a compass and straightedge only, which satisfies $CD = DE$.

2.2 Extension of Archimedes' Proof on some Particular Angles

The above construction in section 2 (Figure 1) stepped outside the restriction imposed by the Greek philosopher Plato (427 BC – 347 BC) that geometers use no instruments besides the compass and straightedge. For this embargo, Archimedes' proof of angle trisection didn't gain the recognition, because he used a marked (only two points on the unmarked ruler) straightedge. But the proof was very beautiful, easy, and reliable. Archimedes was only a single step away from the construction, which satisfies the rules of using only a straightedge and a compass. However, we are trying to locate the desirable points like D and E on the circle and the line OC, respectively, using only a straightedge and a compass which satisfy the conditions that the line AE contains the point D on the circle's locus so that $OD = DE$.

2.2.1 Construction and Proof for Trisection of Angles Sixty, Thirty, and Seventy-Five Degrees

In the verdict of angle trisection, Wantzel mentioned an angle of measure 60° couldn't be trisected . So, first of all, we are trying to trisect a 60° angle. For this, taking an arbitrary length of OA as the same radius, we form an equilateral triangle OAB with the help of two circles, C_1 and C_2 centering O and A in the following figure:

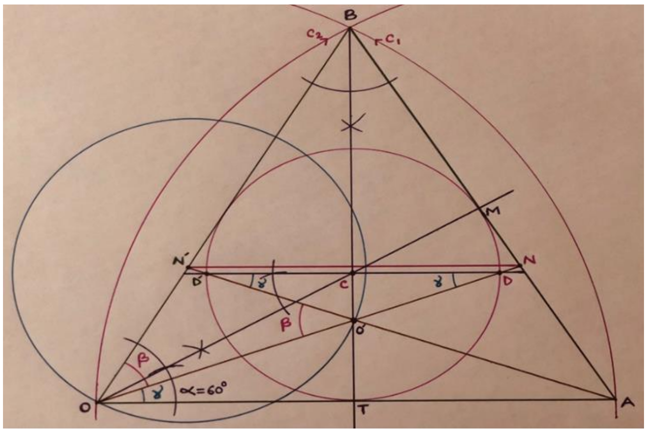

Figure 3: (Proof of the approximate trisection of 60° angle)

We locate the in-center C of the triangle OAB with the help of intersecting point of the bisectors BT and OM of angles $< ABO$ and $< AOB$, respectively. Now we draw a line DCD' passing through the center C and parallel to OA, and also we draw the in-circle of the triangle OAB that intersects line DD' at points D and D'. Extended joining lines OD and AD' intersect AB and OB at points N and N', respectively. We join N and N', then ON and AN' meet at point O'. Since for an equilateral triangle OAB, $BT \perp OA$ and $DD' \parallel OA \parallel NN'$, so obviously, $O'D = O'D'$ and $O'N = O'N'$. At this stage, we complete a circle whose radius is $N'O$ and center at N' that will pass through very near point O'. Therefore, $\Delta O'DD'$ and $\Delta O'ON'$ are isosceles triangles. Let $< AOB = \alpha = 60°$, $< AOD = < AON = \gamma = < O'DD' = < O'NN' = < DD'O' = < NN'O'$, and $< O'ON' = \beta = < OO'N'$. So, $\alpha = \beta + \gamma$ and for $\Delta N'NO'$, since $< N'O'O = \beta$ is the external angle of the extended side $NO'O$ $\therefore \beta = 2\gamma$. Therefore, $\alpha = 3\gamma \therefore \gamma = \frac{1}{3}\alpha = 20°$. Hence, the construction and successful proof of the trisection of 60° angle have been completed at $< AOD = < AON = \gamma = 20°$ by the Archimedean process.

Also, in Figure 3, $< AOM = 30°$ angle has been constructed and trisected at $< NOM = 10°$ and 75° angle has been trisected at $< AOM - \frac{1}{2} < NOM = 25°$, indirectly.

Since the above trisection has been observed on an important and special angle, we want to verify the result of the trisection of 60° with analytical geometry in the figure below:

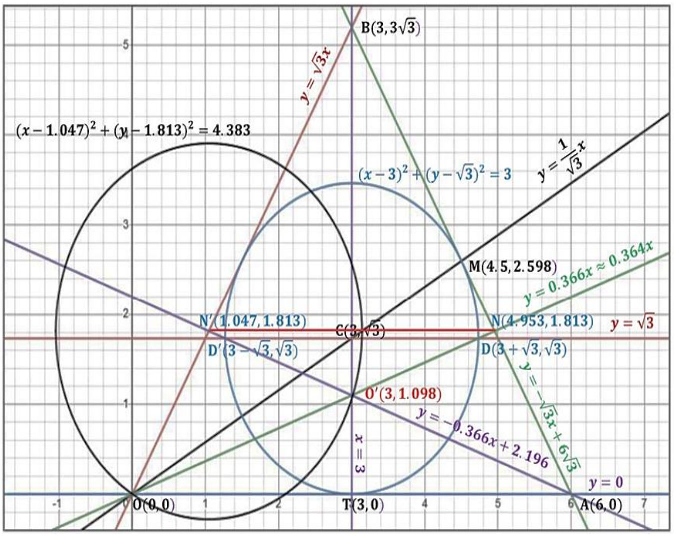

Figure 4: (Verification of trisected value of 60°, Desmos Graphing Calculator, 2011)

Consider $OA = 6$ unit length makes an angle $< AOB = 60°$ and OAB is an equilateral triangle, where equations of OA, OB, and AB are $y = 0$, $y = \sqrt{3}x$, and $y = -\sqrt{3}x + 6\sqrt{3}$. The bisectors BT [By joining B and midpoint $T(3,0)$ of OA] and OM [By joining O and midpoint $M(4.5, 2.598)$ of AB] with the equations $x = 3$ and $y = \frac{1}{\sqrt{3}}x$ meet at point $C(3,\sqrt{3})$, which is the in-center of the triangle OAB. The line with the equation $y = \sqrt{3}$ passes through point C and parallel to OA, intersects the in-circle $(x-3)^2 + (y-\sqrt{3})^2 = 3$ at points $D(3+\sqrt{3},\sqrt{3})$ and $D'(3-\sqrt{3},\sqrt{3})$. Extended joining line OD whose equation is $y = 0.366x$ intersects AB at $N(4.953, 1.813)$ and BT at $O'(3, 1.098)$. Again, extended joining line AD' whose equation is $y = -0.366x + 2.196$ intersects OB at $N'(1.047, 1.813)$ and BT at $O'(3, 1.098)$. Here, each point satisfies all the related equations described in figure 2, approximately. From section 2 (Fig 4), we can say OO' is our required trisector, and it produces the value of trisection from the relation $\theta = tan^{-1}\left(\frac{y}{x}\right)$.

$\therefore$ trisected angle $\theta = tan^{-1}\left(\frac{1.098}{3}\right) = 20.103°$.

We have seen that, although our construction and proof in sub-sub section 2.2.1 are approximately correct, algebraic measurement creates an error of (+0.513%). Here, $N'O = 2.094 \neq (N'O' = O'N = 2.08)$ contains a significant error. However, the error can be minimized followed by the rigorous method in figure 5:

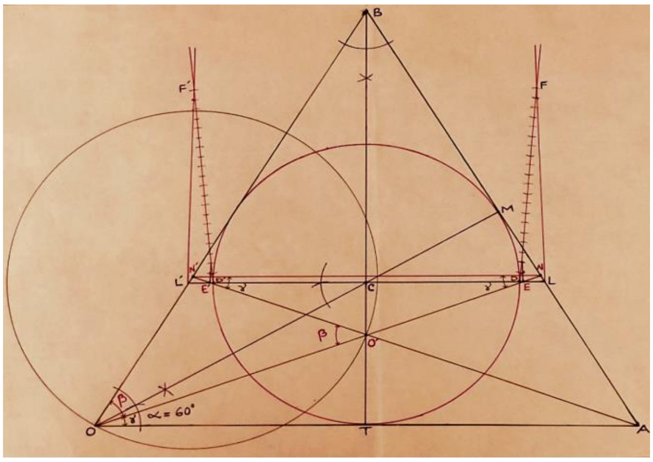

Figure 5: (Rigorous construction of the trisection of 60° angle)

Drawing or Steps:

1. Form an equilateral triangle OAB
2. Draw bisectors BT and OM of the angles OBA and AOB, which meet at point C, and this is the in-center of the triangle OAB
3. 3. Draw a line $LL' \parallel OA$ through point C, which intersects AB and OB at points L and L'
4. Draw the in-circle of the triangle OAB, whose center at C and the radius is CT, which intersects LL' at points D and D'
5. Take first one-tenth division at $E$of DL on the right side of point D using the usual division with the help of line DF. Similarly, take first one-tenth division at E' of $D'L'$ on the left side of point D' with the help of the line $D'F'$
6. Join O and E and extend OE up to N, which meets AB at point N. Similarly, join A and $E'$and extend AE' up to N', which meets OB at point N'. Diagonals ON and AN' meet at point O'
7. Finally, draw a circle whose radius is $N'O$ and center is at N' that passes through point O'. So, $N'OO'$ is an isosceles triangle whose $N'O = N'O'$.

Hence, the trisection of 60° angle has been completed at $< AON = 20°$.

Proof: Following the Archimedes process here, $< AOB = \alpha = 60°$. Let $< AON = < O'NN' = \gamma$ and $< NON' = < N'O'O = \beta$. For an equilateral triangle OAB, $BT \perp OA$ and $LL' \parallel OA \parallel NN'$. So, $O'N = O'N'$. Therefore, $N'O'N$ is an isosceles triangle. $\therefore < O'N'N = \gamma$. We have $\alpha = \beta + \gamma$. In the triangle $O'NN'$, the external angle $\beta = 2\gamma$ is made by the extended side NO'. Therefore, $\alpha = 3\gamma$ or $\gamma = \frac{1}{3}\alpha \therefore < AON = 20°$.

Verification of the trisection of 60° with analytical geometry in the figure below:

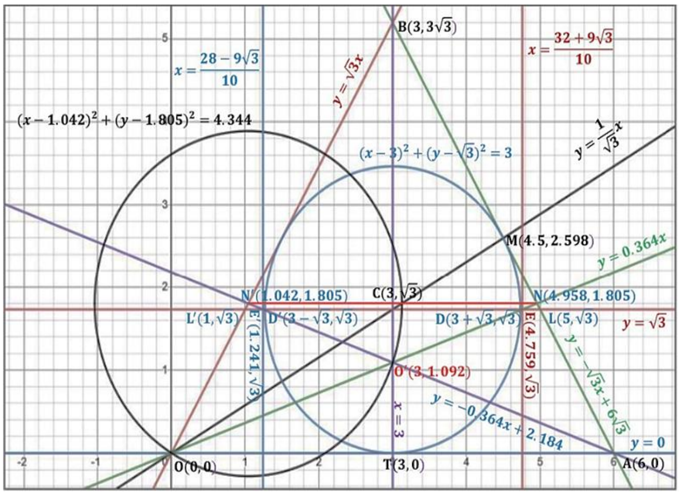

Figure 6: (Verification of the rigorous trisected value of 60°, Desmos Graphing Calculator, 2011)

From the similar description and analysis mentioned in favor of Figure 4, we can say that the previous error, $N'O \neq (N'O' = O'N)$, has been resolved by $N'O = N'O' = O'N = 2.084$ through the construction in figure 5, and the measurements give us the accurate trisected value $< AON = \theta = tan^{-1}\left(\frac{y}{x}\right)$.

Or $\theta = tan^{-1}\left(\frac{1.091893438240872}{3}\right) = 20°$, which is correct.

Trisection of sixty degrees angle from figure 5, in brief, is given below:

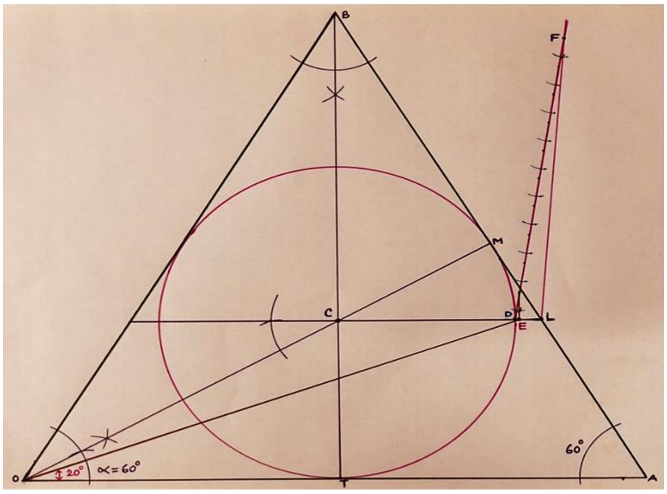

Figure 7: (Construction and trisection of 60° in brief)

Procedure at a glance for figure 7 using only a straightedge and a compass:

1. Form an equilateral triangle OAB
2. Draw bisectors BT and OM of the angles OBA and AOB, which meet at point C
3. Draw a line parallel to OA through point C, which intersects AB at point L
4. Draw the in-circle of the triangle OAB, whose center at C and the radius is CT, which intersects CL at point D
5. Take first one-tenth division at E of DL on the right side of point D using the usual division with the help of line DF
6. Join O and E

$\therefore < AOB = 60°$ angle has been constructed and trisected at $< AOE = 20°$ using classical tools.

2.2.2 *Construction and Proof of The Trisection of Angle Forty-Five Degrees*

Consider the figure below for trisecting the angle of 45°:

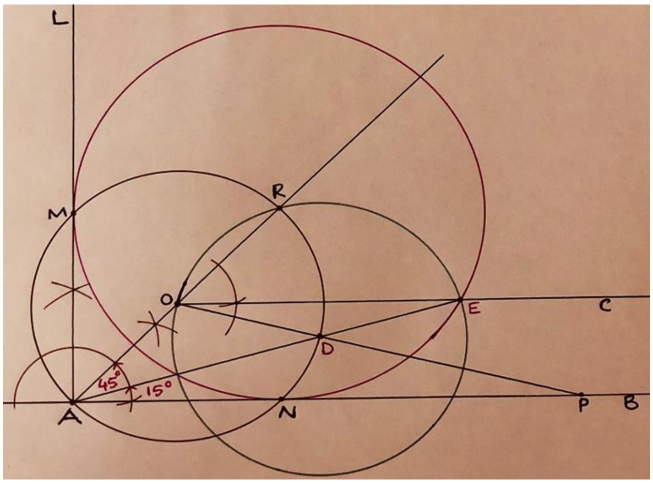

Figure 8: (Construction and trisection of angle 45°)

Steps for the trisection of 45°:

1. Draw $< LAB = 90°$ by the general procedure
2. Bisect $< LAB = 90°$ by the line AR and produce $< BAR = 45°$
3. Take an arbitrary point O on the line AR
4. Draw a circle centering O and radius OA, which intersects AL and AB at M and N, respectively.
5. Draw a line OC through point O and parallel to the line AB
6. Draw another circle whose center is at R and the radius is $AM = AN$ which intersects the line OC at point E
7. Finally, join A and E, which intersects the previous first circle at point D

As per the Archimedean process, since the circle with a radius of OD and center at D passes through point E, i.e., $OD = DE$. So, the trisection of 45° angle has completed at $< OPA = < PAE = 15°$.

Verification of the trisection of 45° with analytical geometry in the figure below:

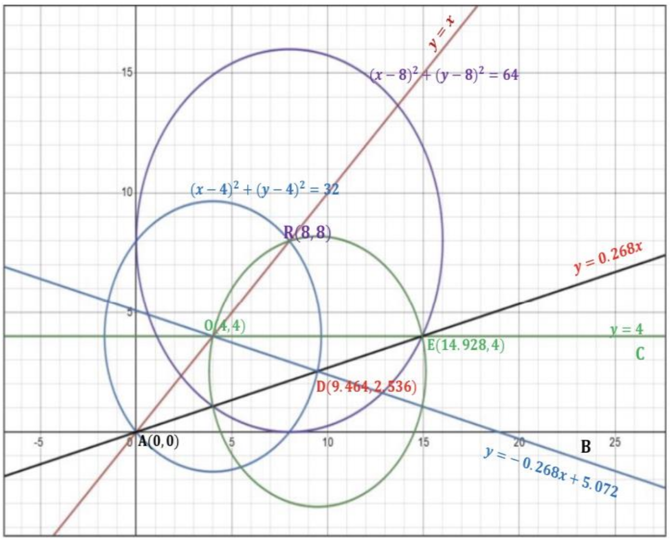

Figure 9: (Verification of the trisection of 45°)

From the analysis of figure 8, we can say that AE is our required trisector, and it produces the value of trisection from the relation $\theta = tan^{-1}\left(\frac{y}{x}\right)$.

$\therefore$ trisected angle $\theta = tan^{-1}\left(\frac{4}{14.92820323027551}\right) = 15°$, which is accurate.

2.2.3 *Location of the points D and E and Proof of the trisection for 90 degrees angle*

We construct a right angle $< BAR = 90°$ with a general drawing in the figure below:

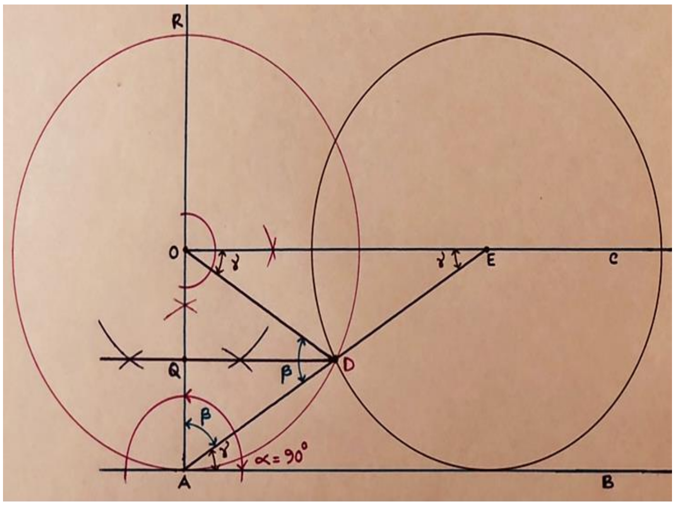

Figure 10: (Construction and Proof of the trisection of 90° angle)

Steps for the trisection of 90°

1. Draw $< BAR = 90°$ by the general procedure
2. Take an arbitrary point O on the perpendicular line AR
3. Draw a circle whose center is at O and the radius is OA
4. Draw a line OC through point O and parallel to the line AB
5. Draw the perpendicular bisector QD to OA, which intersects the circle at point D
6. Finally, join A and D and extend AD up to E, which intersects OC at point E

As per the Archimedean process, since the circle with a radius of OD and center at E passes through point D, i.e., $OD = DE$. So, the trisection of 90° angle has been completed at $< BAE = 30°$.

Verification of the trisection of 90° with analytical geometry in the figure below:

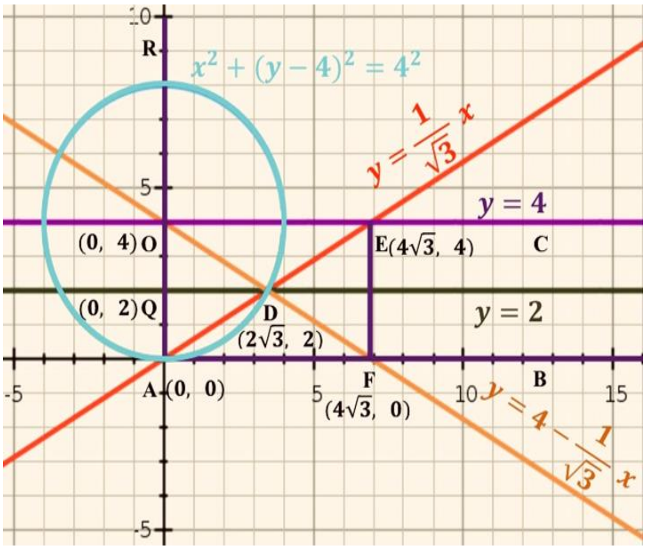

Figure 11: (Verification of the trisection of 90°)

From the analysis of figure 10, we can say that AD is our required trisector, and it produces the value of trisection from the relation $\theta = tan^{-1}\left(\frac{y}{x}\right)$. $\therefore$ trisected angle $\theta = tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) = 30°$, which is accurate.

3. THE TRADITIONAL PROCESS FOR TRISECTION OF SOME PARTICULAR ANGLES

Here we are describing the well-known procedure to trisect the straight angle and the right angle in the figures below:

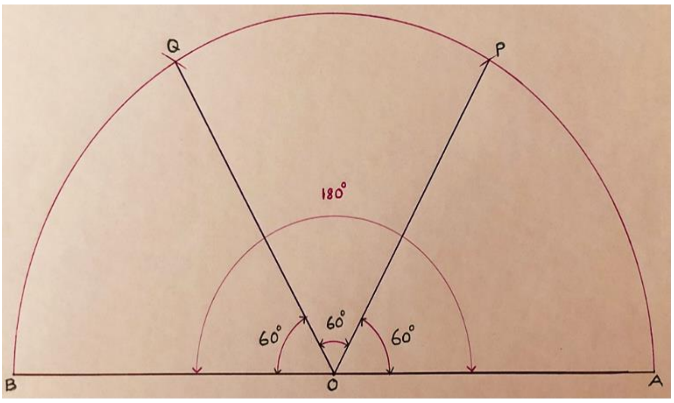

Figure 12: (Trisection of 180°)

For figure 12, $< AOB = 180°$ has been made by drawing a semi-circle with the center O and radius OA. Cutting two equal arcs $AP = OA$, $BQ = OA = OB$ trisects the straight angle, i.e., $< AOP =< POQ =< QOB = 60°$.

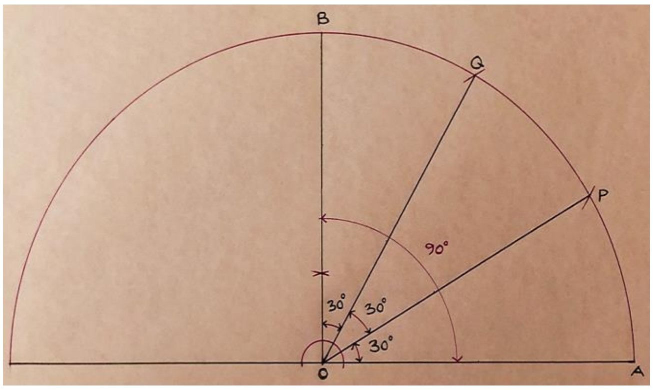

Figure 13: (Trisection of 90°)

For figure 13, $< AOB = 90°$ has been made by drawing a quarter-circle with the center O and radius OA. Cutting two consecutive equal arcs $AQ = OA$, $BP = OA$ trisects the right angle, i.e., $< AOP =< POQ =< QOB = 30°$.

Followed by figures 12, 13, and bisection of the right angle, the trisection of 45° angle is shown in the figure below:

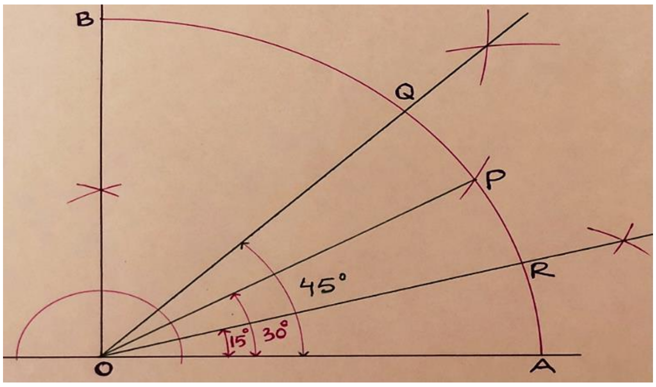

Figure14 : (Trisection of 45°)

Steps for the trisection of 45°

1. Draw $< AOB = 90°$ by the general procedure
2. Take $< BOP = 60°$ by the arc $BP = OA \therefore < AOP = 30°$
3. Bisect $< AOB$ by the line $OQ \therefore < AOQ = 45°$ and $< POQ = 15°$
4. Bisect $< AOP$ by the line $OR \therefore < AOR = 15°$

Therefore, $< AOQ = 45°$ has been trisected $< AOR =< ROP =< POQ = 15°$

Important Note: The above section 3 bears the importance that each equal arc is on the same circular path creating the same angle at the center of that circle.

4. MAIN RESULTS

The reconstruction of the trisection of the angles 180° and 90° is given followed by trisecting the angles' arc below figures in 15 and 16:

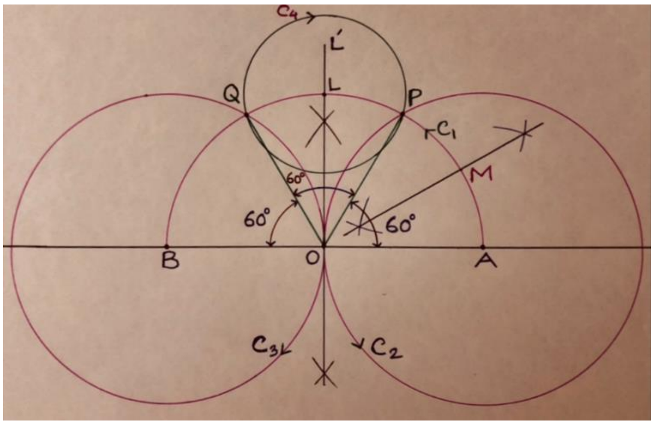

Figure 15: (Reconstruction of the trisection of the angle 180°)

In figure 15, $< AOB = 180°$ has been made by drawing a semi-circle C_1 with the center O and radius OA. Bisect the angle $< AOB$ with the line OL' which meets C_1 at point L. Now draw two circles, C_2 and C_3 centering, at endpoints A and B of the angle $< AOB$ and the same radius OA, which meet C_1 at points P and Q, respectively. Bisect the arc AP at M. Finally, draw circle C_4 centering L and radius AM that passes through points P and Q. Therefore, the angle of 180° has been trisected at $< AOP =< POQ =< QOB = 60°$ followed by the trisection of angle arc AB at $AP = PQ = QB$.

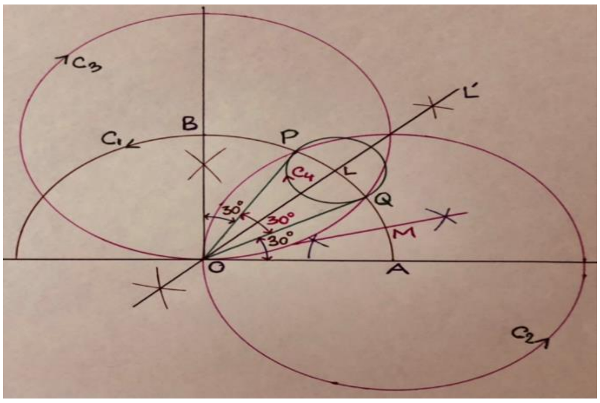

Figure 16: (Reconstruction of the trisection of the angle 90°)

In figure 16, $< AOB = 90°$ has been made by drawing a semi-circle C_1 with the center O and radius OA and bisecting the angle $< AOB$ with the line OB which meets C_1 at point B. Bisect the angle $< AOB$ with the line OL' which meets C_1 at point L. Now draw two circles, C_2 and C_3 centering, at endpoints A and B of the angle $< AOB$ and the same radius OA, which meet C_1 at points P and Q, respectively. Bisect the arc AQ at M. Finally, draw circle C_4 centering L and radius AM that passes through points P and Q. Therefore, the angle of 90° has been trisected at $< AOQ =< QOP =< POB = 30°$ followed by the trisection of angle arc AB at $AQ = QP = PB$.

In the above two figures, radius OA is related to particular angles 60° and 30°, which should not use for trisecting an unknown angle. In that case, we may select the radius in such an amount that there (in-between circles C_2 and C_3 along the arc C_1) exists a minimum equal arc length same as the taken radius for secure trisection of the arc.

Archimedes' angle trisection process is unique, straightforward, and successful, which proves angle trisection is indeed possible. Otherwise, we would not have such beautiful and accurate calculations by drawing, measuring, and reasoning, for which the mathematical world owes its thanks to Archimedes. Since Archimedes took the help of marking two points on the straightedge with a slight deviation from the prohibition of Greek philosophy, we have tried in subsection 2.2 above to fulfill the conditions by taking into account the compass prohibition of the famous philosopher Plato. Also, we have tried our level best by twisting Archimedes' angle trisection method using only a compass and straightedge for certain angles to solve the angle trisection problem successfully and convey these methods to the guardians of today's mathematical world so that it can be recognized. However, although these divisions give correct results for certain angles, the general trisection method for arbitrary angles cannot be achieved in subsection 2.2 and section 3. So, we are trying to discuss a unique process for figuring out a general procedure for the trisection of an arbitrary angle in the below figure

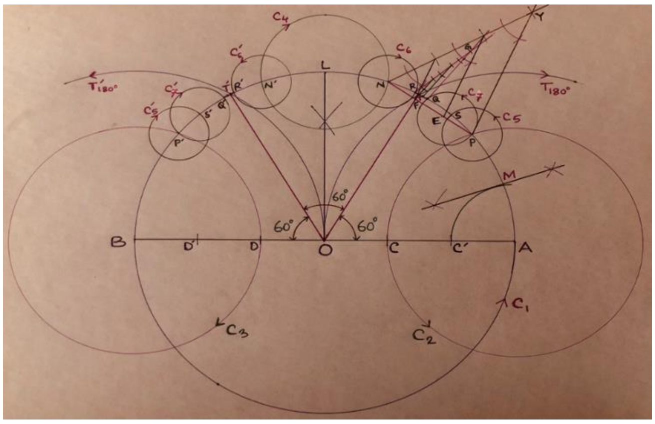

Figure 17: (An alternative process for the trisection of an angle of 180°)

Steps for trisecting 180° angle:

1. Take six equal line segments OC, CC', $C'A$, OD, DD', and DB from point O along line AB using a compass and straightedge
2. Draw a circle C_1 with the center O and radius OA. So, angle $< AOB = 180°$ has formed, and line segment AB has trisected at C and D.
3. Draw a perpendicular OL to AB at point O. $\therefore OL$ is the bisector of trisection-able straight angle
4. Draw two circles, C_2 and C_3, centering at endpoints A and B of the straight angle with the radius r, where $r < \frac{1}{2}AC \leq AC$, i.e., close to AC (here, we have taken $r = AC$). These circles intersect circle C_1 at P and P'
5. Bisect arc AP at M and draw circle C_4 with center L and radius AM, which intersects circle C_1 at N and N'
6. Join P, N and trisect line segment PN with the help of line NY at points E and F
7. Draw two circles, C_5 and $C_5{}'$, centering P and P' and with a radius close to PE and $\leq PE$ (here, we have taken radius $< PE$) that intersects circle C_1 at S and S'
8. Draw other circles, C_6 and $C_6{}'$, centering N and N' and with the same radius PE and $\leq PE$ that intersects circle C_1 at R and R'
9. Draw the final two circles, C_7 and $C_7{}'$ centering S and S' and the same radius close to PE and $< PE$, of which circle C_7 intersect circle C_1 at point Q, and circle $C_7{}'$ intersect circle C_1 at point Q'
10. Again, join Q, R and, Q', R' and take two-thirds of QR and $Q'R'$ with the help of line RG at points T and T'
11. Join O, T, and O, T'

Therefore, our required trisection of 180° angle is $< AOT =< TOT' =< T'OB = 60°$.

Based on figure 17, the reasoning, explanation, and proof of the method described above are as follows:

We have a successful traditional trisection of angle 180° in figure 12, 13, which indicate that if the arc of the relevant angle can be trisected, the angle can also be trisected at the vertex of that angle. For an alternative method to trisect angle 180°, we are trying to trisect arc AB at a straight angle. First of all, we bisect arc AB at L followed by the general procedure of angle bisection. Since arc $AB >$ line segment AB, we trisect line segment AB at C and D with the help of six line segments and cut two equal arcs $AP = AC$ and $BP' = AC$. So, these arcs do not intersect or touch one another. Now if we draw a circle centering L and radius $\frac{1}{6}AB = AC'$, it could not make an equal arc of $AP = BP'$ along circle C_1, or a circle centering A and radius AC' does not pass through the midpoint M of the arc AP. That's why (for making the arc of equal length), bisect arc AP at point M and we draw a circle centering L and radius AM that intersect the arc AB at points N and N'. So, $AP = BP' = NN'$ and $PN = P'N'$. Now, if we can enhance equally the three arcs AP, BP', and NN' in such a way that the sum of the increased parts of the arc is equal to the sum of the arc PN and $P'N'$ on the circle C_1, then the work of reaching our destination, will be done properly. To complete the trisection of arc AB, we are to locate the two-third positions of the arc PN and $P'N'$ separately because each arc AP and BP' should increase by one unit in a single direction, and arc NN' should increase by a single unit in both directions. If arc PN coincides with line segment PN or arc PN and line segment PN are almost the same, we can trisect PN and finalize the trisector of the angle by selecting the two-thirds point of PN. But, in this figure, joining line PN is trisected at E, F, but we can't finalize the trisector because arc $PN >$ line segment PN. Here, identical circles C_5, C_6, C_7 and $C_5{}'$, $C_6{}'$, $C_7{}'$ have been drawn with the same radius close to PE and $< PE$. Circles C_7 and C_6 meet C_1 at Q and R and, similarly, $C_7{}'$ and $C_6{}'$ at Q' and R'. Since arc QR coincides with line segment QR, we trisect QR and $Q'R'$ and finalize the trisection of the angle by selecting the two-thirds points of QR and $Q'R'$ at T and T'. Here, traditional trisectors $T_{180°}$ and $T_{180°}{}'$ of 180° angle pass through points T and T', which prove that our arc AB has been trisected at $AT = TT' = BT'$. Therefore, the trisection of angle 180° has been completed followed by the new method of arc trisection at $< AOT =< TOT' =< T'OB = 60°$.

This successful effort was conducted from a certain angle. Now we will have to describe and verify the method on unknown angles.

The upcoming Scheme of angle trisection followed by trisecting the respective arc creating on the angle as the general procedure for arbitrary angles has been taken based on Figure 17 because it is a verified and successful method for angle trisection. Moreover, our Scheme has been divided here into two parts as follows:

(a) When the trisection-able angle lies between 0° and 180°

(b) When the trisection-able angle lies between 180° and 360°

After selecting the relevant Scheme, eventually, we will have to follow the procedure of Scheme (a).

4.1 Trisection of Arbitrary Acute Angle

4.1.1 *When the acute angle is comparatively small (less than 30 degrees)*

Suppose $< AOB = \theta$ is an unknown smaller acute angle that has been trisected in the figure below:

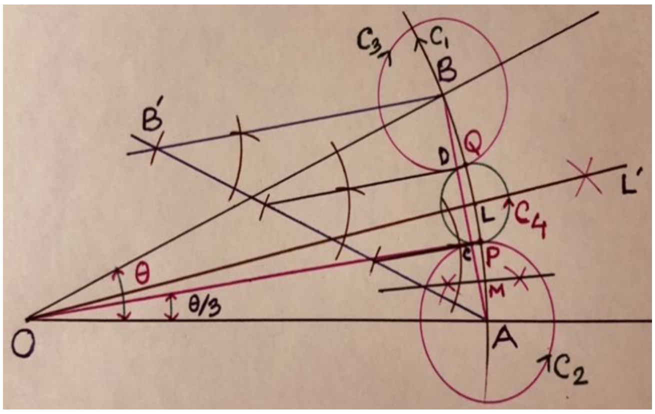

Figure 18: (a) (Trisection of an arbitrarily smaller acute angle)

Steps for trisecting:

1. Draw circle C_1 centering O and radius OA, which intersect the angle's lines at A and B
2. Join A, B and trisect AB with the help of line AB' at C and D
3. Draw circles, C_2 and C_3, with the same radius AC and centering A and B, respectively, which meet C_1 at two points P and Q, maintaining the same arc length $AP = BQ$
4. Bisect angle θ by the line OL' which intersects C_1 at L. Also, bisect arc AP at M
5. Draw circle C_4 with the radius AM and centering L, which passes through points P and Q

So, arc AB is trisected at P and Q, i.e., arc $AP = PQ = QB$. Therefore, $< AOB = \theta$ has trisected at $< AOP = \frac{\theta}{3}$.

4.1.2 When the acute angle is comparatively big (greater than 30 degrees)

Suppose $< AOB = \theta$ is an unknown acute angle that has been trisected in the figure below:

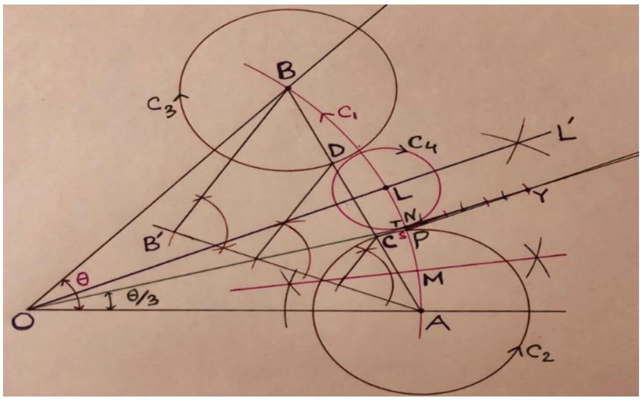

Figure 18: (b) (Trisection of an arbitrarily larger acute angle)

Steps for trisecting acute angle $< AOB = \theta$:

1. Draw circle C_1 centering O and radius OA, which intersects the angle's lines at A and B
2. Join A, B and trisect AB with the help of line AB' at C and D
3. Draw circles, C_2 and C_3, with the same radius AC and centering A and B, respectively, which meet C_1 at two points, maintaining the same arc length. C_2 meets C_1 at P
4. Bisect angle θ by the line OL' which intersects C_1 at L. Also, bisect arc AP at M
5. Draw circle C_4 with the radius AM and centering L, which meets C_1 at N towards point P
6. Since the joining line PN coincides with the arc PN, trisect PN with the help of line NY at points S and T

Therefore, arbitrary angle $< AOB = \theta$ has been trisected by joining line OT at $< AOT = \frac{\theta}{3}$.

4.2 Trisection of Arbitrary Obtuse Angle

Suppose $< AOB = \theta$ is an unknown obtuse angle that has been trisected in the figure below:

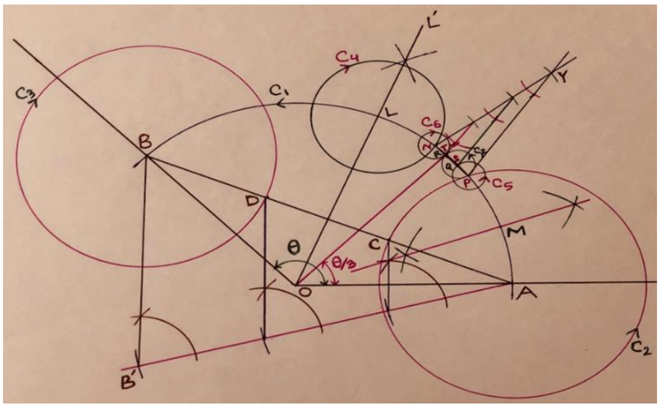

Figure 19: (Trisection of an arbitrary obtuse angle)

Steps for trisecting obtuse angle $< AOB = \theta$:

1. Draw circle C_1 centering O and radius OA, which intersects the angle's lines at A and B
2. Join A, B and trisect AB with the help of line AB' at points C and D
3. Draw circles, C_2 and C_3, with the same radius AC and centering A and B, respectively, which meet C_1 at two points, maintaining the same arc length. C_2 meets C_1 at P
4. Bisect angle θ by the line OL' which intersects C_1 at L. Also, bisect arc AP at M
5. Draw circle C_4 with the radius AM and centering L, which meets C_1 at N towards point P
6. Joining P, N trisect line segment PN with the help of line NY at points Q and R
7. Since arc PN and line segment PN are almost the same, draw two circles, C_5 and C_6, with the same radius PQ and centering P and N, respectively. C_5 and C_6 meet C_1 at S and T in-between PN, respectively
8. Finally, draw circle C_7 centering S and the radius PQ, which passes through the point T

Therefore, arbitrary angle $< AOB = \theta$ has been trisected by joining line OT at $< AOT = \frac{\theta}{3}$.

4.3 Trisection of Arbitrary Reflex Angle, θ

We know the straight angle can be trisected easily, and the trisection result is 60°. Now if we decrease our reflex angle θ by 180° and divide the rest of angle $(\theta - 180°)$ into three equal parts, we can get the final trisection as

$60° + \frac{1}{3}(\theta - 180°)$

4.3.1 *Trisection of Arbitrary Reflex Angle Smaller than 270 Degrees*

Suppose $< AOB = \theta < 270°$ is an unknown reflex angle. The trisection of θ will be $60° + \frac{1}{3}(\theta - 180°)$ that has been trisected in the figure below:

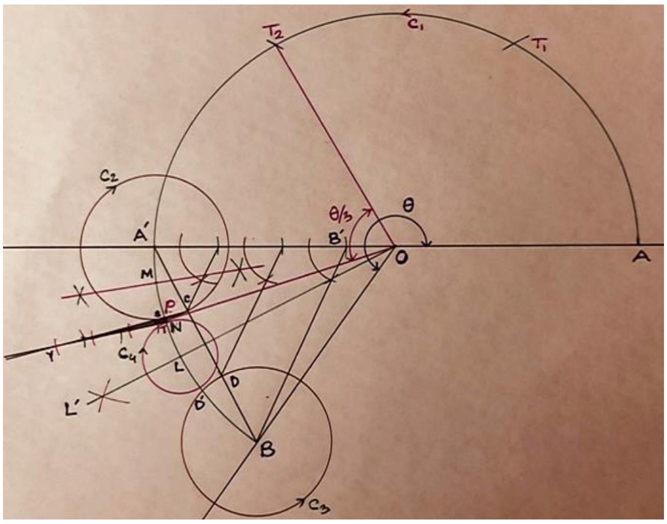

Figure 20: (Trisection of an arbitrary reflex angle smaller than 270°)

Steps for trisecting reflex angle, $< AOB = \theta < 270°$:

1. Draw circle C_1 centering O and radius OA, which intersects the angle's lines at A and B
2. Extend OA up to A' which intersects C_1 at A'
3. Join A', B and trisect $A'B$ with the help of line $A'B'$ at C and D
4. Draw circles, C_2 and C_3, with the same radius $A'C$ and centering A' and B, respectively, which meet C_1 at two points, maintaining the same arc length. C_2 meets C_1 at P
5. Bisect angle $(\theta - 180°)$ by the line OL' which intersects C_1 at L. Also, bisect arc $A'P$ at M
6. Draw circle C_4 with the radius $A'M$ and centering L, which meets C_1 at N towards point P
7. Since the joining line PN coincides with the arc PN, trisect PN with the help of line NY at points S and T

Therefore, arbitrary angle $< AOB = \theta$ has been trisected by joining line OT at $< TOT_2 = \frac{\theta}{3}$, where OT_2 is the second trisector of 180° angle.

4.3.2 *Trisection of Arbitrary Reflex Angle, $270° \leq \theta < 360°$*

Suppose $< AOB = 270° \leq \theta < 360°$ is an unknown reflex angle. The trisection of θ will be $60° + \frac{1}{3}(\theta - 180°)$ that has been trisected in the figure below:

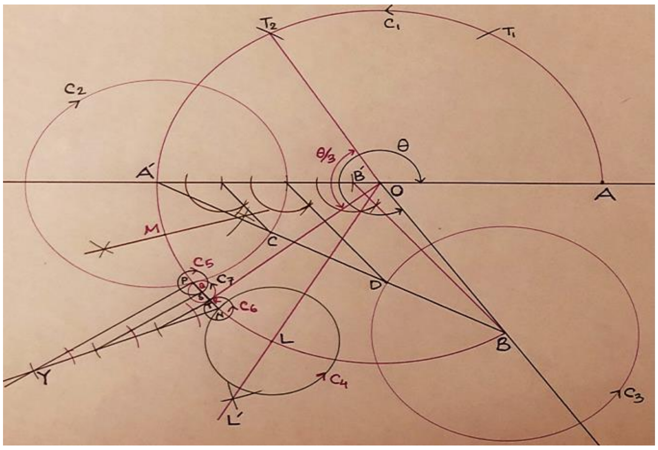

Figure 21: (Trisection of an arbitrary reflex angle $270° \leq \theta < 360°$)

Steps for trisecting reflex angle $< AOB = 270° \leq \theta < 360°$

1. Draw circle C_1 centering O and radius OA, which intersects the angle's lines at A and B
2. Extend OA up to A' which intersects C_1 at A'
3. Join A', B and trisect $A'B$ with the help of line $A'B'$ at C and D
4. Draw circles, C_2 and C_3, with the same radius $A'C$ and centering A' and B, respectively, which meet C_1 at two points, maintaining the same arc length. C_2 meets C_1 at P
5. Bisect angle $(\theta - 180°)$ by the line OL' which intersects C_1 at L. Also, bisect arc $A'P$ at M
6. Draw circle C_4 with the radius $A'M$ and centering L, which meets C_1 at N towards point P
7. Joining P, N trisect line segment PN with the help of line NY at points Q and R
8. Since arc PN and line segment PN are almost the same, draw two circles, C_5 and C_6, with the same radius PQ and centering P and N in-between PN, respectively. C_5 and C_6 meet C_1 at S and T, respectively
9. Finally, draw circle C_7 centering S and the radius PQ, which passes through the point T

Therefore, arbitrary angle $< AOB = \theta$ has been trisected by joining line OT at $< TOT_2 = \frac{\theta}{3}$, where OT_2 is the second trisector of 180° angle.

Desired result: By a new and unique method described in Section 4, we think we have successfully trisected any quadrilateral and arbitrary angle. We have shown the trisection of an arbitrary angle that evolves with different quadrants in this Section 4, where four features of the PN trisection are observed. These features are:

1) If the arc PN and line segment PN are different, the process is taken ahead with a close radius of three equal parts of the PN line segment.
2) If PN does not create, the angle is trisected only by drawing circles C_2, C_3, and C_4, and these circles are touched consecutively.
3) If the arc PN coincides with line segment PN, the respective angle is trisected by two-thirds of line segment PN.
4) If the arc PN seems to coincide with line segment PN, then by taking the trisecting result of line segment PN as radius, three similar circles were drawn and observed they touched themselves consecutively. Meanwhile, the desired result was found by two-thirds of line segment PN.

Now it is to be noted any of these features can happen for any quadrilateral angle, i.e., they are not dependent on the conclusions located in a specific quadrant. Again, if the radius is selected differently in different stages, new experiences are also explored, which will discuss later. However, it can readily say that it is possible to make a trisection of an arbitrary angle by the described method in section 4.

5. DISCUSSION

5.1 Coincidence of the Procedure

We have seen a video on the angle trisection procedure in the web portal, where an esteemed mathematician tried to trisect an arbitrary angle followed by trisecting an arc from where a question may arise that our method coincides with this process, watch please. The procedure has analyzed in the figure below:

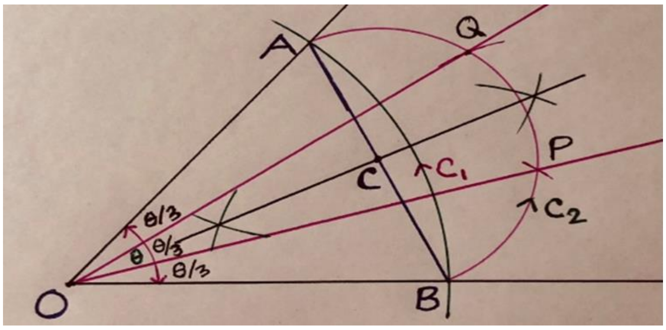

Figure 22: (Procedure of the trisection of angle in)

In the prescribed procedure in figure 22, an arc $AB = C_1$ has been taken from the arbitrary angle $< AOB = \theta$. A semi-circle C_2 has been formed at midpoint C of AB and then trisect C_2 at P and Q by the general procedure. Finally, the process declared OP and OQ as the trisectors of θ, i.e., $< BOP =< POQ =< QOA = \frac{\theta}{3}$. We think this is not a right process because the trisection of arc C_2 should follow to trisect a straight angle at point C. Moreover, arcs C_1 and C_2 have drawn from different points, O and C, respectively. Therefore, arc $AB = C_1$ has not been trisected here at all. So, our established process in section 4 completely contains a different and unique methodology.

5.2 Additional Techniques

The techniques described in section 4 are on unknown angles. We should verify the competency of the applied method for some particular data. That's why we are checking our achieved procedure for some known data.

5.2.1 Trisection of 255° and 75° angles have shown in the figure below

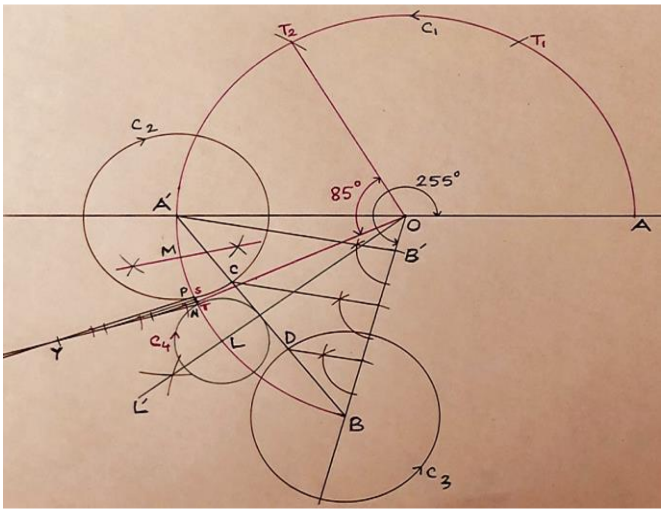

Figure 23: (Trisection of 255° and 75°)

From sub-sub section 4.1.2 and sub-sub section 4.3.1, $< A'OB = 75°$ and $< AOB = 255°$ have been trisected at $< A'OT = 25°$ and $< TOT_2 = 85°$, where OT_2 is the second trisector of 180° angle in figure 23 above, and our protractor gives us accurate results.

5.2.2 Trisection of 330° and 150° angles have shown in the figure below

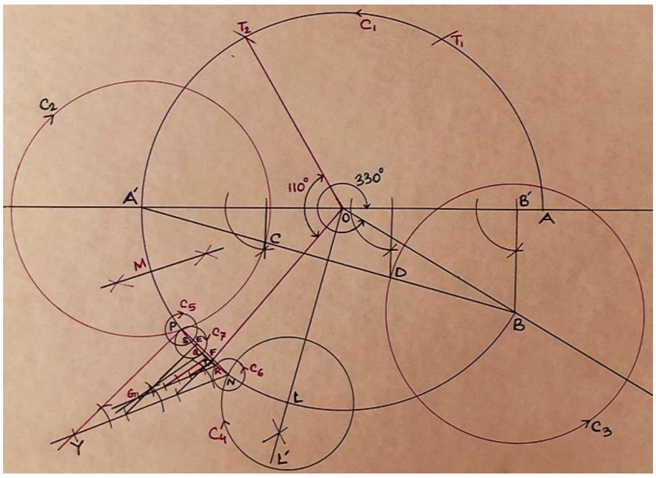

Figure 24: (Trisection of 330° and 150°)

Followed by Figure 17 and sub-subsection 4.3.2, we draw up to step 6, i.e., we complete the drawing up to circle C_4, then

7. Since arc arc $PN >$ line segment PN, we trisect line segment PN at E and F

8. Draw circles, C_5, C_6, centering P, N, and with the same radius close to PE and $< PE$. C_5, C_6 intersect C_1 at S, R in-between PN. Again, draw circle C_7 centering on S and with the same radius, which intersects C_1 at Q towards point R

9. Since arc QR coincides with line segment QR, trisect QR with the help of line RG and take two-thirds point at T

Therefore, $< A'OB = 150°$ and $< AOB = 330°$ have been trisected at $< A'OT = 50°$ and $< TOT_2 = 110°$, where OT_2 is the second trisector of 180° angle in figure 24 above, and our protractor gives us accurate results.

5.2.3 Trisection of 210° and 30° angles have shown in the figure below

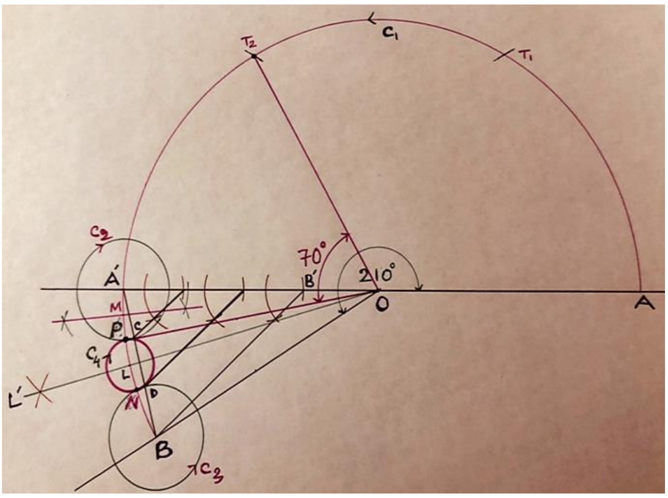

Figure 25: (Trisection of 210° and 30°)

Steps for trisecting:

1. Draw circle C_1 centering O and radius OA, which intersects the angle's lines at A and B
2. Extend OA up to A' which intersects C_1 at A'
3. Join A', B and trisect $A'B$ with the help of line $A'B'$ at C and D
4. Draw circles, C_2 and C_3, with the same radius $A'C$ and centering A' and B, respectively, which meet C_1 at two points, maintaining the same arc length. C_2 meets C_1 at P
5. Bisect angle $(\theta - 180°) = 210° - 180° = 30°$ by the line OL' which intersects C_1 at L. Also, bisect arc $A'P$ at M
6. Draw circle C_4 with the radius $A'M$ and centering L, which meets C_1 and C_2 at the common point P. Also, C_4 meets C_1 and C_3 at the common point N

Therefore, $< A'OB = 30°$ and $< AOB = 210°$ have been trisected at $< A'OP = 10°$ and $< POT_2 = 70°$, where OT_2 is the second trisector of 180° angle in figure 25 above, and our protractor gives us accurate results.

Here in this trisection process, it is seen that arc $A'B$ has been trisected only by drawing circles C_1, C_2, C_3, and C_4. It is mention-able that this is the shortest procedure of angle trisection for our proposed method. While we were doing paperwork for the trisection of 30° angle, our classical tools didn't produce any error. But, the microscopic situation tells us that there exists a very small blank or undivided arc or space in-between circles C_2, C_4, and C_3, C_4. However, by making variations for radius selection, we can avoid the obscured condition, which is shown in the following figures 26 and 27:

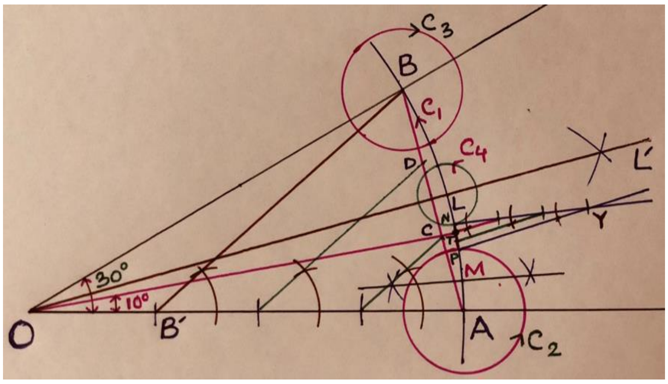

Figure 26: (Trisection of 30°)

Steps

1. Draw circle C_1 centering O and radius OA, which intersects the angle's lines at A and B
2. Join A, B and trisect AB with the help of line AB' at C and D
3. Draw circles, C_2 and C_3, with the same radius close to AC and $< AC$ and centering A and B, respectively, which meet C_1 at two points, maintaining the same arc length. C_2 meets C_1 at P
4. Bisect angle 30° by the line OL' which intersects C_1 at L. Also, bisect arc AP at M
5. Draw circle C_4 with the radius AM and centering L, which meets C_1 at point N towards point P
6. Since arc PN coincides with line segment PN, take the two-thirds point T on C_1 with the help of line NY

Therefore, $< AOB = 30°$ has been trisected at $< AOT = 10°$.

Proof of the trisection of angle 30 degrees through analytical geometry is shown in the figure below:

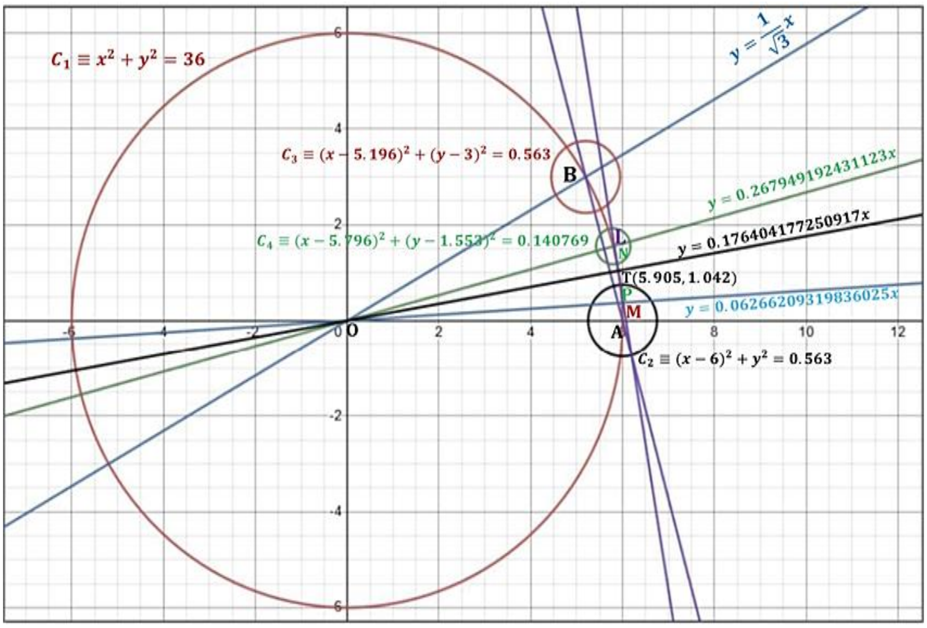

Figure 27: (Analytical-geometric proof of the trisection of the angle of 30°)

Suppose C_1 is a circle drawn on an angle of $< AOB = 30°$ and whose equation of the inclined line OB is $y = \frac{1}{\sqrt{3}}x$. We bisect this angle by the line OL whose equation is $= 0.267949192431123x$. Joining line AB is different from arc AB. According to our established method in Figure 26 under section 5, and using data from the coordinate system of the above graph, we draw circles C_2, C_3, and C_4. Equation of the bisector line of arc AP passes through the point $M(5.9765, 0.3745)$ is $y = 0.06266209319836025x$. C_2 meets C_1 at $P(5.953, 0.749)$, and C_4 meets C_1 at $N(5.881, 1.188)$ towards point P. Now, since again joining line segment PN coincides with arc PN on C_1, PN has been divided at $T(5.905, 1.042)$ as the ratio $2:1$. $\therefore$ OT is the required trisector of the angle $< AOB = 30°$, whose equation is $y = 0.176404177250917x$.

Here, from the slope of line OT, our trisected value of 30° is 10°, which is accurate and the final step for trisecting an angle of 30°.

5.2.4 Verification of the general trisection method with analytical geometry

In section 4 and sub-sub section 5.2.3, we discussed some techniques (e.g., trisecting comparatively long arc PN by circles C_5, C_6, and C_7, small arc PN like line segment PN by the general procedure of trisection of a line segment at S and T, and preliminary arc AB or $A'B$ by circles C_2, C_3, and C_4 only) for trisecting an arbitrary and particular angle and, all these were applied on various angles considering macroscopic measurement on paperwork. Now, we are trying to verify our established method for the trisection of an arbitrary angle with the help of analytical geometry in the figure below:

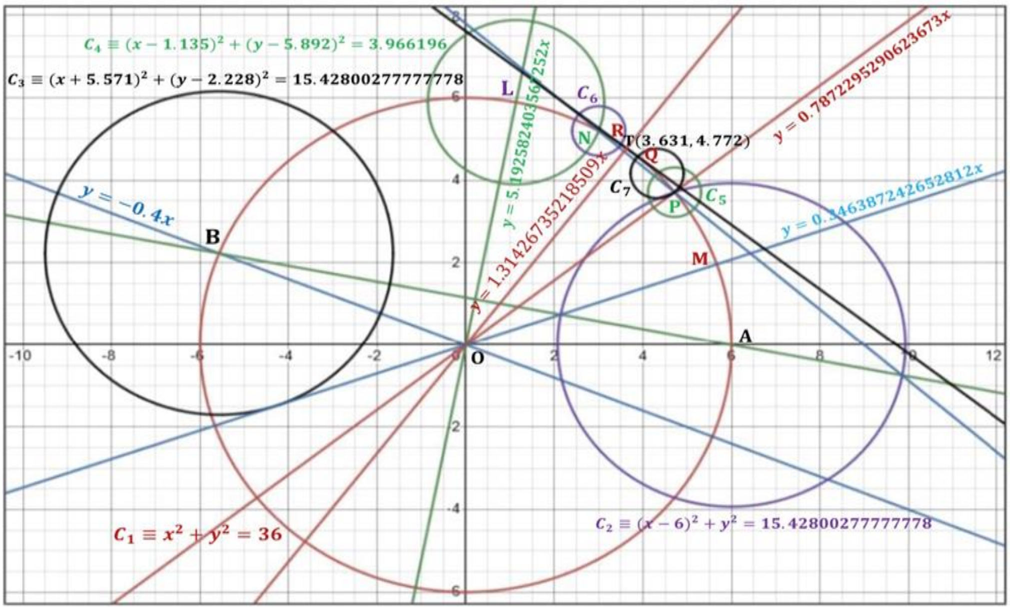

Figure 28: (Verification of the trisection of an arbitrary angle)

Suppose an arbitrary equation $y = -0.4x$ creates an angle $< AOB = \theta$ in the second quadrant. With the help of slope -0.4, we measure angle θ, bisect this angle θ, and form the equation of the bisector as $y = 5.192582403567252x$. Join A, B. According to our established method in section 4 and using data from the coordinate system of the above graph, we draw circles C_1, C_2, and C_3. Equation of the line passes through the intersecting point $P(4.714, 3.711)$ of circles C_1 and C_2 is $y = 0.7872295290623673x$, which creates an angle of 38.21086576725093° with the line OA whose equation of bisector OM is $y = 0.346387242652812x$. Circle C_2, C_4 meet C_1 at P and N. Since arc $PN >$ line segment PN, C_5, C_6, and C_7 have been drawn, where C_7 and C_6 meet C_1 at $Q(3.882, 4.575)$ and $R(3.505, 4.87)$ in-between PN. Now, since again joining line segment QR coincides with arc QR on C_1, QR has been divided at $T(3.631, 4.772)$ as the ratio 2: 1. $\therefore OT$ is the required trisector of the arbitrary angle $< AOB = \theta$, whose equation is $y = 1.31426735218509x$. Here, the arbitrary equation makes an angle of $\theta = 158.199°$ with the x-axis, and our trisected value of θ is 52.733°, which is accurate and the final step for trisecting an angle.

Although a microscopic difference is observed between the line segment QR and the arc length QR in a magnified position, our straightedge and compass do not catch it because the density property of the Real number creates a negligible algebraic variation that is seen in the correct trisection processes of 180 and 90 degrees angles, also. So, from the macroscopic point of view or visual paperwork, we follow the trisecting point T of QR as an accurate value of the angle trisection. Of course, it's a matter of approximation process, which is the beauty of mathematics. If the approximation is not accepted, the imperfection of mathematics will disclose, which is unexpected. For example, if the unit length of a line segment is divided into three equal parts and an arc is bisected without approximation, we can easily observe what happens in the figure below:

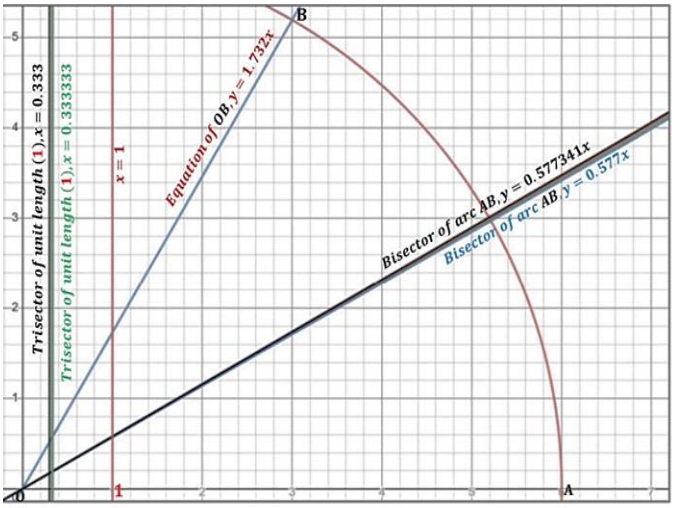

Figure 29: (Microscopic trisection of line segment and bisection of arc)

From the above figure (magnified position or from the microscopic point of view), it is seen that more lines trisect the line segment and bisect the angle. But the existence of the trisector of a line segment and bisector of an arc should be unique in geometry. The traditional results of trisection of a line segment and bisection of an arc are universally accepted for compass-straightedge constructions. So, the actual trisection of the line segment and bisection of the arc from a macroscopic point of view for the above example has given below figure:

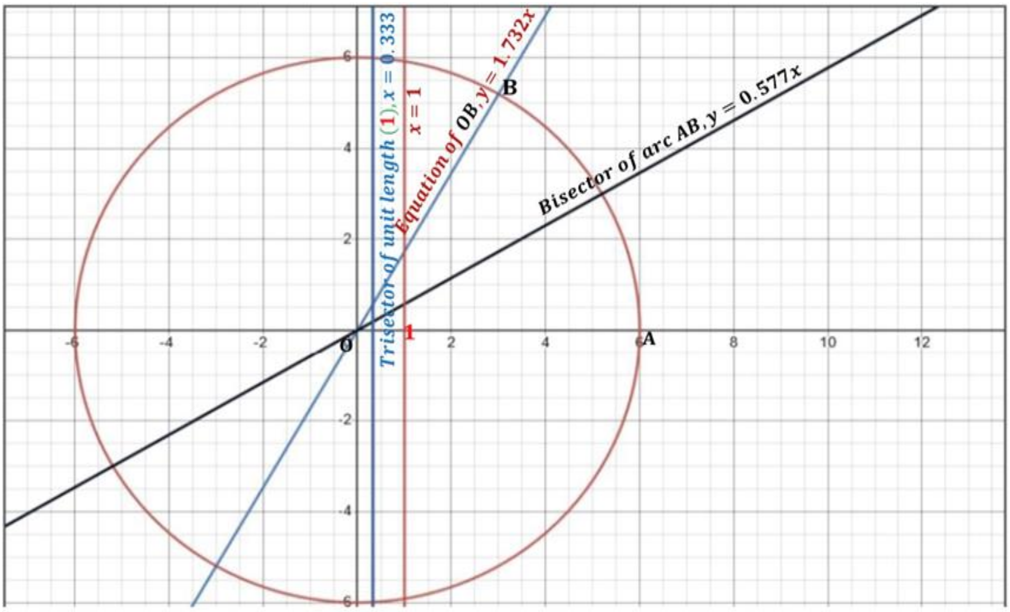

Figure 30: (Macroscopic trisection of line segment and bisection of arc)

Therefore, we think the methods described in section 4 are eligible for correct trisection of an arbitrary angle from a macroscopic point of view. Since by the compass-straightedge process, we have already recognized the trisection of a line segment and bisection of an arc subject to process approximation; we assert that it is logical to give recognition the trisection of an angle or arc by the same rule.

5.3 Additional Information

From section 5 and sub-sub sections 5.2.3, 5.2.4, we can conclude that one of the following can be the required trisector:

i) Drawing up to circle C_4 and taking the intersecting point of circles C_2 and C_4, which is the shortest path of angle trisection.

ii) Drawing up to the trisector of the line segment PN and taking the two-thirds position from PN.

iii) Drawing up to circles C_5, C_6, and C_7, and taking the intersecting point T of C_6 and C_7 as the two-thirds point of PN.

iv) If necessary drawing continue up to the trisector of the line segment QR in-between circles C_7 and C_6 and taking the two-thirds position T from that segment, which is the longest path of angle trisection. In such a case, a small arc on circle C_1 will become a line segment.

5.4 Ideal angle for the trisection of an arbitrary angle

From the experience of trisection drawing through sections 2 to 5 on various known and unknown angles, it has been observed that 60 degrees angle played a vital role in the trisection of an arbitrary angle because 60 degrees measurement has used for the trisection of 180, and 90 degrees angles, which was our starting point for finding a general trisection procedure of an arbitrary angle. An angle of 60° can be considered the turning point of our proposed trisection method. Easily 60° has been trisected in Figure 5, whereas, including Wantzel's verdict, many articles/books said that 60° can't be trisected (Robert and Yates., 2015; Stankova-Frenkel, 2000; AoPS Online,2023; Suzuki, 2008; Smorynski, 2008). Although the successful construction of a sixty-degree angle has been shown several times and divided into three equal parts, the temptation has not been satisfied. That's why the construction and trisection of a 60° angle has been given below as final method in figure 31 again through our general procedure in section 4, see please, for the similarity (Maddox, 2011; Lydon and Barton, 2022.

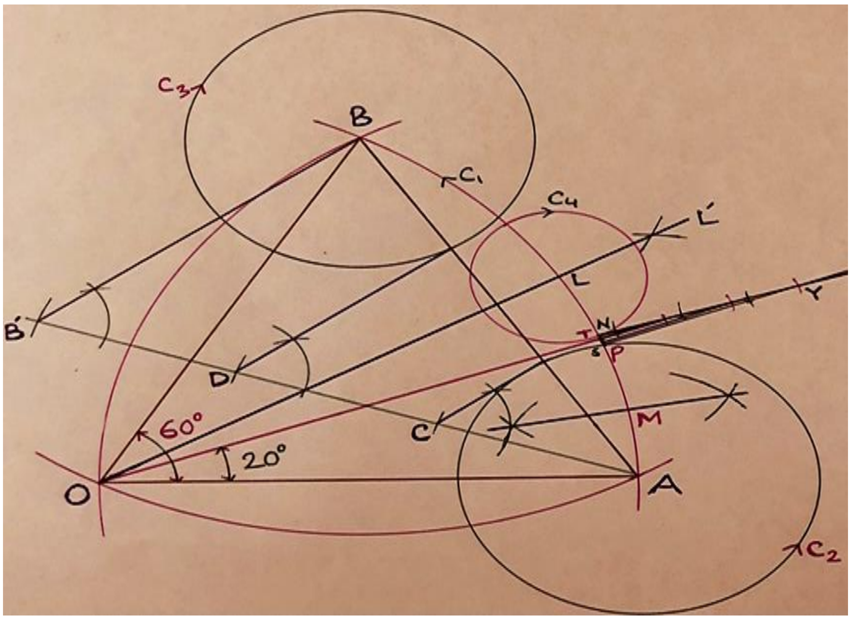

Figure 31: (Trisection of a sixty-degree angle through the general method in section 4)

At first, sixty degrees angle has been formed by three red colored circles with C_1, then from sub-sub section 4.1.2, $< AOB = 60°$ has been trisected at $< AOT = 20°$ in figure 31 above, and our protractor gives us accurate result. Also, we can see the verification of the result in the following Figure:

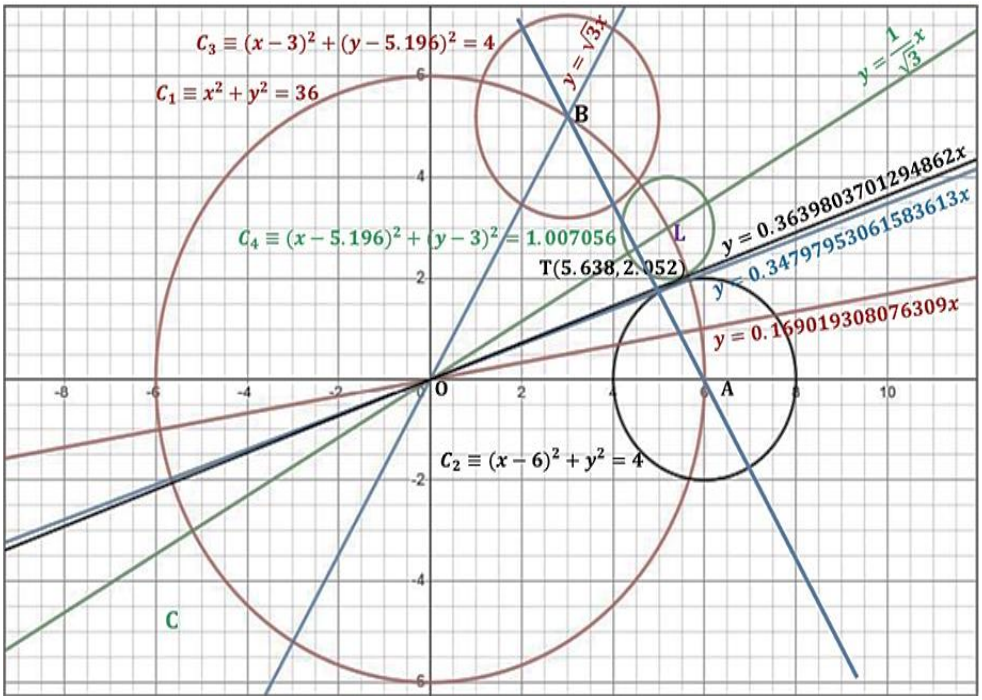

Figure 32: (Verification of the trisection of 60°)

Draw line of equation $y = \sqrt{3}x$, which is inclined line of angle $< AOB = 60°$. Also draw the bisector $y = \frac{1}{\sqrt{3}}x$ of 60° and join A, B. According to our established method in section 5 and using data from the coordinate system of the above graph, we draw circles C_2 and C_3. Equation of the line passes through the intersecting point $(5.667,\ 1.972)$ of circles C_1 and C_2 is $y = 0.34797953061584613x$, which creates an angle of 19.18685047797648° with the horizontal line OA whose bisector is $y = 0.169019308076309x$. Circle C_4 has been drawn, where C_2 and C_4 meet C_1 at $(5.667,\ 1.972)$ and $(5.623,\ 2.092)$ respectively. Now we identify the point $T(5.638,\ 2.052)$, which divides the joining line segment of (5.667, 1.972) and (5.623, 2.092) as the ratio 2: 1.

Therefore, the equation of trisector OT is $y = 0.3639803701294862x$ that is the final step for trisecting an angle of 60° and the correct trisected value is $< AOT = 20°$. Hence the result is verified as true.

5.5 Special Precautions for the Trisection of An Angle

a) Bisection of an angle or arc and trisection of a line segment should complete carefully.

b) For trisecting reflex angle θ, it should be converted into $0° < \theta < 180°$ decreasing by 180°.

c) The radius of circles C_2, C_3, C_5, C_6 and C_7 should not exceed the respective one-third of the line segment.

d) Every undivided arc length in-between C_2 and C_4 (also if there exists in-between C_7 and C_6) on the circle C_1 should be analyzed for correct trisection, however small.

e) We will remember that as if our final trisection-able angle θ lies on $0° < \theta < 180°$.

5.6 Possibility of iteration in the proposed method

In section 4, our described method itself is one kind of iterative method where the conditional radius can be reduced from smaller to smaller by turns. Still, the following Figure is shown, in a different way to repeat the central activity of section 4 as an iteration method for trisecting an arbitrary angle:

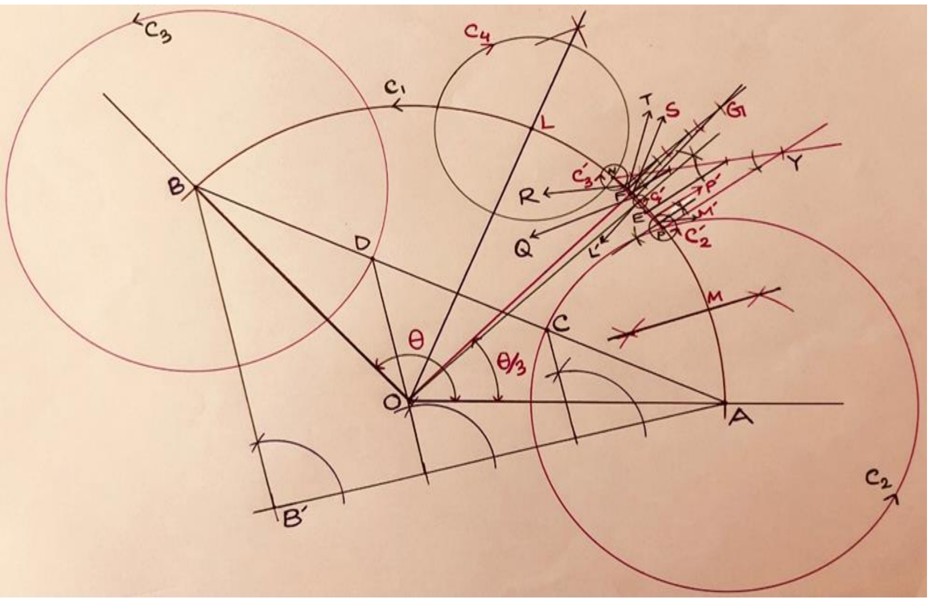

Figure 33: (General trisection method as an iteration)

Steps for trisecting an arbitrary angle $< AOB = \theta$ followed by section 4 as an iteration procedure:

1. Draw a circle C_1 with the center O and radius OA, which intersects the angle's lines at A and B
2. Trisect line segment AB at C and D with the help of line AB'
3. Bisect $< AOB = \theta$ by the line OL that intersects C_1 at L
4. Draw two circles, C_2 and C_3, with the radius AC and centering A and B, respectively, which meet C_1 at two points, maintaining the same arc length. C_2 meets C_1 at P
5. Bisect arc AP at M and draw circle C_4 with center L and radius AM, which intersects circle C_1 at N towards point P
6. Join P, N, and trisect line segment PN with the help of line NY at points E and F
7. Again, draw two circles, $C_2{}'$ and $C_3{}'$, with a radius close to PE and $< PE$ and centering P and N, respectively, maintaining the same arc length in-between PN. $C_2{}'$ meets C_1 at P' towards point N, and $C_3{}'$ meets C_1 at R towards point P
8. Bisect arcs PN and PP' at L' and M', respectively, and draw circle $C_4{}'$ with center L' and radius PM'. $C_4{}'$ meets C_1 at Q towards point R
9. Again, join Q, R and since arc QR coincides with line segment QR, trisect QR with the help of line RG at points S and T
10. Take the one-third point S of QR on C_1, and finally, join O, S

Therefore, our required trisection of angle $< AOB = \theta$ is $< AOS = \frac{\theta}{3}$.

Figure 33 is a little different from the repetition method, and an accurate result has been found from that method. So, on the one hand, the repeat method is a successful application for trisecting angles. Again, the procedure shows that if only proceeding according to Section 4, the trisection of PN is an adjustment point with the result of this method. Therefore, both techniques claim the successful process of angle trisection as an iteration method, and indeed, our general trisection method is unique and appropriate.

5.7 Trisecting procedure of an arbitrary angle at a glance once again

1. For trisection-able angle θ, if $0° < \theta \leq 180°$, then we will have to find $\frac{1}{3}\theta$. But, if $180° < \theta \leq 360°$, we will have to find $60° + \frac{1}{3}(\theta - 180°)$. So, Our trisection-able angle will always be ready for $0° < \theta \leq 180°$.
2. Draw a circle C_1 centering the vertex O of arbitrary angle θ with a radius as large as possible concerning the working paper size, which intersects the angles' lines at A and B.
3. Join A, B, and trisect AB at C and D.
4. Bisect θ, which intersects C_1 at L.
5. Draw two circles, C_2 and C_3, centering A and B with a radius close to AC and $\leq AC$. Circle C_2 intersects circle C_1 at P.
6. Bisect arc AP at M and draw circle C_4 with the center L and the radius AM, which intersects circle C_1 at N and N'. If C_4 touches C_2 and C_3 at N and N', and if N coincides with P, then θ will be trisected at $< AON$ (or $60° + < AON$), and the process is finished. Otherwise, follow the next steps.
7. Join P, N and trisect line segment PN at E and F.
8. If line segment PN coincides with arc PN, the two-thirds point T of arc PN makes the trisection of the arbitrary angle, i.e., θ will be trisected at $< AOT$ (or $60° + < AOT$), and the process is finished. Otherwise, follow the next steps.
9. If line segment PN and arc PN are almost the same, draw two circles, C_5 and C_6, with the radius PE and centering P and N in-between PN, respectively. C_5 and C_6 meet C_1 at S and T, respectively. Finally, draw circle C_7 centering S and with the same radius PE. If C_7 passes through point T, θ will be trisected at $< AOT$ (or $60° + < AOT$), and the process is finished. Otherwise, follow the next steps.
10. If arc PN > line segment PN, draw two circles C_5, C_6 centering P, N and with a radius close to PE and $\leq PE$. Circles C_5 and C_6 meet C_1 at S and R in-between PN. Again, draw another circle C_7 centering S and with the same radius PE and $\leq PE$ that intersects circle C_1 at Q towards point R. If Q coincides with R, the required trisection will be $< AOQ$ (or $60° + < AOQ$), and the process is finished. Otherwise, follow the next steps.
11. Join Q, R and trisect line segment QR at S and T. If line segment QR coincides with arc QR, the two-thirds point T of arc QR makes the trisection of the arbitrary angle, i.e., θ is trisected, at $< AOT$ (or $60° + < AOT$), and the process is finished. Otherwise, follow the next step.
12. If necessary, the processes described above from step 9 can be continued as iterations until and unless we find the finishing step.

It is mentionable that the preliminary radius OA can be chosen as much as small for making small arc AB, which coincides with line segment AB. Also, by this procedure, a smaller angle can be trisected. Although it can be suitable for a smaller angle trisection, a larger one can't be, i.e., it can't apply as a general angle trisection method. Therefore, such a procedure is not suggested here for the trisection of an arbitrary angle.

6. DATA ANALYSIS OF THE CONSTRUCTION STUDY

In some of the processes in this paper, the trisection of many known-unknown angles has been shown adequately. For example, 60, 45, 90, 180, 90, 45, 180, unknown angle θ, (255, 75), (330, 150), (210, 30), 30, 60-degree angles, and unknown angle θ have been trisected in Figures 5, 8, 10, 12, 13, 14, 17, (18(a) - 21), 23, 24, 25, 26, 31, and 33, which proves that the angle trisection is possible and there is no error in construction. Since these processes cannot be applied to arbitrary angles as a specific rule, we established a general angle trisection method in Section 4, which we hope to be acceptable and considerable to the general readers as the correct General angle trisection procedure for arbitrary angles. However, with the overall analysis of our paper, we have been provided a comparative table below with the results of the articles in our reference list in this paper, which seem to be able to disseminate the message of the possibility of the angle trisection to the readers:

Table 1

Reference's Article No.	Percentage (%) of Error in the prescribed constructions for trisecting various known angles					Error for Unknown Angle θ
	30°	45°	60°	90°	150°	
Corte, 2013	0.07	0.18	0.06	0.17	0.73	-
Maddox, J., 2011	-	-	0.218	0.97	-	-
Our Construction In this article	000 (Indirect General Procedure) Fig 25	000 (Fig 8) Discrete Procedure	0.513 (Fig3) Discrete Procedure	000 (Fig10) Discrete Procedure	000 (Indirect General Procedure) Fig 24	000% $0° < \theta < 180°$ General Procedure
	000 (Direct General Procedure) Fig 26	000 (Fig14) Traditional Procedure	000 (Fig 5) Discrete Procedure 000 (Fig 31) General Procedure	000 (Fig13) Traditional Procedure	-	000% $180° < \theta < 360°$ General Procedure

The author of says that may Wantzel is correct, but his (author's) trisection result is approximately close and higher in accuracy, i.e., the author is in doubt regarding the possibility of angle trisection (Corte, 2013).

The prominent mathematicians say it looks like a very close result and a better version of the trisection result (Maddox, 2011). So, perhaps they are enthusiasts of the possibility of angle trisection.

Analyzing the data from Table 1, we can claim that our construction accuracy indicates angle trisection is possible and follows the general rule.

7. RESPONSE TO WANTZEL'S DECLARATION OF IMPOSSIBILITY

The logic behind the impossibility of the trisection of an arbitrary angle declared by Pierre Laurent Wantzel in 1837 based on the 60° angle, and was as below:

If 60° could be trisected, the degree of a minimal polynomial of $cos\ 20°$ over Q would be a power of two. From the trigonometric identity, $4cos^3\theta = 3cos\theta + cos3\theta$ for, $\theta = 20°$, $4cos^3 20° = 3cos20° + \frac{1}{2}$ or $8cos^3 20° - 6cos20° - 1 = 0$ or $(2cos20°)^3 - 3(2cos20°) - 1 = 0$. After substituting $x = 2cos20°$, his formed cubic equation became $x^3 - 3x - 1 = 0$, where he expected that the root of this equation will be $x = \pm 1$ by using the theorem of the rational roots. But none of these is a root. Then he concluded that the equation $x^3 - 3x - 1 = 0$ is irreducible over by Q, and the minimal polynomial for $cos20°$ is of degree 3. So, $cos20°$ is not constructible and hence, an angle of measure 60° cannot be trisected. Therefore, it is impossible to trisect an arbitrary angle.

In the article, the writer says (Suzuki, 2008).

Equivalently, let r be the root of an irreducible polynomial $f(x)$. If the degree of f is not equal to 2^n, then r is not constructible. This proves the impossibility of duplicating the cube or trisecting an arbitrary angle. In the first case, $\sqrt[3]{2}$ is the root of $x^3 - 2 = 0$, which is irreducible but not of degree 2^n. So "Wantzel's Theorem alone is insufficient to prove the impossibility of squaring the circle, though it does lay the groundwork for a proof. If $\sqrt{\pi}$ is a constructible number, it must be the root of an irreducible equation of degree 2^n."

In another book, the writer quoted from S. G. Shanker' (Smorynski, C.,2008)

"*And herein lies the key to Wantzel's failure to make any impact on the history of mathematics. Had Wantzel's proof been instrumental in bringing this new algebraic framework to the attention of this peers, his proof is undoubtedly had received considerable attention. But, of course, this was far from being the case. Indeed, Wantzel himself seems to have regarded his proof as little more than an offshoot of the real issue at stake: the proof that the general equation of degree $n > 4$ cannot be solved algebraically. Certainly, there is no evidence to suggest that he had been led into the field of modern algebra by his desire to solve the trisection problem. Wantzel's proof may have been the final step in the search to provide an adequate explanation for the impossibility of the trisection problem – thereby closing one of the most prolonged episodes in the history of mathematics – but it had only come about as a result of the developments in the theory of equations.*"

The Ancient Greek impossibility problem of angle trisection was a classical problem involving a straightedge-compass construction in the field of plane geometry, i.e., in the two-dimensional Euclidean space. But, we are still admitting the verdict on angle trisection from a different unrelated point of view. Impossibility imposed from the analysis of cubic equation $x^3 - 3x - 1 = 0$ followed by $x = 2cos20°$, which is an analysis of trigonometry, abstract algebra of field extensions (pure algebra, theory of equations), and its branch analytical geometry/Cartesian geometry (where the polynomial curve meets the X-axis).

As general readers of mathematics, we think it will not be fair to expect $x = \pm 1$ as the roots of the equation $x^3 - 3x - 1 = 0$ because $cos20° = \frac{x}{2}$, $0 < cos20° < 1$, and $cos30° = \frac{1}{2}$. Let us see the graph of $f(x) = x^3 - 3x - 1$ and solutions of the equation $x^3 - 3x - 1 = 0$ as follows:

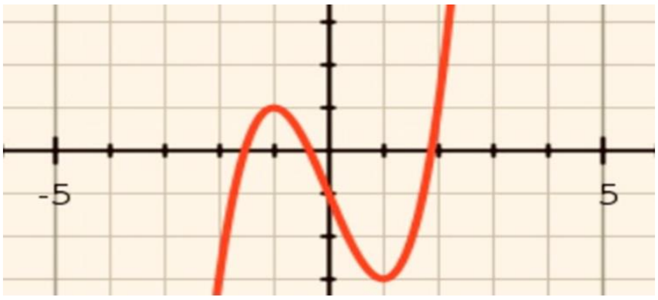

Figure 34: (Graph of $f(x) = x^3 - 3x - 1$, performed at internet,(Suzuki, 2008)

From the above graph, using the bisection method, we get the solutions (up to five decimal places) to the equation are $x = -1.53028$, -0.34729, and 1.87938. All of these satisfy the equation $x^3 - 3x - 1 = 0$ and only $x = 1.87938$ satisfies $x = 2cos20°$. On the one hand, $x = -1.53028$, -0.34729 do not satisfy the supposition of $x = 2cos20°$. So these are the byproduct of whom and why these have been created here? On the other hand, the presentation of expressing a constant number $2cos20°$ by variable x is not conventional. So perhaps, we should not treat $(2cos20°)^3 - 3(2cos20°) - 1 = 0$ as an equation of $2cos20°$. The particular value of θ can use for verifying the formula $4cos^3\theta = 3cos\theta + cos3\theta$. One thing is obscured here, Has the equation $4cos^3\theta = 3cos\theta + cos3\theta$ three roots for $\theta = 20°$? Since any formula is an identity, it is valid for any real value of θ. Again, if we take $\theta = 90°,\ 180°$, then our equations will be $(2cos30°)^3 - 3(2cos30°) - 0 = 0$ and $(2cos60°)^3 - 3(2cos60°) + 2 = 0$, which need not convert into the equation of x, and of course, 90° and 180° can be trisected. Cannot we say that 90° and 180° are arbitrary? So, using $4cos^3\theta = 3cos\theta + cos3\theta$ as the scale to prove angle trisection is impossible seems unrelated because it is not a suitable pure algebraic equation in this case. Rather than as it is an identity, it is satisfied by any real value of θ. Hope, the proof of the impossibility requires a non-ambiguous explanation for universal acceptance.

Construction of 20° angle or transcendence of the number $cos20°$ concerning algebraic numbers seems cannot affect the geometrical trisection of an arbitrary angle despite the existing verdict on squaring the circle by Ferdinand von Lindemann because the mathematical world is trying to prove π has algebraic character. Algebraic properties of irrational numbers and transcendental numbers are a matter of approximation. Otherwise, mathematical imperfections will arise that will hamper the beauty of mathematics (Roy, 2021). So we think the verdict of the impossibility of the angle trisection should reconsider properly.

Using only a compass and straightedge, we have drawn more than 200 figures (most of them were approximately accurate trisected values) to determine the general method of the angle trisection. In many of these drawings, all not attached in this article, it seems the compass-straightedge process is giving the correct result. But, when algebra-coordinate geometry is applied to verify the correctness, the algebra sometimes hampers the results of compass-straightedge. We think that the algebraic approximation in the various stages of the solution process makes such a difference. So, if we ignore the minor unrelated algebraic difference with the result of the compass-straightedge, we can say that the angle trisection can be possible by compass-straightedge. Therefore, the negligible algebraic difference is not vital as a barrier in this regard, for similar, please, see (Sivasubramanian and Kalimuthu, 2021).

The compass and straightedge don't give visible wrong results while drawing. The scope of the microscopic error does not determine by the compass-straightedge. The original problem was related to compass-straightedge, but we are taking the help of other instruments (after magnifying figures or receiving subtle algebraic accounts) to find out the errors, which seems reluctant. If we want to admit the microscopic difference, can we mention the amount of that error by compass-straightedge? If correctness is measurable, the error also should be measurable, using compass-straightedge only. If necessary and there is a visible error in the paperwork, it will not be unreasonable that we can remove the error as an iteration method, as described method in sections 4 and 5. We shouldn't leave anything declared impossible without presenting the appropriate logic. If there is a light to solve the way, the possibilities should expose what science is to do.

8. FUTURE RESEARCH AND DRAWBACKS OF THE STUDY

Endeavor for finding solutions to three impossibilities of antiquity in mathematics or the activities will continue until we find the acceptable solution or explanation. We have seen that Euler's identity-based verdict of Ferdinand von Lindemann could not stop the efforts of squaring the circle. Due to the successful trisection of 60-degree angles, we continued similarly on a wide range of Angles. Many mathematicians have taken the trisection result to near enough. We have shown successful work on many known angles and have invented a General method for trisecting known and unknown angles. Therefore, math readers who are new to this topic and more skilled readers hope they will follow the descriptions of the given methods mentioned in this paper, which will inspire them and concerned will able to find more beautiful solutions to the trisection of the arbitrary angle by further research.

We do not want to claim our displayed work is the highest transparent because in the last step in the process, the minor arc has to think equivalent to the straight line segment, and we had to reach the results of the sanctuary, which did not meet the desire of 100 percent. This situation can be considered a limitation in our study.

9. CONCLUSION

Mathematics learners have established many more techniques over the centuries for solving the angle trisection problem. We also tried to represent some techniques for trisecting some particular angles by extending the way of Archimedes in this article. Also, we have established a general method in sections 4 and 5 to trisect arbitrary angles. We started our effort by trisecting the angle of 60°, and this angle has been trisected 3 times in different positions, of which two are correct trisection processes and one is the general method. It is a matter of hope that we have achieved an accurate one-third angle of 20° for the trisection of 60°. Also, it was verified as correct with the protractor and coordinate geometry. We want to claim the successful trisection of 60° angle using only compass-straightedge as an easier way, which can appeal to the world's mathematical authorities for re-thinking the geometric validity of Wantzel's verdict. Together with arbitrary angles through sections 4 and 5, by turns, we have trisected particular angles 30°, 45°, 60°, 75°, 90°, 150°, 180°, 210°, 255°, and 330° in this paper using classical tools straightedge-compass only. Of course, a mathematics learner must know the cause of the impossibility of any mathematical phenomenon but should not retreat from searching the light of possibility to disclose new techniques. We think, no work is insignificant, no endeavor is a trifle, and a waste of time. A day may come when math enthusiasts will grasp the possibilities of successful solutions to the three famous impossibilities left by the Greek mathematicians of antiquity.

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