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RESEARCH ARTICLE

A COMPREHENSIVE APPROACH TO EVALUATING SOLUTIONS OF BESSEL'S FUNCTION OF THE FIRST KIND OF ORDER N

D.O. Akpootu[a]*, G. Bello[b]

[a]Department of Physics, Usmanu Danfodiyo University, Sokoto, Nigeria
[b]Sultan Abdurrahaman College of Health Technology Gwadabawa, Sokoto State
*Corresponding Author Email: davidson.odafe@udusok.edu.ng

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ABSTRACT

This study establishes mathematically a comprehensive approach for evaluating the solutions of Bessel's function of the first kind for different values of the order n. The study revealed that the knowledge of gamma function, Maclaurin's series and basic laws of indices plays a significant role in evaluating the Bessel's function of order n. The results showed that the solutions, $J_{-1}(x) = J_{-2}(x) = J_{-3}(x) = 0$ for x = 0, 1, 2, 3, 4…….12; the pattern of variation for the various graphs of $J_0(x)$, $J_{1/2}(x)$, $J_{-1/2}(x)$, $J_1(x)$, $J_{3/2}(x)$, $J_{-3/2}(x)$, $J_2(x)$, $J_{5/2}(x)$, $J_{-5/2}(x)$ and $J_3(x)$ were investigated and the results showed that there is a slight decrease in the values of $J_0(x)$ from x = 0 to 3 which then increase steadily from x = 4 to 12, for $J_{1/2}(x)$, $J_{3/2}(x)$, $J_2(x)$ and $J_3(x)$, the values increases from x = 0 to 12. The values of $J_{-1/2}(x)$ increases steadily from x = 0 to 12 in the form of a parabola. For $J_1(x)$, the graph increase from x = 0 to 2 then decreases at x = 3 and increases continuously from x = 4 to 12. The figure depicting $J_{-3/2}(x)$ showed that when x = 0, $J_{-3/2}(0) = \infty$, negative values were obtained at x = 1 and 2, the values then increases steadily from x = 4 to 12. The figure for $J_{5/2}(x)$ shows that the values increases from x = 0 to 2 and decreases negatively from x = 3 to 8 and increases steadily from x = 9 to 12. The figure for $J_{-5/2}(x)$ depicts that when x = 0, $J_{-5/2}(0) = \infty$, the values decrease from x = 1 to 2 and decreases further but negatively from x = 3 to 12.

KEYWORDS

Bessel's function, gamma function, laws of indices, Maclaurin's series, order n

1. Introduction

Bessel functions are named for Friedrich Wilhelm Bessel (1784 - 1846), although, Daniel Bernoulli was given the credit as being the first to bring in the concept of Bessels functions in 1732. He employed the function of zero order as a solution to the problem of an oscillating chain suspended at one end. Finding solutions of differential equations has been a difficult task in pure mathematics since the development of calculus by Newton and Leibniz in the 17th century. Apart from this, these equations have been found useful in other disciplines such as engineering, economics biology, and physics. Bessel functions are solutions of a particular differential equation, called Bessel's equation (García, 2015). The theory of Bessel functions is a rich subject due to its essential role in providing solutions for differential equations associated with many applications (Mahmoud et al., 2021).

The theory of special functions is a significant branch of modern mathematical analysis. In the past three decades, numerous new classes of special functions have been proposed as solutions of fractional differential equations (FDEs). No other special functions have received such detailed treatment in readily obtainable treatises as Bessel functions. The exploration of such functions is an important problem in fractional calculus, which has earned much attention as real-life problems can be analyzed well. Fractional calculus emerges in many branches of science, such as medicine, electromagnetics, material sciences, and fluid mechanics (Podlubny, 1998; Kilbas et al., 2006; Dalir and Bashour, 2010; Qureshi, 2020).

The importance of Bessel's Function in Mathematical Physics is revealed by their application to the modern solutions of problems in Wave-Theory, Elasticity, Hydrodynamics and Optics. The designation of Bessel's Functions as cylindrical function has its source in the use of these functions to express solutions of such physical problems as flow of heat or electricity in solid circular cylinder (West, 1957). Researchers have been carried out studies on Bessel's function. A group researchers introduced a version k of Bessel function of first kind (Romero et al., 2012). They studied some basic properties and present a relationship connecting this function with the functions k-Mittag Leffler and k-Wright. García reported that Bessel functions of integer order can also be seen as the coefficients of a Laurent series (García, 2015).

The aim of this study is to employ the gamma function, Maclaurin series and basic laws of indices in a comprehensive form to obtain solutions of Bessel's function of the first kind for different values of n and to investigate the different patterns of the graphs depicted for the solutions, $J_n(x)$.

2. Methodology

The Bessel's differential equation is of the form (Gupta, 2010).

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0 \tag{1}$$

In order to integrate it in a series of ascending power of x, its series solution are assumed to be:

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \tag{2}$$

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$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1}$ (3)

$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2}$ (4)

Substituting equations (2), (3) and (4) into equation (1), the identity equation was obtained which is of the form (Gupta, 2010).

$\sum_{r=0}^{\infty}[\{(k+r)^2 - n^2\}x^{k+r-2} + x^{k+r}]\, a_r \equiv 0$
(5)

Equating to zero the coefficient of the general term, that is x^{k+r} in equation (5) we obtained

$a_{r+2} = -\frac{a_r}{(k+r+2-n)(k+r+2+n)}$ (6)

When $k = +n$ by substituting $r = 0, 1, 2, 3 \ldots\ldots$ in equation (6), $a_1, a_2, a_3, a_4, a_5, a_6 \ldots$.. can be obtained. Hence, a series solution of the form is obtained which is expressed as:

$y = a_0\left[x^n - \frac{x^{n+2}}{2(2n+2)} + \frac{x^{n+4}}{2(2n+2)\times 4(2n+4)} - \cdots\right]$ (X)

$y = a_0x^n\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2(2n+2)\times 4(2n+4)} - \cdots \frac{(-1)^r x^{2r}}{2\times 4 \ldots\ldots 2r(2n+2)\ldots\ldots(2n+2r)} + \cdots\right]$

$y = a_0x^n \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^r r! 2^r (n+1)\ldots\ldots(n+r)}$ (7)

If $a_0 = \frac{1}{2^n\Gamma(n+1)}$ (Y)

This solution is called $J_n(x)$ and is given as (Gupta, 2010).

$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)} \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^{2r} r!(n+1)(n+2)\ldots\ldots(n+r)}$

$J_n(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\Gamma(n+r+1)}$ (8)

Equation (8) is known as the Bessel's function of the first kind of order n. Using equations X and Y, this can simply be expressed as (Gupta, 2010).

$J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2(2n+2)\times 4(2n+4)} - \cdots\right]$
(9)

The Gamma function (Γ) plays a role when defining most Bessel functions (Garcia, 2015). The gamma function $\Gamma(x)$ is defined by the integral

$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$ (10)

Which is convergent for $x > 0$. It follows from equation (10) that

$\Gamma(x+1) = \int_0^{\infty} t^x e^{-t}dt$

Integrating by parts, we have that

$$\Gamma(x+1) = \left[t^x\left(\frac{e^{-t}}{-1}\right)\right]_0^{\infty} + x\int_0^{\infty} t^{x-1}e^{-t}dt$$

$\Gamma(x+1) = x\Gamma(x)$ (11)

If $x = n$, a positive integer, i.e., if $n \geq 1$ it follows that

$\Gamma(n+1) = n\Gamma(n)$ (12)

The gamma functions of positive whole numbers can be obtained by obtaining $\Gamma(1)$ as

$\Gamma(1) = \int_0^{\infty} t^0 e^{-t}dt = [-e^{-t}]_0^{\infty} = 1$ Therefore

$\Gamma(n+1) = n!$ (13)

It follows that

$\Gamma(7) = 6! = 720$ and so on.

Negative values of x can be obtained by following the same fashion, since

$\Gamma(x) = \frac{\Gamma(x+1)}{x}$, then as $x \to 0$, $\Gamma(x) \to \infty$ this implies that $\Gamma(0) \to \infty$. Also, $\Gamma(-1) = \frac{\Gamma(0)}{-1} = \infty$. Other negative values of x can be obtained following similar procedure.

It has been shown that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (Gupta, 2010).

The Maclaurin's series of $\sin x$ and $\cos x$ are given by (Stoud, 2001).

$\sin x = \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$ (14)

and

$\cos x = \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$ (15)

The knowledge and ability to apply the equation of Bessel's function of the first kind of order n given in equation (9), gamma functions, Maclaurin's series and the basic laws of indices, will help in evaluating Bessel's function for different values of n. The laws of indices are found (Olowofeso, 2019). This study seek to employ these methods in a comprehensive form to obtain solutions of Bessel's function of the first kind for different values of n and to investigate the different patterns of the graphs depicted for $J_n(x)$.

3. Results and Discussion

The thirteen (13) solutions of Bessel's function for n = 0, ½, -½, 1, -1, 3/2, -3/2, 2, -2, 5/2, -5/2, 3 and -3 are evaluated using the comprehensive approach.

Obtain the solution for the Bessel's function $J_0(x)$

Here $n = 0$ from

$J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2(2n+2)\times 4(2n+4)} - \cdots\right]$

$$J_0(x) = \left(\frac{x}{2}\right)^0 \frac{1}{\Gamma(0+1)}\left[1 - \frac{x^2}{2(2\times 0+2)} + \frac{x^4}{2(2\times 0+2)\times 4(2\times 0+4)}\right]$$

$$J_0(x) = \frac{1}{\Gamma(1)}\left[1 - \frac{x^2}{4} + \frac{x^4}{64}\right]$$

But $\Gamma(1) = 1$

$J_0(x) = \left[1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots\right]$ (16)

The Bessel's function of $J_0(x)$ for values ranging from x = 0, 1, 2, 3, 4.......12 were plotted and the graph is as shown below:

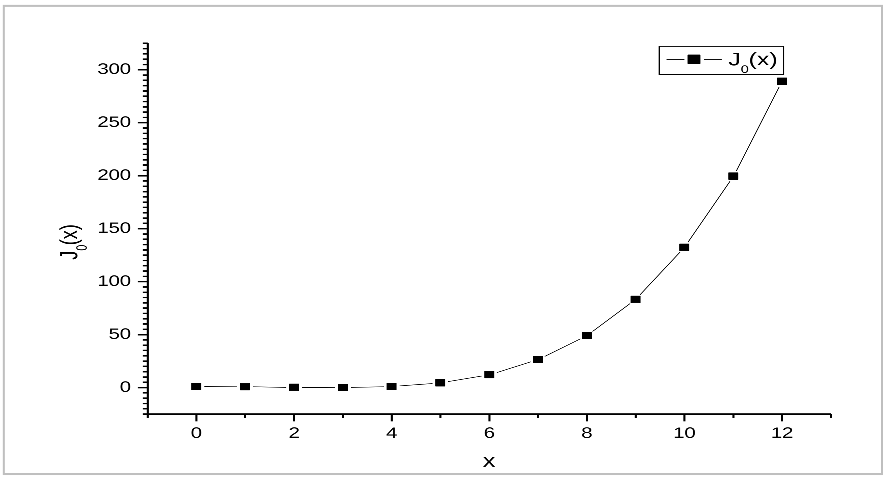

Figure 1: Bessel's function for $J_0(x)$

Figure 1 shows the values of $J_0(x)$ for x = 0, 1, 2, 3, 4…….12 based on the solution of the Bessel's function given in equation 16 when $x = 0$ *and* 4, the value of $J_0(x)$ is unity. The graph showed that there is a slight decrease in the values of $J_0(x)$ from $x = 0$ *to* 3 which then increase steadily from $x = 4$ *to* 12.

Obtain the solution for the Bessel's function $J_{1/2}(x)$

Here $n = \frac{1}{2}$ from

$$J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2(2n+2)\times 4(2n+4)} - \cdots\right]$$

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \frac{1}{\Gamma\left(\frac{1}{2}+1\right)}\left[1 - \frac{x^2}{2\left(2\times\frac{1}{2}+2\right)} + \frac{x^4}{2\left(2\times\frac{1}{2}+2\right)\times 4\left(2\times\frac{1}{2}+4\right)} - \cdots\right]$$

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \frac{1}{\Gamma\left(\frac{3}{2}\right)}\left[1 - \frac{x^2}{2(3)} + \frac{x^4}{2(3)\times 4(5)} - \cdots\right]$$

Now evaluate $\Gamma\left(\frac{3}{2}\right)$

From $\Gamma(n) = \frac{\Gamma(n+1)}{n}$, this implies that $n + 1 = \frac{3}{2}$, therefore, $n = \frac{1}{2}$

Thus, $\Gamma\left(\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}+1\right)}{\frac{1}{2}} = \frac{\Gamma\left(\frac{3}{2}\right)}{\frac{1}{2}}$

But $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ Therefore

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\times\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \frac{1}{\frac{\sqrt{\pi}}{2}}\left[1 - \frac{x^2}{2(3)} + \frac{x^4}{2(3)\times 4(5)} - \cdots\right]$$

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \times \frac{2}{\sqrt{\pi}} \times \frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

But $\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] = \sin x$ therefore by applying the law of indices and rearranging we have that

$$J_{1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\sin x \qquad (17)$$

The Bessel's function of $J_{1/2}(x)$ for values ranging from x = 0, 1, 2, 3, 4…….12 were plotted and the graph is as shown below.

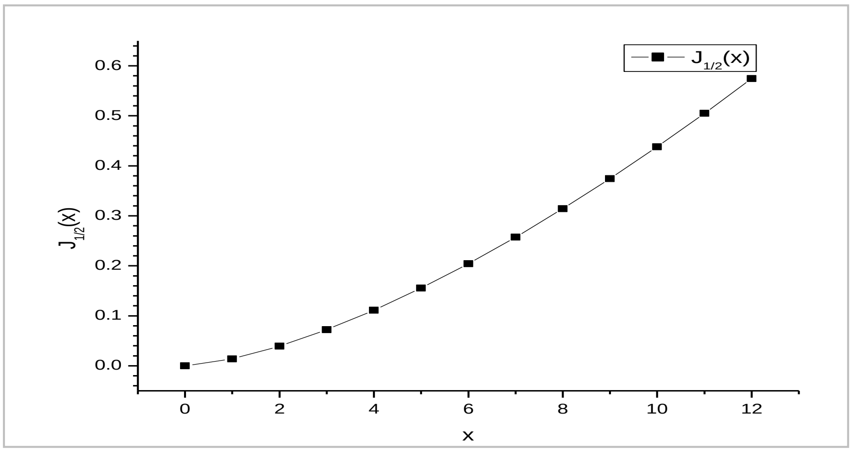

Figure 2: Bessel's function for $J_{1/2}(x)$

Figure 2 shows the values of $J_{1/2}(x)$ for x = 0, 1, 2, 3, 4…….12 based on the solution of the Bessel's function given in equation 17. The graph shows that the values increases from $x = 0$ *to* 12.

Obtain the solution for the Bessel's function $J_{-1/2}(x)$

Here $n = -1/2$ from

$$J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2(2n+2)\times 4(2n+4)} - \cdots\right]$$

$$J_{-1/2}(x) = \left(\frac{x}{2}\right)^{-1/2} \frac{1}{\Gamma\left(-\frac{1}{2}+1\right)}\left[1 - \frac{x^2}{2\left(2\times-\frac{1}{2}+2\right)} + \frac{x^4}{2\left(2\times-\frac{1}{2}+2\right)\times 4\left(2\times-\frac{1}{2}+4\right)} - \cdots\right]$$

$$J_{-1/2}(x) = \left(\frac{x}{2}\right)^{-1/2} \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[1 - \frac{x^2}{2(1)} + \frac{x^4}{2(1)\times 4(3)} - \cdots\right]$$

$$J_{-1/2}(x) = \left(\frac{x}{2}\right)^{-1/2} \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[1 - \frac{x^2}{2(1)} + \frac{x^4}{2(1)\times 4(3)} - \cdots\right]$$

But $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$J_{-1/2}(x) = \left(\frac{x}{2}\right)^{-1/2} \frac{1}{\sqrt{\pi}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

But $\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right] = \cos x$ therefore by applying the law of indices and rearranging we have that

$$J_{-1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\cos x \qquad (18)$$

The Bessel's function of $J_{-1/2}(x)$ for values ranging from x = 0, 1, 2, 3, 4…….12 were plotted and the graph is as shown below:

Figure 3 shows the values of $J_{-1/2}(x)$ for x = 0, 1, 2, 3, 4…….12 based on the solution of the Bessel's function given in equation 18. The graph showed that the values of $J_{-1/2}(x)$ increases steadily from $x = 0$ *to* 12 in the form of a parabola.

Obtain the solution for the Bessel's function $J_1(x)$

Here $n = 1$ from

$$J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2(2n+2)\times 4(2n+4)} - \cdots\right]$$

$$J_1(x) = \left(\frac{x}{2}\right)^1 \frac{1}{\Gamma(1+1)}\left[1 - \frac{x^2}{2(2\times 1+2)} + \frac{x^4}{2(2\times 1+2)\times 4(2\times 1+4)} - \cdots\right]$$

$$J_1(x) = \left(\frac{x}{2}\right)^1 \frac{1}{\Gamma(2)}\left[1 - \frac{x^2}{2(4)} + \frac{x^4}{2(4)\times 4(6)} - \cdots\right]$$

$\Gamma(2) = 1! = 1$ from equation (13)

$$J_1(x) = \left(\frac{x}{2}\right)\left[1 - \frac{x^2}{8} + \frac{x^4}{192} - \cdots\right] \qquad (19)$$

The Bessel's function of $J_1(x)$ for values ranging from x = 0, 1, 2, 3, 4…….12 were plotted and the graph is as shown below

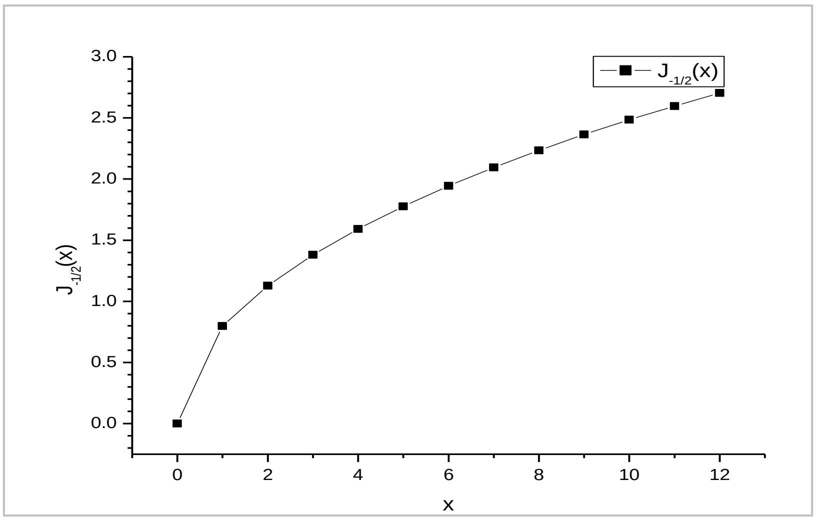

Figure 3: Bessel's function for $J_{-1/2}(x)$

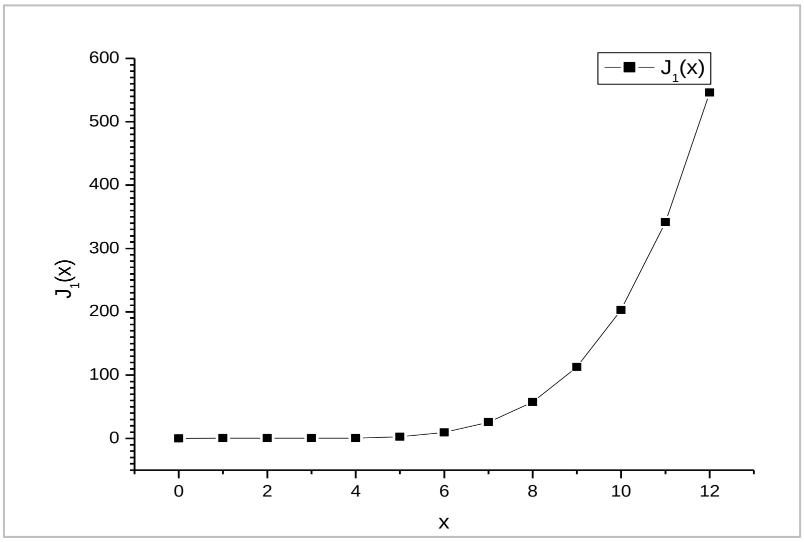

Figure 4: Bessel's function for $J_1(x)$

Figure 4 shows the values of $J_1(x)$ for x = 0, 1, 2, 3, 4…….12 based on the solution of the Bessel's function given in equation 19. The graph increase from $x = 0\ to\ 2$ then decreases at $x = 3$ and increases continuously from $x = 4\ to\ 12$.

Obtain the solution for the Bessel's function $J_{-1}(x)$

Here $n = -1$ from

$$J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2(2n+2)\times 4(2n+4)} - \cdots\right]$$

$$J_{-1}(x) = \left(\frac{x}{2}\right)^{-1} \frac{1}{\Gamma(-1+1)}\left[1 - \frac{x^2}{2(2\times -1+2)} + \frac{x^4}{2(2\times -1+2)\times 4(2\times -1+4)} - \cdots\right]$$

$$J_{-1}(x) = \left(\frac{x}{2}\right)^{-1} \frac{1}{\Gamma(0)}\left[1 - \frac{x^2}{2(0)} + \frac{x^4}{2(0)\times 4(2)} - \cdots\right]$$

$\Gamma(0) = \infty$

$$J_{-1}(x) = \left(\frac{2}{x}\right) \times \infty[1 - \infty + \infty - \cdots]$$

$J_{-1}(x) = 0$

Obtain the solution for the Bessel's function $J_{3/2}(x)$

Here $n = {}^{3}\!/_{2}$ from

$$J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2(2n+2)\times 4(2n+4)} - \cdots\right]$$

$$J_{3/2}(x) = \left(\frac{x}{2}\right)^{3/2} \frac{1}{\Gamma\left(\frac{3}{2}+1\right)}\left[1 - \frac{x^2}{2\left(2\times\frac{3}{2}+2\right)} + \frac{x^4}{2\left(2\times\frac{3}{2}+2\right)\times 4\left(2\times\frac{3}{2}+4\right)} - \cdots\right]$$

$$J_{3/2}(x) = \left(\frac{x}{2}\right)^{3/2} \frac{1}{\Gamma\left(\frac{5}{2}\right)}\left[1 - \frac{x^2}{2(5)} + \frac{x^4}{2(5)\times 4(7)} - \cdots\right]$$

Now evaluate $\Gamma\left(\frac{5}{2}\right)$

From $\Gamma(n) = \frac{\Gamma(n+1)}{n}$, this implies that $n + 1 = \frac{5}{2}$, therefore, $n = \frac{3}{2}$

Thus, $\Gamma\left(\frac{3}{2}\right) = \frac{\Gamma\left(\frac{3}{2}+1\right)}{\frac{3}{2}} = \frac{\Gamma\left(\frac{5}{2}\right)}{\frac{3}{2}}$

$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \times \Gamma\left(\frac{3}{2}\right)$ But $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \times \frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4}$$

$$J_{3/2}(x) = \left(\frac{x}{2}\right)^{3/2} \frac{1}{\frac{3\sqrt{\pi}}{4}}\left[1 - \frac{x^2}{10} + \frac{x^4}{280} - \cdots\right]$$

$$J_{3/2}(x) = \frac{1}{3}\sqrt{\left(\frac{2}{\pi}\right)}\, x^{3/2}\left[1 - \frac{x^2}{10} + \frac{x^4}{280} - \cdots\right] \quad (20)$$

The Bessel's function of $J_{3/2}(x)$ for values ranging from x = 0, 1, 2, 3, 4…….12 were plotted and the graph is as shown below:

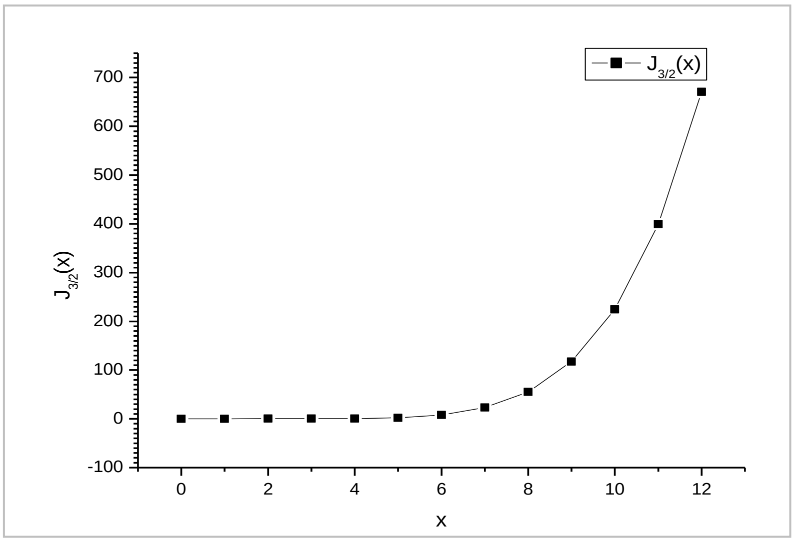

Figure 5: Bessel's function for $J_{3/2}(x)$

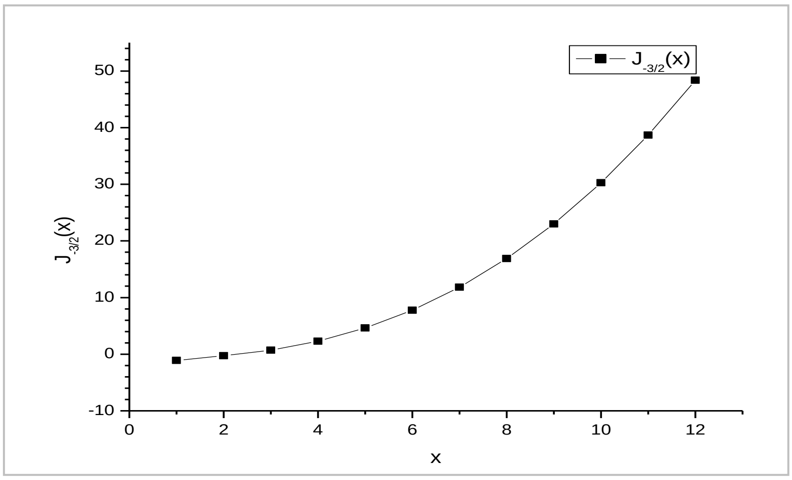

Figure 6: Bessel's function for $J_{-3/2}(x)$

Figure 5 shows the values of $J_{3/2}(x)$ for x = 0, 1, 2, 3, 4.......12 based on the solution of the Bessel's function given in equation 20. The graph shows that the values increases from $x = 0 \; to \; 12$.

Obtain the solution for the Bessel's function $\boldsymbol{J_{-3/2}(x)}$

Here $n = {}^{-3}/_{2}$ from

$$J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2(2n+2)\times 4(2n+4)} - \cdots\right]$$

$$J_{-3/2}(x) = \left(\frac{x}{2}\right)^{-3/2} \frac{1}{\Gamma\left(-\frac{3}{2}+1\right)}\left[1 - \frac{x^2}{2\left(2\times -\frac{3}{2}+2\right)} + \frac{x^4}{2\left(2\times -\frac{3}{2}+2\right)\times 4\left(2\times -\frac{3}{2}+4\right)} - \cdots\right]$$

$$J_{-3/2}(x) = \left(\frac{x}{2}\right)^{-3/2} \frac{1}{\Gamma\left(-\frac{1}{2}\right)}\left[1 - \frac{x^2}{2(-1)} + \frac{x^4}{2(-1)\times 4(1)} - \cdots\right]$$

Now evaluate $\Gamma\left(-\frac{1}{2}\right)$

From $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

Thus, $\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}}$

$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$

$$J_{-3/2}(x) = \left(\frac{x}{2}\right)^{-3/2} \frac{1}{-2\sqrt{\pi}}\left[1 + \frac{x^2}{2} - \frac{x^4}{8} + \cdots\right]$$

$$J_{-3/2}(x) = -\sqrt{\left(\frac{2}{\pi}\right)}\,(x)^{-3/2}\left[1 + \frac{x^2}{2} - \frac{x^4}{8} + \cdots\right] \qquad (21)$$

The Bessel's function of $J_{-3/2}(x)$ for values ranging from x = 0, 1, 2, 3, 4.......12 were plotted and the graph is as shown below

Figure 6 shows the values of $J_{-3/2}(x)$ for x = 0, 1, 2, 3, 4.......12 based on the solution of the Bessel's function given in equation 21. The figure showed that when $x = 0$, $J_{-3/2}(0) = \infty$, negative values were obtained at $x = 1 \; and \; 2$, the values then increases steadily from $x = 4 \; to \; 12$.

Obtain the solution for the Bessel's function $\boldsymbol{J_2(x)}$

Here $n = 2$ from

$$J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2(2n+2)\times 4(2n+4)} - \cdots\right]$$

$$J_2(x) = \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(2+1)}\left[1 - \frac{x^2}{2(2\times 2+2)} + \frac{x^4}{2(2\times 2+2)\times 4(2\times 2+4)} - \cdots\right]$$

$$J_2(x) = \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(3)}\left[1 - \frac{x^2}{2(6)} + \frac{x^4}{2(6)\times 4(8)} - \cdots\right]$$

$\Gamma(3) = 2! = 2$ from equation (13)

$$J_2(x) = \left(\frac{x}{2}\right)^2 \left(\frac{1}{2}\right)\left[1 - \frac{x^2}{12} + \frac{x^4}{384} - \cdots\right]$$

$$J_2(x) = \frac{1}{8}x^2\left[1 - \frac{x^2}{12} + \frac{x^4}{384} - \cdots\right] \qquad (22)$$

The Bessel's function of $J_2(x)$ for values ranging from x = 0, 1, 2, 3, 4.......12 were plotted and the graph is as shown below

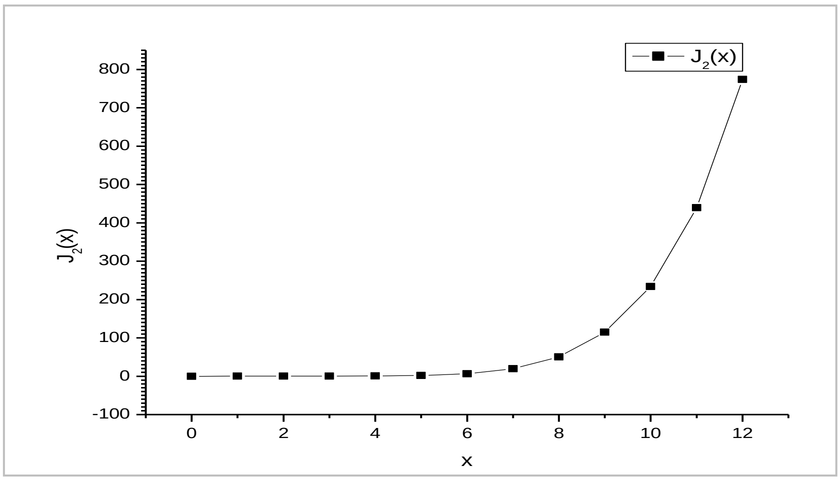

Figure 7: Bessel's function for $J_2(x)$

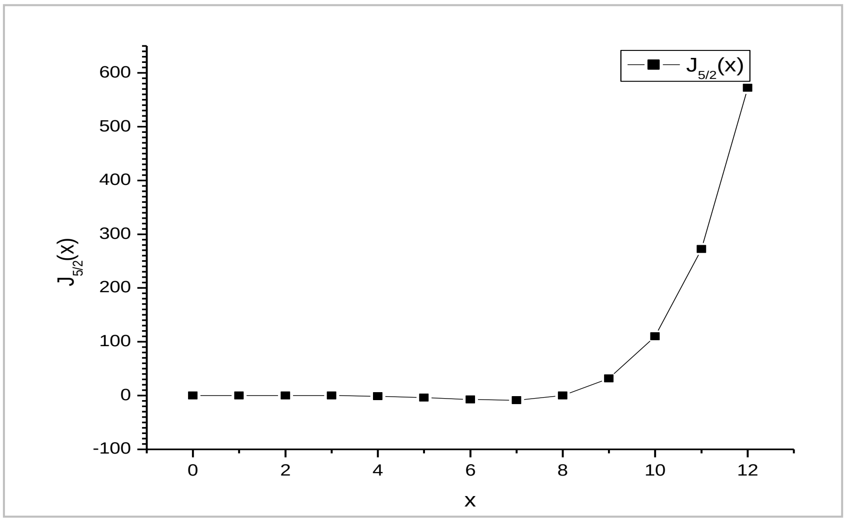

Figure 8: Bessel's function for $J_{5/2}(x)$

Figure 7 shows the values of $J_2(x)$ for x = 0, 1, 2, 3, 4…….12 based on the solution of the Bessel's function given in equation 22. The graph shows that the values increases from $x = 0\ to\ 12$.

Obtain the solution for the Bessel's function $\boldsymbol{J_{-2}(x)}$

Here $n = -2$ from

$$J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2(2n+2)\times 4(2n+4)} - \cdots\right]$$

$$J_{-2}(x) = \left(\frac{x}{2}\right)^{-2} \frac{1}{\Gamma(-2+1)}\left[1 - \frac{x^2}{2(2\times -2+2)} + \frac{x^4}{2(2\times -2+2)\times 4(2\times -2+4)} - \cdots\right]$$

$$J_{-2}(x) = \left(\frac{x}{2}\right)^{-2} \frac{1}{\Gamma(-1)}\left[1 - \frac{x^2}{2(-2)} + \frac{x^4}{2(-2)\times 4(0)} - \cdots\right]$$

$$\Gamma(-1) = \infty$$

$$J_{-2}(x) = \left(\frac{x}{2}\right)^{-2} \infty\left[1 + \frac{x^2}{4} + \frac{x^4}{0} - \cdots\right]$$

$$J_{-2}(x) = \left(\frac{x}{2}\right)^{-2} \infty\left[1 + \frac{x^2}{4} + \infty - \cdots\right]$$

$$J_{-2}(x) = 0$$

Obtain the solution for the Bessel's function $\boldsymbol{J_{5/2}(x)}$

Here $n = {}^5/_2$ from

$$J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2(2n+2)\times 4(2n+4)} - \cdots\right]$$

$$J_{5/2}(x) = \left(\frac{x}{2}\right)^{5/2} \frac{1}{\Gamma\left(\frac{5}{2}+1\right)}\left[1 - \frac{x^2}{2\left(2\times\frac{5}{2}+2\right)} + \frac{x^4}{2\left(2\times\frac{5}{2}+2\right)\times 4\left(2\times\frac{5}{2}+4\right)} - \cdots\right]$$

$$J_{5/2}(x) = \left(\frac{x}{2}\right)^{5/2} \frac{1}{\Gamma\left(\frac{7}{2}\right)}\left[1 - \frac{x^2}{2(7)} + \frac{x^4}{2(7)\times 4(9)} - \cdots\right]$$

Now evaluate $\Gamma\left(\frac{7}{2}\right)$

From $\Gamma(\text{n}) = \frac{\Gamma(\text{n}+1)}{\text{n}}$, this implies that $n + 1 = \frac{7}{2}$, therefore, $n = \frac{5}{2}$

Thus, $\Gamma\left(\frac{5}{2}\right) = \frac{\Gamma\left(\frac{5}{2}+1\right)}{\frac{5}{2}} = \frac{\Gamma\left(\frac{7}{2}\right)}{\frac{5}{2}}$

$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\times\Gamma\left(\frac{5}{2}\right)$ But $\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$

$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\times\frac{3\sqrt{\pi}}{4} = \frac{15\sqrt{\pi}}{8}$$

$$J_{5/2}(x) = \left(\frac{x}{2}\right)^{5/2} \frac{1}{\frac{15\sqrt{\pi}}{8}}\left[1 - \frac{x^2}{7} + \frac{x^4}{504} - \cdots\right]$$

$$J_{5/2}(x) = \frac{1}{15}\sqrt{\left(\frac{2}{\pi}\right)}\, x^{5/2}\left[1 - \frac{x^2}{7} + \frac{x^4}{504} - \cdots\right] \quad (23)$$

The Bessel's function of $J_{5/2}(x)$ for values ranging from x = 0, 1, 2, 3, 4…….12 were plotted and the graph is as shown below:

Figure 8 shows the values of $J_{5/2}(x)$ for x = 0, 1, 2, 3, 4…….12 based on the solution of the Bessel's function given in equation 23. The figure shows that the values increases from $x = 0$ *to* 2 and decreases negatively from $x = 3$ *to* 8 and increases steadily from $x = 9$ *to* 12.

Obtain the solution for the Bessel's function $J_{-5/2}(x)$

Here $n = {}^{-5}/_{2}$ from

$$J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2(2n+2)\times 4(2n+4)} - \cdots\right]$$

$$J_{-5/2}(x) = \left(\frac{x}{2}\right)^{-5/2} \frac{1}{\Gamma\left(-\frac{5}{2}+1\right)}\left[1 - \frac{x^2}{2\left(2\times -\frac{5}{2}+2\right)} + \frac{x^4}{2\left(2\times -\frac{5}{2}+2\right)\times 4\left(2\times -\frac{5}{2}+4\right)} - \cdots\right]$$

$$J_{-5/2}(x) = \left(\frac{x}{2}\right)^{-5/2} \frac{1}{\Gamma\left(-\frac{3}{2}\right)}\left[1 - \frac{x^2}{2(-3)} + \frac{x^4}{2(-3)\times 4(-1)} - \cdots\right]$$

Now evaluate $\Gamma\left(-\frac{3}{2}\right)$

From $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

Thus, $\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}}$

$\Gamma\left(-\frac{3}{2}\right) = -\frac{2}{3}\,\Gamma\left(-\frac{1}{2}\right)$

But $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$

$\Gamma\left(-\frac{3}{2}\right) = -\frac{2}{3} \times -2\sqrt{\pi}$ Therefore

$\Gamma\left(-\frac{3}{2}\right) = \frac{4\sqrt{\pi}}{3}$

$$J_{-5/2}(x) = \left(\frac{x}{2}\right)^{-5/2} \frac{1}{\frac{4\sqrt{\pi}}{3}}\left[1 + \frac{x^2}{6} - \frac{x^4}{24} + \cdots\right]$$

$$J_{-5/2}(x) = 3x^{-5/2}\sqrt{\left(\frac{2}{\pi}\right)}\left[1 + \frac{x^2}{6} - \frac{x^4}{24} + \cdots\right] \quad (24)$$

The Bessel's function of $J_{-5/2}(x)$ for values ranging from x = 0, 1, 2, 3, 4…….12 were plotted and the graph is as shown below

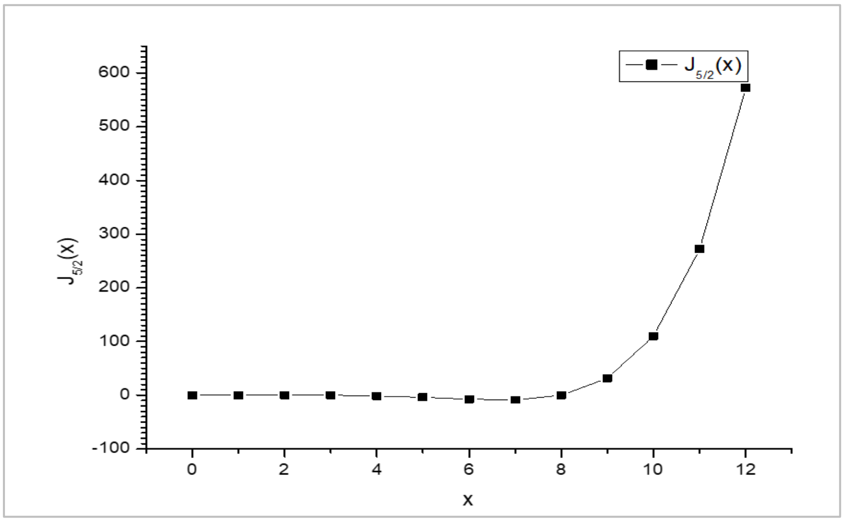

Figure 8: Bessel's function for $J_{5/2}(x)$

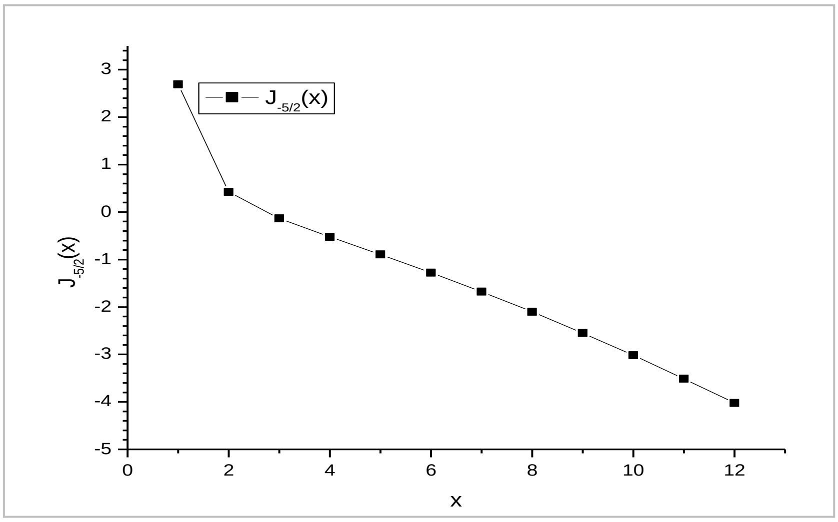

Figure 9: Bessel's function for $J_{-5/2}(x)$

Figure 9 shows the values of $J_{-5/2}(x)$ for x = 0, 1, 2, 3, 4…….12 based on the solution of the Bessel's function given in equation 24. The figure depicts that when $x = 0, J_{-5/2}(0) = \infty$, the values decrease from $x =$ 1 *to* 2 and decreases further but negatively from $x = 3$ *to* 12

Obtain the solution for the Bessel's function $J_3(x)$

Here $n = 3$ from

$$J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2(2n+2)\times 4(2n+4)} - \cdots\right]$$

$$J_3(x) = \left(\frac{x}{2}\right)^3 \frac{1}{\Gamma(3+1)}\left[1 - \frac{x^2}{2(2\times 3+2)} + \frac{x^4}{2(2\times 3+2)\times 4(2\times 3+4)} - \cdots\right]$$

$$J_3(x) = \left(\frac{x}{2}\right)^3 \frac{1}{\Gamma(4)}\left[1 - \frac{x^2}{2(8)} + \frac{x^4}{2(8)\times 4(10)} - \cdots\right]$$

$\Gamma(4) = 3! = 6$ from equation (13)

$$J_3(x) = \left(\frac{x}{2}\right)^3 \left(\frac{1}{6}\right)\left[1 - \frac{x^2}{16} + \frac{x^4}{640} - \cdots\right]$$

$$J_3(x) = \frac{1}{48}x^3\left[1 - \frac{x^2}{16} + \frac{x^4}{640} - \cdots\right] \quad (25)$$

The Bessel's function of $J_3(x)$ for values ranging from x = 0, 1, 2, 3, 4.......12 were plotted and the graph is as shown below

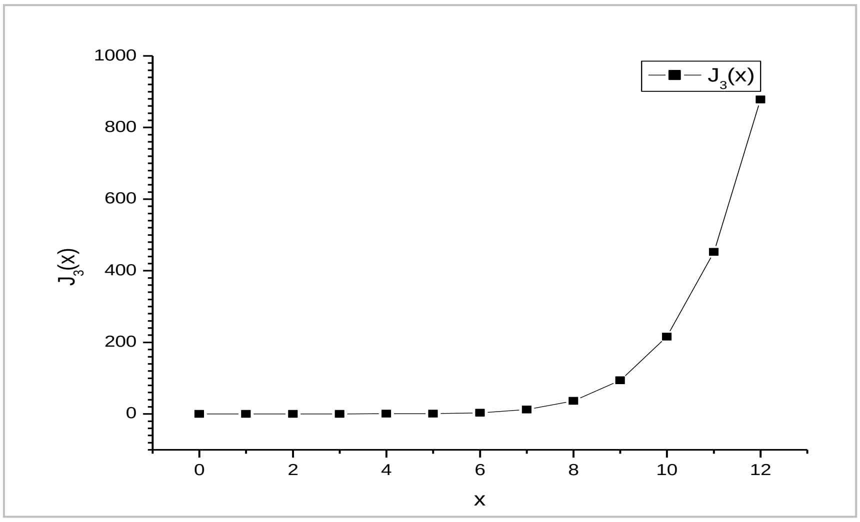

Figure 10: Bessel's function for $J_3(x)$

Figure 10 shows the values of $J_3(x)$ for x = 0, 1, 2, 3, 4.......12 based on the solution of the Bessel's function given in equation 25. The graph shows that the values increases from $x = 0$ *to* 12.

Obtain the solution for the Bessel's function $J_{-3}(x)$

Here $n = -3$ from

$$J_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2(2n+2)\times 4(2n+4)} - \cdots\right]$$

$$J_{-3}(x) = \left(\frac{x}{2}\right)^{-3} \frac{1}{\Gamma(-3+1)}\left[1 - \frac{x^2}{2(2\times -3+2)} + \frac{x^4}{2(2\times -3+2)\times 4(2\times -3+4)} - \cdots\right]$$

$$J_{-3}(x) = \left(\frac{x}{2}\right)^{-3} \frac{1}{\Gamma(-2)}\left[1 - \frac{x^2}{2(-4)} + \frac{x^4}{2(-4)\times 4(-2)} - \cdots\right]$$

$\Gamma(-2) = \infty$

$$J_{-3}(x) = \left(\frac{x}{2}\right)^{-3} \infty \left[1 + \frac{x^2}{8} + \frac{x^4}{64} - \cdots\right]$$

$J_{-3}(x) = 0$

4. Conclusion

This paper has addressed the issue of simplification of the solutions of Bessel's function of the first kind of order n through a comprehensive approach by employing the gamma function, Maclaurin's series and basic laws of indices. Thirteen solutions of Bessel's functions of order n were determined and the graphs showing the pattern of variation for eleven solutions were investigated. The solutions of the various values of n are given in equations (16), (17), (18), (19), (20), (21), (22) (23), (24) and (25). The results showed that the solutions, $J_{-1}(x) = J_{-2}(x) = J_{-3}(x) = 0$ for x = 0, 1, 2, 3, 4.......12; the results showed that there is a slight decrease in the values of $J_0(x)$ from x = 0 to 3 which then increase steadily from x = 4 to 12, the results for $J_0(x)$ further revealed that the solution is unity at x = 0 and 4. For $J_{1/2}(x)$, $J_{3/2}(x)$, $J_2(x)$ and $J_3(x)$ the values increases steadily from x = 0 to 12. The values of $J_{-1/2}(x)$ increases steadily from x = 0 to 12 in the form of a parabola. For $J_1(x)$, the graph increase from x = 0 to 2 then decreases at x = 3 and increases continuously from x = 4 to 12. The figure depicting $J_{-3/2}(x)$ showed that when x = 0, $J_{-3/2}(0) = \infty$, negative values were obtained at x = 1 and 2, the values then increases steadily from x = 4 to 12. The figure for $J_{5/2}(x)$ shows that the values increases from x = 0 to 2 and decreases negatively from x = 3 to 8 and increases steadily from x = 9 to 12. The figure for $J_{-5/2}(x)$ depicts that when x = 0, $J_{-5/2}(0) = \infty$, the values decrease from x = 1 to 2 and decreases further but negatively from x = 3 to 12.

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