ZIBELINE INTERNATIONAL™ PUBLISHING

ISSN: 2521-0831 (Print)
ISSN: 2521-084X(Online)
CODEN: MSMAD

Matrix Science Mathematic (MSMK)

DOI: http://doi.org/10.26480/msmk.02.2023.95.102

CrossMark

RESEARCH ARTICLE

A COMPREHENSIVE APPROACH TO EVALUATING SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS USING LAPLACE TRANSFORM

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ARTICLE DETAILS

Article History:

Received 11 June 2023
Revised 15 August 2023
Accepted 29 September 2023
Available online 02 October 2023

ABSTRACT

This paper employed the use of Laplace transform to obtain solutions of differential equations of the first and second order with given boundary conditions, also the study investigated the pattern of variation exhibited graphically by the various solutions of differential equations at x = 0 to 12. The results in this study revealed that the solutions obtained usually contain exponential functions, while the differential equations with sine and cosine functions has solution that contains exponential and sine function. Furthermore, the differential equations with cosine function has solution that contain sine function only while differential equations that contain a perfect square multiple of f(x) in which the differential equation is equated to zero may have a solution that contain a sine function only. The graphically representation shows different pattern of variation depending on the solutions of the evaluated differential equations.

KEYWORDS

Boundary condition, differential equations, Laplace transforms, partial fraction, solutions

1. INTRODUCTION

The relationship between an independent variable, x, a dependent variable, y, and one or more derivatives of y with respect to x is referred to as differential equation (Stroud, 2001). Differential equations represent dynamic relationship, that is, quantities that vary, and are therefore found in scientific and engineering problems. The order of differential equation is given by the highest derivative found in the equation (Stroud, 2001). All Differential Equation (DE) have solutions containing a number of unknown integration constants. The values of these constants are usually found by applying boundary conditions to the solution, a procedure that can be often prove to be tedious. Fortunately, for a certain type of differential equation there is a method of obtaining the solution where these unknown integration constants are evaluated during the process of solution.

Furthermore, rather than employing integration as the way of unraveling the differential equation, straightforward algebra are used. The method hinges on what is known as the Laplace transform (Stroud, 2001). The Laplace transform which is a part of the new – growing topic known as operational calculus is easily and effectively applicable to the boundary value problems of differential equations arising in Physics, Mathematics and Engineering. The subject was mainly originated in the work of Heaviside who found it useful to solve equation of electromagnetic theory in the end of the nineteenth century (Gupta, 2010).

Several studies have been carried out to investigate the solutions of Laplace transformations. Kexue and Jigen gave sufficient condition in guaranteeing the rationality of solving constant coefficient fractional differential equation using Laplace transform method (Kexue and Jigen, 2011). In another study, the exact solution of some linear fractional differential equations by Laplace transform was investigated by Saheed where Laplace transform was applied in solving linear fractional-order differential equation (Saheed, 2014). It was observed that the Laplace transform is powerful and efficient for obtaining analytic solution of linear fractional differential equations. Other studies include that of to mention but a few (Mohamed and Torky, 2013; Gupta et al., 2015; Dinesh, 2018; Usman et al., 2021). This paper aimed at unavailing a comprehension approach of evaluating the solutions of differential equations for different boundary conditions using the Laplace transform and to investigate graphically the pattern of variation for various values of x ranging from 0 – 12.

2. METHODOLOGY

If $F(t)$ be a function of t defined for all positive values of t, that is, $t \geq 0$, then the Laplace transform of $F(t)$ denoted by $\mathcal{L}\{f(x)\}$ is defined by the expression (Gupta, 2010).

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}F(t)dt \tag{1}$$

where s is a parameter (real or complex). If the integral $\int_0^\infty e^{-st}F(t)dt$ converges for some value of s, then the Laplace transform of $F(t)$ is said to exist, otherwise it does not exist (Gupta, 2010).

The Laplace of 1 can be obtained by employing equation (1), where $f(t) = 1$, we have that

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}F(t)dt$$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \times 1\, dt$$

$$\therefore \mathcal{L}\{1\} = \frac{1}{s} \tag{2}$$

It follows therefore that the $\mathcal{L}\{2\} = \frac{2}{s}$, $\mathcal{L}\{3\} = \frac{3}{s}$, $\mathcal{L}\{4\} = \frac{4}{s}$ and so on.

Quick Response Code	Access this article online	
	Website: www.matrixsmathematic.com	**DOI:** 10.26480/msmk.02.2023.95.102

Cite The Article: G. Bello, and D. O. Akpootu (2023). A Comprehensive Approach to Evaluating Solutions of Differential Equations with Boundary Conditions Using Laplace Transform. *Matrix Science Mathematic, 7(2): 95-102.*

The Laplace of t can be obtained by employing equation (1), where $f(t) = t$, we have that

$\mathcal{L}\{t\} = \int_0^\infty e^{-st} \times t\, dt$

By applying integration by part $\int u dv = uv - \int v du$ and use the priority that t should be assigned u while exponential function dv (Stroud, 2001), we have that

$$\mathcal{L}\{t\} = \frac{1}{s^2} \quad (3)$$

By similar argument, we have that

$\mathcal{L}\{t^2\} = \frac{2}{s^3}$, $\mathcal{L}\{t^3\} = \frac{3}{s^4}$, $\mathcal{L}\{t^4\} = \frac{4}{s^5}$ and so on.

The Laplace of e^{at} can be obtained by employing equation (1), where $f(t) = e^{at}$, we have that

$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \times e^{at}\, dt$

$$\therefore \mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad (4)$$

The Laplace of te^{-at} can be obtained by employing equation (1), where $f(t) = te^{-at}$, we have that

$\mathcal{L}\{te^{-at}\} = \int_0^\infty e^{-st} \times te^{-at}\, dt$

By applying integration by part $\int u dv = uv - \int v du$ and use the priority that t should be assigned u while exponential function dv (Stroud, 2001), we have that

$$\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2} \quad (5)$$

The Laplace of $\sin at$ can be obtained by employing equation (1), where $f(t) = \sin at$, we have that

$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \times \sin at\; dt$

By applying integration by part $\int u dv = uv - \int v du$ and use the priority that exponential function should be assigned u while trigonometry function dv (Stroud, 2001), we have that

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \quad (6)$$

The Laplace of $\cos at$ can be obtained by employing equation (1), where $f(t) = \cos at$, we have that

$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st} \times \cos at\; dt$

By applying integration by part $\int u dv = uv - \int v du$ and use the priority that exponential functions should be assigned u while trigonometry function dv (Stroud, 2001), we have that

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2} \quad (7)$$

If $F(s)$ is the Laplace transform of $f(x)$, then $f(x)$ is the inverse Laplace transform of $F(s)$. Therefore, we can simply write the expression (Gupta, 2010).

$$f(x) = \mathcal{L}^{-1}\{F(s)\} \quad (8)$$

There is no simple integral definition of the inverse transform; the idea now is to obtain it by working backwards. For example, if $f(x) = 7$ then the Laplace transform $\mathcal{L}\{f(x)\} = F(s) = \frac{7}{s}$, if $F(s) = \frac{7}{s}$ then the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\} = f(x) = 7$. It is this ability to find the Laplace transform of an expression and then reverse it that makes the Laplace transform so useful in the solution of differential equation (Gupta, 2010).

From the foregoing $\mathcal{L}^{-1}\{F(s)\} = f(x)$, this implies that

$$\mathcal{L}\{f(x)\} = F(s) \quad (9)$$

So that

$$\mathcal{L}\{f'(x)\} = sF(s) - f(0) \quad (10)$$

This can be proven further that

$$\mathcal{L}\{f''(x)\} = s^2F(s) - sf(0) - f'(0) \quad (11)$$

By similar argument, the following are expressed (Gupta, 2010).

$$\mathcal{L}\{f'''(x)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0) \quad (12a)$$

$$\mathcal{L}\{f'^{v}(x)\} = s^4F(s) - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0) \quad (12b)$$

The Laplace transform is used to solve differential equation of the form

$$af'(x) + bf(x) = g(x) \quad (13)$$

Given that $f(0) = k$ where a, b and k are known constants and $g(x)$ is a known expression in x.

Similarly, Laplace transform is used to differential equation of the form

$$a_n f^{(n)}(x) + a_{n-1} f^{(n-1)}(x) + \cdots + a_2 f''(x) + a_1 f'(x) + a_0 f(x) = g(x) \quad (14)$$

where a_n, $a_{n-1} \dots a_2$, a_1, a_0 are known constants $g(x)$ is a known expression in x and the values of $f(x)$ and its derivatives are known at $x = 0$. This type of equation is called a linear, constant – coefficient, inhomogeneous differential equation and the values of $f(x)$ and its derivatives at $x = 0$ are called boundary conditions (Stroud, 2001).

Differential equations in the form of (13) and (14) with their respective boundary conditions are solved using Laplace transform with the following laid down procedures.

(i) Take the Laplace transform of both sides of the differential equation.

(ii) Substitute the given boundary conditions into the Laplace transform.

(iii) Find the expression $F(s) = \mathcal{L}\{f(x)\}$ in the form of an algebraic fraction

(iv) Separate $F(s)$ into its partial fractions, if the expression for $F(s)$ cannot directly give the solution of the differential equation.

(v) Take the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$ to obtain the solution $f(x)$ of the differential equation.

In evaluating partial fractions, it is important to note that, to effect the partial fraction breakdown of a rational algebraic expression it is necessary for the degree of the numerator to be less than the degree of the denominator (Stroud, 2001). Numerical problems and solutions to different partial fractions are found in Stroud.

3. Results and Discussion

Obtain the solution of the DE using Laplace transform

$f'(x) - f(x) = 3$ where $f(0) = 0$

We start by taking the Laplace transform on both sides of the equation.

$\mathcal{L}\{f'(x)\} - \mathcal{L}\{f(x)\} = \mathcal{L}\{3\}$

$sF(s) - f(0) - F(s) = \frac{3}{s}$

Now substitute the boundary condition that $f(0) = 0$

$sF(s) - 0 - F(s) = \frac{3}{s}$

Now find the expression for $F(s)$

$F(s)\{s-1\} = \frac{3}{s}$

$F(s) = \frac{3}{s\{s-1\}}$

Now separate $F(s)$ into its partial fraction, we have

$F(s) = \frac{3}{s\{s-1\}} = \frac{P}{s} + \frac{Q}{s-1}$

$P(s-1) + Q(s) = 3$

To obtain P substitute $s = 0$

$P(0-1) + Q(0) = 3$

$\therefore P = -3$

To obtain Q substitute $s = 1$

$P(1-1) + Q(1) = 3$

$\therefore Q = 3$

$F(s) = \frac{3}{s\{s-1\}} = \frac{-3}{s} + \frac{3}{s-1}$

Now take the inverse transform which gives the solution of the DE

$f(x) = -3 + 3e^x = 3e^x - 3$

$$\therefore f(x) = 3(e^x - 1) \quad (15)$$

Figure 1 shows the graph of the solution of the differential equation with its respective boundary condition $f'(x) - f(x) = 3$ where $f(0) = 0$

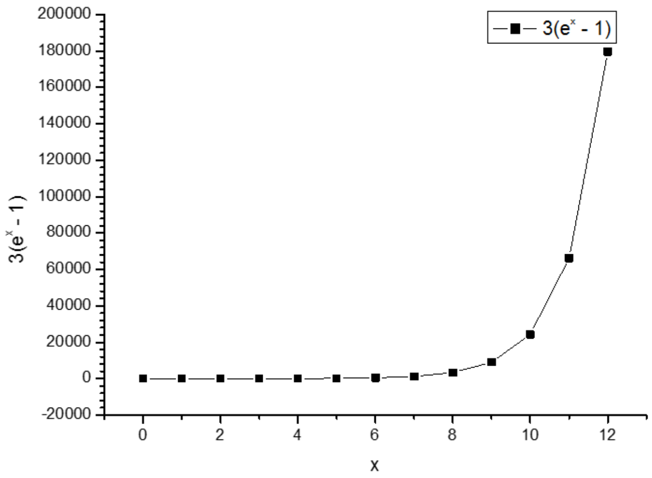

Figure 1: Solution of $f'(x) - f(x) = 3$ at x = 0 to 12

Figure 1 shows the graphical representation of the differential equation $f'(x) - f(x) = 3$ with boundary condition of $f(0) = 0$ at x = 0 to 12. The values of the solution obtained increases from x = 12

Obtain the solution of the DE using Laplace transform

$f'(x) + f(x) = e^{-x}$ where $f(0) = 0$

We start by taking the Laplace transform on both sides of the equation.

$\mathcal{L}\{f'(x)\} + \mathcal{L}\{f(x)\} = \mathcal{L}\{e^{-x}\}$

$sF(s) - f(0) + F(s) = \frac{1}{s+1}$

Now substitute the boundary condition that $f(0) = 0$

$sF(s) - 0 + F(s) = \frac{1}{s+1}$

Now find the expression for $F(s)$

$F(s)\{s + 1\} = \frac{1}{s+1}$

$F(s) = \frac{1}{(s+1)^2}$

Now take the inverse transform which gives the solution of the DE

$f(x) = xe^{-x}$ (16)

Figure 2 shows the graph of the solution of the differential equation with its respective boundary condition $f'(x) + f(x) = e^{-x}$ where $f(0) = 0$

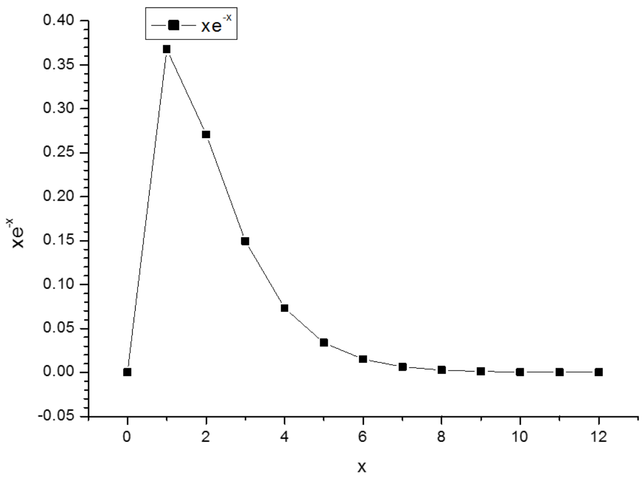

Figure 2: Solution of $f'(x) + f(x) = e^{-x}$ at x = 0 to 12

Figure 2 shows the graphical representation of the differential equation $f'(x) + f(x) = e^{-x}$ with boundary condition of $f(0) = 0$ at x = 0 to 12. The values of the solution obtained shows that when x = 0, the solution $xe^{-x} = 0$, which then increases from x = 0 to 1, and decrease further from x = 1 to 12. When x = 12, the solution gives a negative value.

Obtain the solution of the DE using Laplace transform

$f'(x) - f(x) = e^{-2x}$ where $f(0) = 1$

We start by taking the Laplace transform on both sides of the equation.

$\mathcal{L}\{f'(x)\} - \mathcal{L}\{f(x)\} = \mathcal{L}\{e^{-2x}\}$

$sF(s) - f(0) - F(s) = \frac{1}{s+2}$

Now substitute the boundary condition that $f(0) = 1$

$sF(s) - 1 - F(s) = \frac{1}{s+2}$

Now find the expression for $F(s)$

$F(s)\{s - 1\} = \frac{1}{s+2} + 1$

$F(s) = \frac{1}{(s+2)(s-1)} + \frac{1}{(s-1)}$

$F(s) = \frac{1+s+2}{(s+2)(s-1)} = \frac{s+3}{(s+2)(s-1)}$

Now separate $F(s)$ into its partial fraction, we have

$F(s) = \frac{s+3}{(s+2)(s-1)} = \frac{P}{(s+2)} + \frac{Q}{(s-1)}$

$P(s - 1) + Q(s + 2) = s + 3$

To obtain P substitute $s = -2$

$P(-2 - 1) + Q(-2 + 2) = -2 + 3$

$-3P = 1$

$\therefore P = -\frac{1}{3}$

To obtain Q substitute $s = 1$

$P(1 - 1) + Q(1 + 2) = 1 + 3$

$3Q = 4$

$\therefore Q = \frac{4}{3}$

$F(s) = \frac{s+3}{(s+2)(s-1)} = -\frac{1}{3}\frac{1}{(s+2)} + \frac{4}{3}\frac{1}{(s-1)}$

Now take the inverse transform which gives the solution of the DE

$f(x) = -\frac{1}{3}e^{-2x} + \frac{4}{3}e^{x}$

$\therefore f(x) = \frac{4}{3}e^{x} - \frac{1}{3}e^{-2x}$ (17)

Figure 3 shows the graph of the solution of the differential equation with its respective boundary condition $f'(x) - f(x) = e^{-2x}$ where $f(0) = 1$

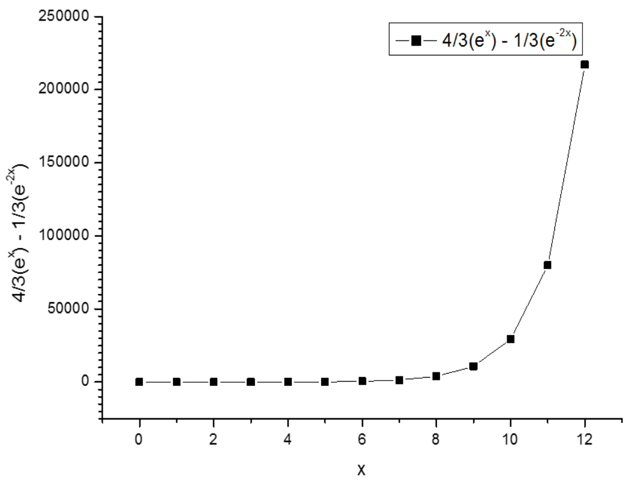

Figure 3: Solution of $f'(x) - f(x) = e^{-2x}$ at x = 0 to 12

Figure 3 shows the graphical representation of the differential equation $f'(x) - f(x) = e^{-2x}$ with boundary condition of $f(0) = 1$ at x = 0 to 12. The values of the solution obtained shows that when x = 0, the solution $\frac{4}{3}e^{x} - \frac{1}{3}e^{-2x} = 0$, which then increases from x = 1 to 12.

Obtain the solution of the DE using Laplace transform

$f'(x) - f(x) = e^{2x}$ where $f(0) = 1$

We start by taking the Laplace transform on both sides of the equation.

$\mathcal{L}\{f'(x)\} - \mathcal{L}\{f(x)\} = \mathcal{L}\{e^{2x}\}$

$sF(s) - f(0) - F(s) = \frac{1}{s-2}$

Now substitute the boundary condition that $f(0) = 1$

$sF(s) - 1 - F(s) = \frac{1}{s-2}$

Now find the expression for $F(s)$

$F(s)\{s-1\} = \frac{1}{s-2} + 1$

$F(s) = \frac{1}{(s-2)(s-1)} + \frac{1}{(s-1)}$

$F(s) = \frac{s-2+1}{(s-1)(s-2)} = \frac{s-1}{(s+2)(s-1)}$

$F(s) = \frac{1}{(s-2)}$

Now take the inverse transform which gives the solution of the DE

$$f(x) = e^{2x} \tag{18}$$

Figure 4 shows the graph of the solution of the differential equation with its respective boundary condition $f'(x) - f(x) = e^{2x}$ where $f(0) = 1$

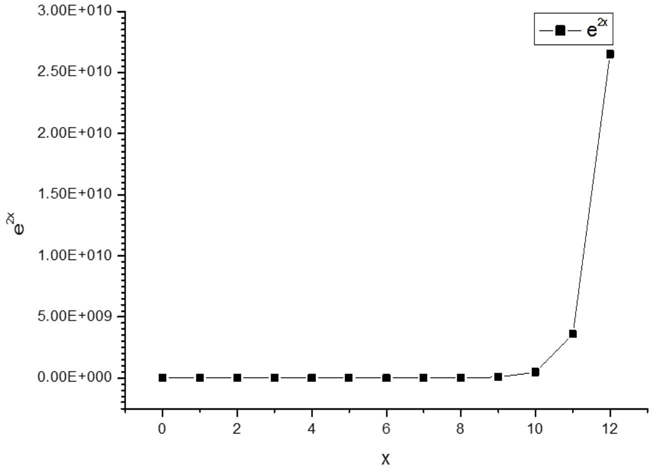

Figure 4: Solution of $f'(x) - f(x) = e^{2x}$ at x = 0 to 12

Figure 4 shows the graphical representation of the differential equation $f'(x) - f(x) = e^{2x}$ with boundary condition of $f(0) = 1$, at x = 0 to 12. The values of the solution obtained shows that when x = 0, the solution $e^{2x} = 1$, which then increases further from x = 1 to 12.

Obtain the solution of the DE using Laplace transform

$3f'(x) - 2f(x) = 4e^{-x} + 2$ where $f(0) = 0$

We start by taking the Laplace transform on both sides of the equation.

$3\mathcal{L}\{f'(x)\} - 2\mathcal{L}\{f(x)\} = 4\mathcal{L}\{e^{x}\} + \mathcal{L}\{2\}$

$3[sF(s) - f(0)] - 2F(s) = \frac{4}{s+1} + \frac{2}{s}$

Now substitute the boundary condition that $f(0) = 0$

$3[sF(s) - 0] - 2F(s) = \frac{4}{s+1} + \frac{2}{s}$

$3sF(s) - 2F(s) = \frac{4}{s+1} + \frac{2}{s}$

Now find the expression for $F(s)$

$F(s)\{3s-2\} = \frac{4}{s+1} + \frac{2}{s}$

$F(s)\{3s-2\} = \frac{4s+2(s+1)}{s(s+1)}$

$F(s)\{3s-2\} = \frac{6s+2}{s(s+1)}$

$F(s) = \frac{6s+2}{s(s+1)\{3s-2\}}$

Now separate $F(s)$ into its partial fraction, we have

$F(s) = \frac{6s+2}{s(s+1)\{3s-2\}} = \frac{P}{s} + \frac{Q}{s+1} + \frac{R}{3s-2}$

$P(s+1)(3s-2) + Q(s)(3s-2) + R(s)(s+1) = 6s + 2$

To obtain P substitute $s = 0$

$P(0+1)(3\times 0-2) + Q(0)(3\times 0-2) + R(0)(0+1) = 6\times 0 + 2$

$P(1)(-2) = 2$

$-2P = 2$

$\therefore P = -1$

To obtain Q substitute $s = -1$

$P(-1+1)(3\times -1-2) + Q(-1)(3\times -1-2) + R(-1)(-1+1) = 6\times -1 + 2$

$Q(-1)(-5) = -6 + 2$

$5Q = -4$

$\therefore Q = -\frac{4}{5}$

To obtain R substitute $s = \frac{2}{3}$

$P\left(\frac{2}{3}+1\right)\left(3\times\frac{2}{3}-2\right) + Q\left(\frac{2}{3}\right)\left(3\times\frac{2}{3}-2\right) + R\left(\frac{2}{3}\right)\left(\frac{2}{3}+1\right) = 6\times\frac{2}{3} + 2$

$R\left(\frac{2}{3}\right)\left(\frac{5}{3}\right) = 4 + 2$

$R\left(\frac{10}{9}\right) = 6$

$\therefore R = \frac{27}{5}$

$F(s) = \frac{6s+2}{s(s+1)\{3s-2\}} = \frac{P}{s} + \frac{Q}{s+1} + \frac{R}{3s-2}$

$F(s) = \frac{6s+2}{s(s+1)\{3s-2\}} = -\frac{1}{s} - \frac{4}{5}\frac{1}{(s+1)} + \frac{27}{5}\frac{R}{(3s-2)}$

$F(s) = \frac{6s+2}{s(s+1)\{3s-2\}} = -\frac{1}{s} - \frac{4}{5}\frac{1}{(s+1)} + \frac{27}{5}\frac{R}{3(s-\frac{2}{3})}$

Now take the inverse transform which gives the solution of the DE

$f(x) = -1 - \frac{4}{5}e^{-x} + \frac{9}{5}e^{2x}/_3$

$$\therefore f(x) = \frac{9}{5}e^{2x}/_3 - \frac{4}{5}e^{-x} - 1 \tag{19}$$

Figure 5 shows the graph of the solution of the differential equation with its respective boundary condition $3f'(x) - 2f(x) = 4e^{-x} + 2$ where $f(0) = 0$

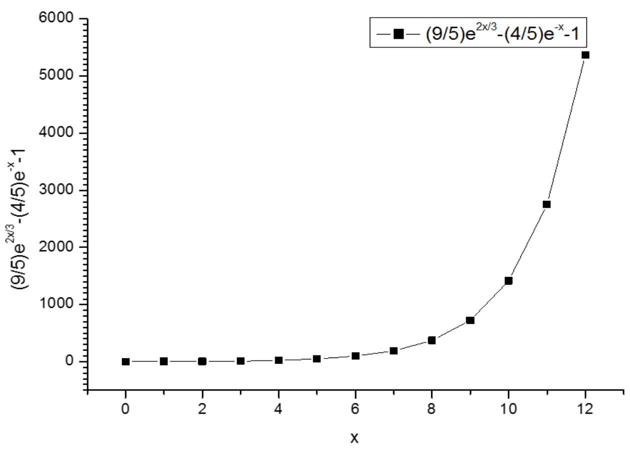

Figure 5: Solution of $3f'(x) - 2f(x) = 4e^{-x} + 2$ at x = 0 to 12

Figure 5 shows the graphical representation of the differential equation $3f'(x) - 2f(x) = 4e^{-x} + 2$ with boundary condition of $f(0) = 0$, at x = 0 to 12. The values of the solution obtained increases from x = 0 to 12.

Obtain the solution of the DE using Laplace transform

$f''(x) + 5f'(x) + 6f(x) = 2e^{-x}$ where $f(0) = 0$ and $f'(0) = 0$

We start by taking the Laplace transform on both sides of the equation.

$\mathcal{L}\{f''(x)\} + 5\mathcal{L}\{f'(x)\} + 6\mathcal{L}\{f(x)\} = 2\mathcal{L}\{e^{-x}\}$

$[s^2F(s) - sf(0) - f'(0)] + 5[sF(s) - f(0)] + 6F(s) = \frac{2}{s+1}$

Now substitute the boundary condition that $f(0) = 0$ and $f'(0) = 0$

$[s^2F(s) - s(0) - (0)] + 5[sF(s) - 0] + 6F(s) = \frac{2}{s+1}$

$s^2F(s) + 5sF(s) + 6F(s) = \frac{2}{s+1}$

Now find the expression for $F(s)$

$F(s)\{s^2 + 5s + 6\} = \frac{2}{s+1}$

Now factorize $s^2 + 5s + 6$ we have $(s + 2)\ (s + 3)$ Therefore

$F(s)\{(s + 2)\ (s + 3)\} = \frac{2}{s+1}$

$F(s) = \frac{2}{(s+1)\{(s+2)\ (s+3)\}}$

Now separate $F(s)$ into its partial fraction, we have

$F(s) = \frac{2}{(s+1)\{(s+2)\ (s+3)\}} = \frac{P}{(s+1)} + \frac{Q}{(s+2)} + \frac{R}{(s+3)}$

$P(s + 2)(s + 3) + Q(s + 1)(s + 3) + R(s + 1)(s + 2) = 2$

To obtain P substitute $s = -1$

$P(-1 + 2)(-1 + 3) + Q(-1 + 1)(-1 + 3) + R(-1 + 1)(-1 + 2) = 2$

$P(1)(2) = 2$

$2P = 2$

$\therefore P = 1$

To obtain Q substitute $s = -2$

$P(-2 + 2)(-2 + 3) + Q(-2 + 1)(-2 + 3) + R(-2 + 1)(-2 + 2) = 2$

$Q(-1)(1) = 2$

$-Q = 2$

$\therefore Q = -2$

To obtain R substitute $s = -3$

$P(-3 + 2)(-3 + 3) + Q(-3 + 1)(-3 + 3) + R(-3 + 1)(-3 + 2) = 2$

$R(-2)(-1) = 2$

$R(2) = 2$

$\therefore R = 1$

$F(s) = \frac{2}{(s+1)\{(s+2)\ (s+3)\}} = \frac{P}{(s+1)} + \frac{Q}{(s+2)} + \frac{R}{(s+3)}$

$F(s) = \frac{2}{(s+1)\{(s+2)\ (s+3)\}} = \frac{1}{(s+1)} - \frac{2}{(s+2)} + \frac{1}{(s+3)}$

Now take the inverse transform which gives the solution of the DE

$\therefore f(x) = e^{-x} - 2e^{-2x} + e^{-3x}$ (20)

Figure 6 shows the graph of the solution of the differential equation with its respective boundary condition $f''(x) + 5f'(x) + 6f(x) = 2e^{-x}$ where $f(0) = 0$ and $f'(0) = 0$

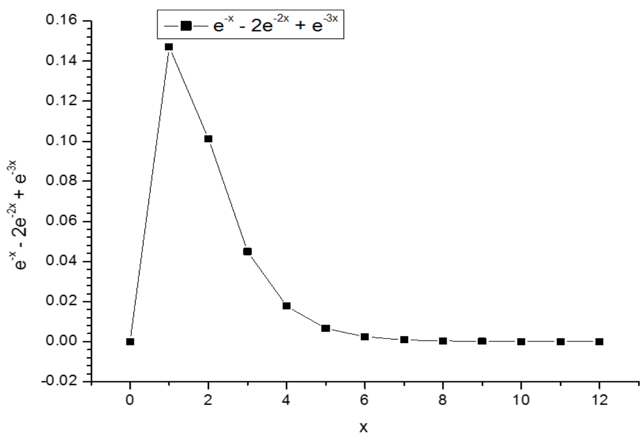

Figure 6: Solution of $f''(x) + 5f'(x) + 6f(x) = 2e^{-x}$ at x = 0 to 12

Figure 6 shows the graphical representation of the differential equation $f''(x) + 5f'(x) + 6f(x) = 2e^{-x}$ with boundary condition of $f(0) = 0$ and $f'(0) = 0$ at x = 0 to 12. The values of the solution obtained shows that when x = 0, the solution $e^{-x} - 2e^{-2x} + e^{-3x} = 0$, which then increases from x = 0 to 1, and decrease further from x = 1 to 12 with negative values from x = 10 to12.

Obtain the solution of the DE using Laplace transform

$f''(x) - 4f(x) = \sin 2x$ where $f(0) = 1$ and $f'(0) = -2$

We start by taking the Laplace transform on both sides of the equation.

$\mathcal{L}\{f''(x)\} - 4\mathcal{L}\{f(x)\} = \mathcal{L}\{\sin 2x\}$

$[s^2F(s) - sf(0) - f'(0)] - 4F(s) = \frac{2}{s^2+2^2}$ where $\mathcal{L}\{\sin ax\} = \frac{a}{s^2+a^2}$

Now substitute the boundary condition that $f(0) = 1$ and $f'(0) = -2$

$[s^2F(s) - s(1) - (-2)] - 4F(s) = \frac{2}{s^2+2^2}$

$s^2F(s) - s + 2 - 4F(s) = \frac{2}{s^2+2^2}$

Now find the expression for $F(s)$

$F(s)\{s^2 - 4\} = \frac{2}{s^2+2^2} + s - 2$

$F(s) = \frac{2}{(s^2+2^2)\{s^2-4\}} + \frac{s-2}{\{s^2-4\}}$

$F(s) = \frac{2+(s-2)(s^2+4)}{(s^2+4)\{s^2-4\}}$

$F(s) = \frac{s^3-2s^2+4s-6}{(s^2+4)\{s^2-4\}} = \frac{s^3-2s^2+4s-6}{(s^2+4)(s-2)(s+2)}$

Now separate $F(s)$ into its partial fraction, we have

$F(s) = \frac{s^3-2s^2+4s-6}{(s^2+4)(s-2)(s+2)} = \frac{P}{(s^2+4)} + \frac{Q}{(s-2)} + \frac{R}{(s+2)}$

$P(s - 2)(s + 2) + Q(s^2 + 4)(s + 2) + R(s^2 + 4)(s - 2) = s^3 - 2s^2 + 4s - 6$

To obtain R substitute $s = -2$

$P(-2 - 2)(-2 + 2) + Q(-2^2 + 4)(-2 + 2) + R(-2^2 + 4)(-2 - 2) = -2^3 - 2 \times -2^2 + 4 \times -2 - 6$

$R(8)(-4) = -8 - 8 - 8 - 6$

$-32P = -30$

$\therefore R = \frac{15}{16}$

To obtain Q substitute $s = 2$

$P(2 - 2)(2 + 2) + Q(2^2 + 4)(2 + 2) + R(2^2 + 4)(2 - 2) = 2^3 - 2 \times 2^2 + 4 \times 2 - 6$

$Q(8)(4) = 8 - 8 + 8 - 6$

$32Q = 2$

$\therefore Q = \frac{1}{16}$

P can be obtained by comparing the coefficients of s^2 of the given equations.

$P(s - 2)(s + 2) + Q(s^2 + 4)(s + 2) + R(s^2 + 4)(s - 2) = s^3 - 2s^2 + 4s - 6$

$P(s^2 - 4) + Q(s^3 + 2s^2 + 4s + 8) + R(s^3 - 2s^2 + 4s - 8) = s^3 - 2s^2 + 4s - 6$

After expanding and equating the coefficients of s^2 we have

$Ps^2 + 2Qs^2 - 2Rs^2 = -2s^2$

$(P + 2Q - 2R)s^2 = -2s^2$

$P + 2Q - 2R = -2$

$P + 2\left(\frac{1}{16}\right) - 2\left(\frac{15}{16}\right) = -2$

$P + \left(\frac{1}{8}\right) - \left(\frac{15}{8}\right) = -2$

$\therefore P = -\frac{1}{4}$

$F(s) = -\frac{1}{4}\frac{1}{(s^2+4)} + \frac{1}{16}\frac{1}{(s-2)} + \frac{15}{16}\frac{1}{(s+2)}$

$F(s) = -\frac{1}{8}\frac{2}{(s^2+4)} + \frac{1}{16}\frac{1}{(s-2)} + \frac{15}{16}\frac{1}{(s+2)}$

Now take the inverse transform which gives the solution of the DE

$\therefore f(x) = -\frac{1}{8}\sin 2x + \frac{1}{16}e^{2x} + \frac{15}{16}e^{-2x}$

$\therefore f(x) = \frac{1}{16}e^{2x} + \frac{15}{16}e^{-2x} - \frac{1}{8}\sin 2x$ (21)

Figure 7 shows the graph of the solution of the differential equation with its respective boundary condition $f''(x) - 4f(x) = \sin 2x$ where $f(0) = 1$ and $f'(0) = -2$

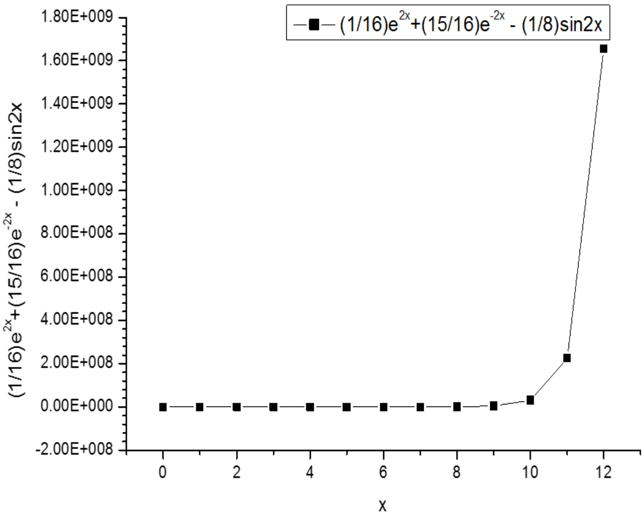

Figure 7: Solution of $f''(x) - 4f(x) = \sin 2x$ at x = 0 to 12

Figure 7 shows the graphical representation of the differential equation $f''(x) - 4f(x) = \sin 2x$ with boundary condition of $f(0) = 1$ and $f'(0) = -2$ at x = 0 to 12. The values of the solution obtained shows that when x = 0, the solution, $\frac{1}{16}e^{2x} + \frac{15}{16}e^{-2x} - \frac{1}{8}\sin 2x = 1$, which then decrease from x = 0 to 1, and then increases further from x = 1 to 12.

Obtain the solution of the DE using Laplace transform

$2f''(x) - f'(x) - f(x) = e^{-3x}$ where $f(0) = 2$ and $f'(0) = 1$

We start by taking the Laplace transform on both sides of the equation.

$2\mathcal{L}\{f''(x)\} - \mathcal{L}\{f'(x)\} - \mathcal{L}\{f(x)\} = \mathcal{L}\{e^{-3x}\}$

$2[s^2F(s) - sf(0) - f'(0)] - [sF(s) - f(0)] - F(s) = \frac{1}{s+3}$

Now substitute the boundary condition that $f(0) = 2$ and $f'(0) = 1$

$2[s^2F(s) - s(2) - (1)] - [sF(s) - 2] - F(s) = \frac{1}{s+3}$

$2s^2F(s) - 4s - 2 - sF(s) + 2 - F(s) = \frac{1}{s+3}$

Now find the expression for $F(s)$

$F(s)\{2s^2 - s - 1\} = \frac{1}{s+3} + 4s$

Now factorize $2s^2 - s - 1$ we have $(s-1)\,(2s+1)$ Therefore

$F(s)\{(s-1)\,(2s+1)\} = \frac{1}{s+3} + 4s$

$F(s)\{(s-1)\,(2s+1)\} = \frac{1+4s(s+3)}{s+3}$

$F(s)\{(s-1)\,(2s+1)\} = \frac{4s^2+12s+1}{s+3}$

$F(s) = \frac{4s^2+12s+1}{(s+3)\{(s-1)\,(2s+1)\}}$

Now separate $F(s)$ into its partial fraction, we have

$F(s) = \frac{4s^2+12s+1}{(s+3)\{(s-1)\,(2s+1)\}} = \frac{P}{(s+3)} + \frac{Q}{(s-1)} + \frac{R}{(2s+1)}$

$P(s-1)(2s+1) + Q(s+3)(2s+1) + R(s+3)(s-1) = 4s^2 + 12s + 1$

To obtain P substitute $s = -3$

$P(-3-1)(2\times -3+1) + Q(-3+3)(2\times -3+1) + R(-3+3)(-3-1) = 4\times -3^2 + 12\times -3 + 1$

$P(-4)(-5) = 36 - 36 + 1$

$20P = 1$

$\therefore P = \frac{1}{20}$

To obtain Q substitute $s = 1$

$P(1-1)(2\times 1+1) + Q(1+3)(2\times 1+1) + R(1+3)(1-1) = 4\times 1^2 + 12\times 1 + 1$

$Q(4)(3) = 4 + 12 + 1$

$12Q = 17$

$\therefore Q = \frac{17}{12}$

To obtain R substitute $s = -\frac{1}{2}$

$P\left(-\frac{1}{2}-1\right)\left(2\times -\frac{1}{2}+1\right) + Q\left(-\frac{1}{2}+3\right)\left(2\times -\frac{1}{2}+1\right) + R\left(-\frac{1}{2}+3\right)\left(-\frac{1}{2}-1\right) = 4\times\left(-\frac{1}{2}\right)^2 + 12\times -\frac{1}{2} + 1$

$R\left(\frac{5}{2}\right)\left(-\frac{3}{2}\right) = 4\times\frac{1}{4}\times -6 + 1$

$R\left(-\frac{15}{4}\right) = -4$

$\therefore R = \frac{16}{15}$

$F(s) = \frac{4s^2+12s+1}{(s+3)\{(s-1)\,(2s+1)\}} = \frac{P}{(s+3)} + \frac{Q}{(s-1)} + \frac{R}{(2s+1)}$

$F(s) = \frac{1}{20}\frac{1}{(s+3)} + \frac{17}{12}\frac{1}{(s-1)} + \frac{16}{15}\frac{1}{(2s+1)}$

$F(s) = \frac{1}{20}\frac{1}{(s+3)} + \frac{17}{12}\frac{1}{(s-1)} + \frac{16}{15}\times\frac{1}{2\left(s+\frac{1}{2}\right)}$

Now take the inverse transform which gives the solution of the DE

$\therefore f(x) = \frac{1}{20}e^{-3x} + \frac{17}{12}e^{x} + \frac{8}{15}e^{\frac{1}{2}x}$ (22)

Figure 8 shows the graph of the solution of the differential equation with its respective boundary condition $2f''(x) - f'(x) - f(x) = e^{-3x}$ where $f(0) = 2$ and $f'(0) = 1$

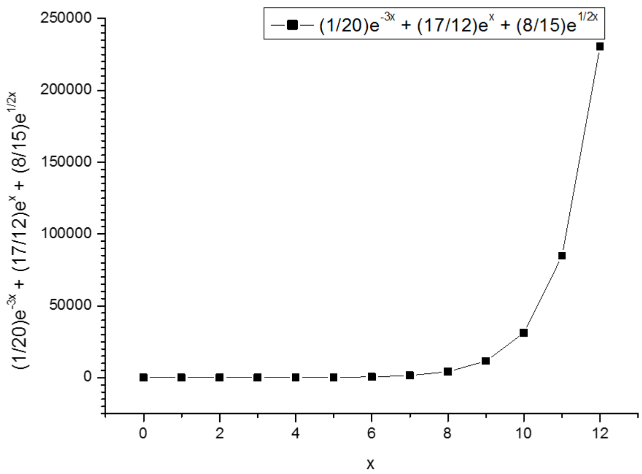

Figure 8: Solution of $2f''(x) - f'(x) - f(x) = e^{-3x}$ at x = 0 to 12

Figure 8 shows the graphical representation of the differential equation $2f''(x) - f'(x) - f(x) = e^{-3x}$ with boundary condition of $f(0) = 2$ and $f'(0) = 1$, at x = 0 to 12. The values of the solution obtained shows that the values increases from x = 1 to 12.

Obtain the solution of the DE using Laplace transform

$2f''(x) - f'(x) - f(x) = \sin x - \cos x$ where $f(0) = 0$ and $f'(0) = 0$

We start by taking the Laplace transform on both sides of the equation.

$2\mathcal{L}\{f''(x)\} - \mathcal{L}\{f'(x)\} - \mathcal{L}\{f(x)\} = \mathcal{L}\{\sin x\} - \mathcal{L}\{\cos x\}$

$2[s^2F(s) - sf(0) - f'(0)] - [sF(s) - f(0)] - F(s) = \frac{1}{s^2+1^2} - \frac{s}{s^2+1^2}$

where $\mathcal{L}\{\sin ax\} = \frac{a}{s^2+a^2}$ and $\mathcal{L}\{\cos ax\} = \frac{s}{s^2+a^2}$

Now substitute the boundary condition that $f(0) = 0$ and $f'(0) = 0$

$2[s^2F(s) - s(0) - (0)] - [sF(s) - (0)] - F(s) = \frac{1}{s^2+1^2} - \frac{s}{s^2+1^2}$

$2s^2F(s) - sF(s) - F(s) = \frac{1-s}{s^2+1^2}$

Now find the expression for $F(s)$

$F(s)\{2s^2 - s - 1\} = \frac{1-s}{s^2+1^2}$

Now factorize $2s^2 - s - 1$ we have $(s-1)\,(2s+1)$ Therefore

$F(s)\{(s-1)\,(2s+1)\} = \frac{1-s}{s^2+1^2}$

$F(s) = \frac{1-s}{(s^2+1^2)\{(s-1)\,(2s+1)\}}$

$F(s) = \frac{-(s-1)}{(s^2+1^2)\{(s-1)\,(2s+1)\}}$

$F(s) = \frac{-1}{(s^2+1^2)\{\,(2s+1)\}}$

Now separate $F(s)$ into its partial fraction, we have

$F(s) = \frac{-1}{(s^2+1^2)\{\,(2s+1)\}} = \frac{P}{(s^2+1^2)} + \frac{Q}{(2s+1)}$

$P(2s+1) + Q(s^2+1) = -1$

To obtain Q substitute $s = -\frac{1}{2}$

$P\left(2 \times -\frac{1}{2} + 1\right) + Q\left(\left(-\frac{1}{2}\right)^2 + 1\right) = -1$

$Q\left(\frac{5}{4}\right) = -1$

$\therefore Q = -\frac{4}{5}$

P can be obtained by comparing the coefficients of the constant terms from the expression.

$P(2s+1) + Q(s^2+1) = -1$

$2Ps + P + Qs^2 + Q = -1$ Therefore by equating the terms and substituting $Q = -\frac{4}{5}$ we have

$P + Q = -1$

$P - \frac{4}{5} = -1$

$\therefore P = -\frac{1}{5}$

$F(s) = -\frac{1}{5}\frac{1}{(s^2+1)} - \frac{4}{5}\frac{1}{(2s+1)}$

$F(s) = -\frac{1}{5}\frac{1}{(s^2+1)} - \frac{4}{5} \times \frac{1}{2\left(s+\frac{1}{2}\right)}$

Now take the inverse transform which gives the solution of the DE

$$\therefore f(x) = -\frac{1}{5}\sin x - \frac{2}{5}e^{-\frac{1}{2}x} \quad (23)$$

Figure 9 shows the graph of the solution of the differential equation with its respective boundary condition $2f''(x) - f'(x) - f(x) = \sin x - \cos x$ where $f(0) = 0$ and $f'(0) = 0$

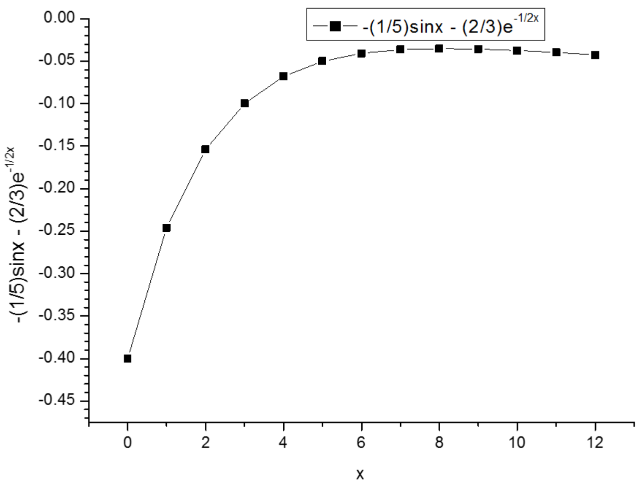

Figure 9: Solution of $2f''(x) - f'(x) - f(x) = \sin x - \cos x$ at x = 0 to 12

Figure 9 shows the graphical representation of the differential equation $2f''(x) - f'(x) - f(x) = \sin x - \cos x$ with boundary condition of $f(0) = 0$ and $f'(0) = 0$, at x = 0 to 12. The values of the solution obtained shows that the values increases negatively from when x = 0 to 12.

Obtain the solution of the DE using Laplace transform

$f''(x) + 16f(x) = 0$ where $f(0) = 0$ and $f'(0) = 4$

We start by taking the Laplace transform on both sides of the equation.

$\mathcal{L}\{f''(x)\} + 16\mathcal{L}\{f(x)\} = \mathcal{L}\{0\}$

$[s^2F(s) - sf(0) - f'(0)] + 16F(s) = 0$

Now substitute the boundary condition that $f(0) = 0$ and $f'(0) = 4$

$[s^2F(s) - s(0) - (4)] + 16F(s) = 0$

$s^2F(s) - 4 + 16F(s) = 0$

Now find the expression for $F(s)$

$F(s)\{s^2 + 16\} = 4$

$F(s) = \frac{4}{\{s^2+16\}}$ From $\mathcal{L}\{\sin ax\} = \frac{a}{s^2+a^2}$

$F(s) = \frac{4}{\{s^2+4^2\}}$

Now take the inverse transform which gives the solution of the DE

$$\therefore f(x) = \sin 4x \quad (24)$$

Figure 10 shows the graph of the solution of the differential equation with its respective boundary condition $f''(x) + 16f(x) = 0$ where $f(0) = 0$ and $f'(0) = 4$

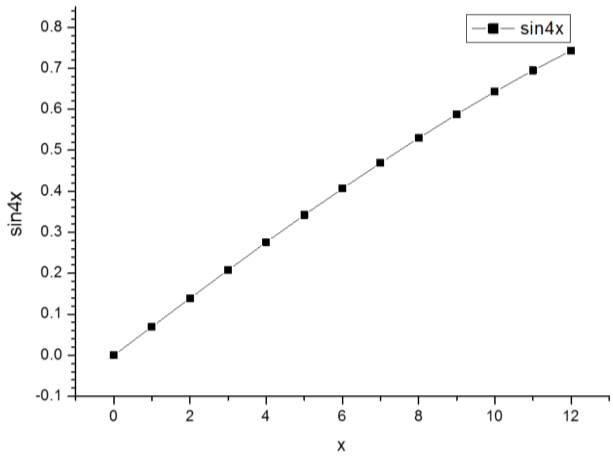

Figure 10: $f''(x) + 16f(x) = 0$ at x = 0 to 1

Figure 10 shows the graphical representation of the differential equation $f''(x) + 16f(x) = 0$ with boundary condition of $f(0) = 0$ and $f'(0) = 4$, at x = 0 to 12. The values of the solution obtained increases from x = 0 to 12 in almost a straight line.

Obtain the solution of the DE using Laplace transform

$f''(x) - 4f(x) = \cos 2x$ where $f(0) = 1$ and $f'(0) = -2$

We start by taking the Laplace transform on both sides of the equation.

$\mathcal{L}\{f''(x)\} - 4\mathcal{L}\{f(x)\} = \mathcal{L}\{\cos 2x\}$

$[s^2F(s) - sf(0) - f'(0)] - 4F(s) = \frac{s}{s^2+2^2}$

where $\mathcal{L}\{\cos ax\} = \frac{s}{s^2+a^2}$

Now substitute the boundary condition that $f(0) = 1$ and $f'(0) = -2$

$[s^2F(s) - s(1) - (-2)] - 4F(s) = \frac{s}{s^2+2^2}$

$s^2F(s) - s + 2 - 4F(s) = \frac{s}{s^2+4}$

Now find the expression for $F(s)$

$F(s)\{s^2 - 4\} = \frac{s}{s^2+4} + s - 2$

$F(s) = \frac{s+(s-2)(s^2+4)}{(s^2+4)\{s^2-4\}}$

$F(s) = \frac{s^3-2s^2+5s-8}{(s^2+4)\{s^2-4\}}$

Now separate $F(s)$ into its partial fraction, we have

$F(s) = \frac{s^3-2s^2+5s-8}{(s^2+4)\{s^2-4\}} = \frac{P}{(s^2+4)} + \frac{Q}{\{s^2-4\}}$

$P\{s^2-4\} + Q(s^2+4) = s^3 - 2s^2 + 5s - 8$

To obtain Q substitute $s = 2$

$P\{s^2-4\} + Q(s^2+4) = s^3 - 2s^2 + 5s - 8$

$P\{2^2-4\} + Q(2^2+4) = 2^3 - 2 \times 2^2 + 5 \times 2 - 8$

$8Q = 8 - 8 + 10 - 8$

$\therefore Q = \frac{1}{4}$

P can be obtained by comparing the coefficients of the constant terms from the expression.

$P\{s^2-4\} + Q(s^2+4) = s^3 - 2s^2 + 5s - 8$

$Ps^2 - 4P + Qs^2 + 4Q = s^3 - 2s^2 + 5s - 8$ Equating the terms and substituting $Q = \frac{1}{4}$ we have

$-4P + 4Q = -8$

$-4P + 4 \times \frac{1}{4} = -8$

$\therefore P = \frac{9}{4}$

$F(s) = \frac{9}{4}\frac{1}{(s^2+4)} + \frac{1}{4}\frac{1}{\{s^2-4\}}$

$F(s) = \frac{9}{4}\frac{1}{(s^2+2^2)} + \frac{1}{4}\frac{1}{\{s^2-2^2\}}$

$F(s) = \frac{9}{8}\frac{2}{(s^2+2^2)} + \frac{1}{8}\frac{2}{\{s^2-2^2\}}$

Now take the inverse transform which gives the solution of the DE

$$\therefore f(x) = \frac{9}{8}\sin(2x) + \frac{1}{8}\sin(-2x) \quad (25)$$

Figure 11 shows the graph of the solution of the differential equation with its respective boundary condition $f''(x) - 4f(x) = \cos 2x$ where $f(0) = 1$ and $f'(0) = -2$

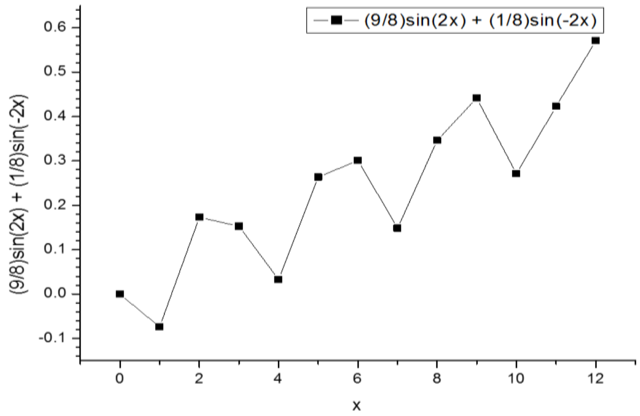

Figure 11: Solution of $f''(x) - 4f(x) = \cos 2x$ at x = 0 to 12

Figure 11 shows the graphical representation of the differential equation $f''(x) - 4f(x) = \cos 2x$ with boundary condition of $f(0) = 1$ and $f'(0) = -2$, at x = 0 to 12. The values of the solution obtained shows that the solution decreases from x = 0 to 1, increase from x = 1 to 2, decrease from x = 2 to 4, increase from x = 4 to 6, decrease from x = 6 to 7, increase from x = 7 to 9, decrease from x = 9 to 10 and finally increases further from x = 10 to 12.

4. Conclusion

This study have established a comprehensive approach in using the Laplace transforms to find the solutions of differential equations of the first and second order with boundary conditions and to investigate the pattern of variation exhibited graphically by the various evaluated solutions of the differential equations at x = 0 to 12. The results obtained contain solutions of exponential function, exponential and sine functions, and sine functions only. The results of the graphical representation of the solutions of differential equations at x = 0 to 12 shows different patterns of variation depending on the evaluated differential equations. This study revealed that the Laplace transform can effectively be used to find solutions of differential equations at their respective boundary conditions.

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