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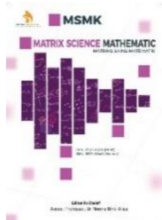

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RESEARCH ARTICLE

A NEW SOFT SET OPERATION: COMPLEMENTARY SOFT BINARY PIECEWISE PLUS (+) OPERATION

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ABSTRACT

Soft set theory, introduced by Molodtsov, is as an important mathematical tool to deal with uncertainty and it has been applied to many fields both as theoretical and application aspects. Since 1999, different kinds of soft set operations has been defined and used in various types. In this paper, we define a new kind of soft set operation called, complementary soft binary piecewise plus operation and investigate its basic algebraic properties. Moreover, by examining the distribution rules, we contribute to the soft set literature by obtaining the relationships between this new soft set operation and some other types of soft set operations such as soft restrcited, extended, soft binary piecewise, and complementary soft binary piecewise operations. As proposing new soft set operations and obtaining their algebraic properties and implementations opens up new avenues for handling parametric data challenges in terms of decision-making methods and new cryptography approaches, and analyzing the algebraic structure of soft sets from the standpoint of new soft set operations offers a thorough understanding of the algebraic structure of soft sets, this paper can be regarded as both theoretical and application study.

KEYWORDS

Soft sets, Soft Set Operations, Conditional Complements

1. INTRODUCTION

There are three well-known basic theories that we can consider as a mathematical tool to deal with uncertainties in the problems of many fields such as economics, environmental and health sciences, engineering, which prevents us from using classical methods to solve the problems successfully. These well-known theories are Probability Theory, Fuzzy Set Theory and Interval Mathematics. But since all these theories have their own shortcomings, Molodtsov (1999) introduced Soft Set Theory as a mathematical tool to overcome these uncertainties. Since then, this theory was applied to many fields including information systems, decision making (Yang and Yao, 2020; Petchimuthu et al., 2020; Zorlutuna, 2021), optimization theory, game theory, operations research, measurement theory, soft equality relation (Alshamı and Mohammed, 2020; Ali et al., 2022) and so on. Studies on fuzzy modeling such as Linear Diophantine Fuzzy Sets (Riaz and Hashimi, 2019; Ayub et al.,2021), Linear Diophantine Fuzzy aggregation operators (Riaz et al., 2023), Spherical Linear Diophantine Fuzzy Sets (Riaz et al., 2021) etc. are also some top recent topics as novel mathematical approachs to model vagueness and uncertainty in decision-making problems. First contributions as regards soft set operations were made by Maji et al., 2003 and Pei and Miao, 2005. After then, Ali et al. (2009) introduced and examined several soft set operations such as restricted and extended soft set operations. Sezgin and Atagün (2011) discussed the basic properties of soft set operations and illustrated all the interconnections of soft set operations with each others. They also defined the notion of restricted symmetric difference of soft sets and investigate its properties. Sezgin et al. (2019) defined new soft set operation called extended difference of soft sets and Stojanovic (2021) introduced extended symmetric difference of soft sets and investigated its properties. When the studies are examined, we see that the operations in soft set theory proceed under two main headings, as restricted soft set operations and extended soft set operations.

Çağman (2021) proposed inclusive complement and exclusive complement and explored the relationships between them. By the inspiration of this study, some new complements of sets were defined by Sezgin et al. (2023c). They also transferred these complements to soft set theory, and Aybek (2024) defined some new restricted soft set operations and extended soft set operations. Demirci, 2024; Sarıalioğlu, 2024; Akbulut, 2024 defined a new type of extended operation by changing the form of extended soft set operations using the complement at the first and second row of the piecewise function of extended soft set operations and studied the basic properties of them in detail. Moreover, a new type of soft difference operations was defined by Eren and Çalışıcı (2019) and by being inspired this study Yavuz (2024) defined some new soft set opeations, which they call soft binary piecewise operations and they studied their basic properties in detail, too. Also, Sezgin and Sarıalioğlu, 2023; Sezgin and Demirci, 2023; Sezgin and Yavuz, 2023; Sezgin and Aybek, 2023; Sezgin et al. 2023a, Sezgin et al. 2023b continued their work on soft set operations by defining a new type of soft binary piecewise operation. They changed the form of soft binary piecewise operation by using the complement at the first row of the soft binary piecewise operations.

The aim of this study is to contribute to the literature of soft set theory by describing a new soft set operation which we call "complementary soft binary piecewise plus operation". For this purpose, definiton of the operation and its example are given, the algebraic properties, such as closure, associativity, unit and inverse element and abelian property of this new operation are examined in detail. Especially it is aimed to contribute to the soft set literature by obtaining the distributions of the complementary soft binary piecewise plus operation over as extended and soft binary piecewise intersection and union, complementary extended and complementary soft binary piecewise theta and star and restricted intersection, restricted union, restricted theta and restricted star.

2. Preliminaries

In this section, some basic concepts as regard soft set theory are given.

Definition 2.1. Let U be the universal set, E be the parameter set, P(U) be the power set of U and $L \subseteq E$. A pair (X, L) is called a soft set over U where X is a set-valued function such that $X: L \rightarrow P(U)$. (Molodtsov, 1999)

Throughout this paper, the set of all the soft sets over U is designated by $S_E(U)$. Let A be a fixed subset of E and $S_A(U)$ be the collection of all soft sets over U with the fixed parameter set A. Clearly $S_A(U)$ is a subset of $S_E(U)$.

Definition 2.2. (X, L) is called a relative null soft set (with respect to the parameter set L), denoted by $\emptyset_L$, if $D(ö) = \emptyset$ for all ö∈Ʊ and (D, Ʊ) is called a relative whole soft set (with respect to the parameter set Ʊ), denoted by $U_Ʊ$ if $D(ö) = U$ for all ö∈Ʊ. The relative whole soft set U_E with respect to the universe set of parameters E is called the absolute soft set over U. We shall denote by $\emptyset_\emptyset$ the unique soft set over U with an empty parameter set, which is called the empty soft set over U. Note that by $\emptyset_\emptyset$ and by $\emptyset_A$ are different soft sets over U (Ali et.al., 2009).

Definition 2.3. For two soft sets (X, L) and (G, Ʊ), we say that (X, L) is a soft subset of (G, Ʊ) and it is denoted by $(X, L) \widetilde{\subseteq} (G, Ʊ)$, if $L \subseteq Ʊ$ and $X(ö) \subseteq G(ö), \forall ö \in L$. Two soft sets (X, L) and (G, Ʊ) are said to be soft equal if (X, L) is a soft subset of (G, Ʊ) and (G, Ʊ) is a soft subset of (X, L) (Pei and Miao, 2005).

Definition 2.4. The relative complement of a soft set (X, L), denoted by $(X, L)^r$, is defined by $(X, L)^r = (X^r, L)$, where $X^r: L \rightarrow P(U)$ is a mapping given by $(X, L)^r = U - X(ö)$ for all $ö \in L$ (Ali et.al., 2009). From now on, $U - X(ö)=[X(ö)]'$ will be designated by $X'(t)$ for the sake of designation.

Çağman (2021) defined two conditional complements of sets as a new concept of set theory, that is, inclusive complement and exclusive complement. For the ease of illustration, we show these complements as + and θ, respectively. These complements are binary operations and are defined as follows: Let P and R be two subsets of U. R-inclusive complement of P is defined by, $P+R=P'\cup R$ and J-Exlusive complement of P is defined by $P\theta R = P'\cap R'$. Here, U refers to a universe, P' is the complement of P over U. For more information, we refer to Çağman (2021).

Sezgin et al. (2023c) examined the relations between these two complements in detail and they also introduced such new three complements as binary operations of sets as follows: Let P and R be two subsets of U. Then, $P*R=P'\cup R'$, $P\gamma R= P'\cap R$, $P\lambda R=Ʊ\cup R'$ (Sezgin et al., 2023c). Aybek (2023) conveyed these set operations to soft sets and they defined some new restricted and extended soft set operations and examined their properties.

As a summary for soft set operations, we can categorize all types of soft set operations as follows: Let "∇" be used to represent the set operations (i.e., here ∇ can be ∩, ∪,\, Δ, +,θ, *, λ,γ), then restricted operations, extended operations, complementary extended operations, soft binary piecewise operations, complementary soft binary piecewise operations are defined in soft set theory as follows:

Definition 2.5. Let (X, L) and (G, Ʊ) be soft sets over U. The restricted ∇ operation of (X, L) and (G, Ʊ) is the soft set (Y, S), denoted by $(X, L)\nabla_R(G, Ʊ) = (Y, S)$, where $S = L \cap Ʊ \neq \emptyset$ and $\forall ö \in S$, $Y(t) = X(ö) \nabla G(ö)$. Here note that if $K \cap T = \emptyset$, then $(W, K)\Theta_R(Ş, T) = \emptyset_\emptyset$ [17]. (Ali et. al., 2009; Sezgin and Atagün, 2011; Aybek, 2023)

Definition 2.6. Let (X, L) and (G, Ʊ) be soft sets over U. The extended ∇ operation of (X, L) and (G, Ʊ) is the soft set (Y, S), denoted by $(X, L)\nabla_\varepsilon(G, Ʊ) = (Y, S)$, where $S = L \cup Ʊ$ and $\forall ö \in S$,

$$Y(ö) = \begin{cases} X(ö), & ö \in L - Ʊ, \\ G(ö), & ö \in Ʊ - L, \\ X(ö) \nabla G(ö), & ö \in L \cap Ʊ. \end{cases}$$

(Maji et.al., 2003; Ali et. al., 2009; Sezgin et. al., 2019; Stojanovic, 2021; Aybek, 2024)

Definition 2.7. Let (X, L) and (G, Ʊ) be soft sets over U. The complementary extended ∇ opetation of (X, L) and (G, Ʊ) is the soft set (Y, S), denoted by $(X, L) \overset{*}{\nabla}_\varepsilon (G, Ʊ) = (Y, S)$, where $S = L \cup Ʊ$ and $\forall ö \in S$,

$$Y(ö) = \begin{cases} X'(ö), & ö \in L - Ʊ \\ G'(ö), & ö \in Ʊ - L, \\ X(ö) \nabla G(ö), & ö \in L \cap Ʊ. \end{cases}$$

(Sarıalioğlu, 2024; Demirci, 2024; Akbulut, 2024)

Definition 2.8. Let (X, L) and (G, Ʊ) be soft sets over U. The soft binary piecewise ∇ operation of (X, L) and (G, Ʊ) is the soft set (Y, S), denoted by, $(X, L) \underset{\nabla}{\sim} (G, Ʊ) = (Y, L)$, where ∀ö∈L,

$$Y(t)= \begin{cases} X(ö), & ö\in L-Ʊ \\ X(ö) \nabla G(ö), & ö\in L\cap Ʊ \end{cases}$$

(Eren and Çalışıcı, 2019; Yavuz, 2024)

Definition 2.9. Let (X, L) and (G, Ʊ) be soft sets over U. The complementary soft binary piecewise ∇ operation of (X, L) and (G, Ʊ) is the soft set (Y, L), denoted by, $(X, L) \underset{\nabla}{\overset{*}{\sim}} (G, Ʊ) = (Y, L)$, where ∀ö∈L;

$$Y(ö)= \begin{cases} X'(ö), & ö\in L-Ʊ \\ X(ö) \nabla G(ö), & ö\in L\cap Ʊ \end{cases}$$

(Sezgin and Sarıalioğlu, 2024; Sezgin and Demirci, 2023; Sezgin and Aybek, 2023; Sezgin et al., 2023a, Sezgin et al., 2023b; Sezgin and Yavuz, 2023; Sezgin and Çağman, 2024)

3. Complementary Soft Binary Piecewise Plus (+) Operation and its Properties

Definition 3.1. Let (X, L) and (G, Ʊ) be soft sets over U. The complemetary soft binary piecewise plus (+) operation of (X, L) and (G, Ʊ) is the soft set (H, L), denoted by, $(X, L) \underset{+}{\overset{*}{\sim}} (G, Ʊ) = (H, L)$, where ∀ö∈L,

$$H(ö)= \begin{cases} X'(ö), & ö\in L-Ʊ \\ X'(ö)\cup G(ö), & ö\in L\cap Ʊ \end{cases}$$

Example 3.2. Let $E=\{e_1,e_2,e_3,e_4\}$ be the parameter set $L=\{e_1, e_3\}$ and $P=\{e_2, e_3, e_4\}$ be the subsets of E and $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the initial universe set. Assume that (X,L) and (G,Ʊ) are the soft sets over U defined as follows:

$(X,L) =\{(e_1, \{h_2, h_5\}), (e_3,\{h_1,h_2,h_5\})\}$

$(G,Ʊ)=\{(e_2,\{h_1 ,h_4,h_5\}), (e_3,\{h_2,h_3,h_4\}),(e_4,\{ h_3 ,h_5\}) \}$.

Let $(X, L) \underset{+}{\overset{*}{\sim}} (G,Ʊ)=(H,L)$. Then,

$$H(ö)= \begin{cases} X'(ö), & ö\in L-Ʊ \\ X'(ö)\cup G(ö), & ö\in L\cap Ʊ \end{cases}$$

Since $L=\{e_1, e_3\}$ and $L-Ʊ=\{e_1\}$, so $H(e_1) = X'(e_1)=\{ h_1 ,h_3,h_4\}$. And since $L\capƱ=\{e_3\}$ so $H(e_3)= X'(e_3) \cup G(e_3)=\{h_3,h_4\}\cup\{h_2,h_3,h_4\}=\{h_2, h_3,h_4\}$. Thus,

$(X, L) \underset{+}{\overset{*}{\sim}} (G,Ʊ)=\{(e_1,\{h_1 ,h_3,h_4\}), (e_3,\{h_2, h_3,h_4\})\}$.

3.1. Algebraic Properties

1) The set $S_E(U)$ is closed under the operation $\underset{+}{\overset{*}{\sim}}$. That is, when (X,L) and

$(G,\mho)$ are two soft sets over U, then so is $(X,L)\underset{+}{\overset{*}{\sim}}(G,\mho)$.

Proof: It is clear that $\underset{+}{\overset{*}{\sim}}$ is a binary operation in $S_E(U)$. Hence, the set $S_E(U)$ is closed under the operation $\underset{+}{\overset{*}{\sim}}$. The set $S_A(U)$, where A is a fixed subset of E, is closed under the operation $\underset{+}{\overset{*}{\sim}}$, too.

2) $[(X,L)\underset{+}{\overset{*}{\sim}}(G,L)]\underset{+}{\overset{*}{\sim}}(H,L)\neq(X,L)\underset{+}{\overset{*}{\sim}}[(G,L)\underset{+}{\overset{*}{\sim}}(H,L)]$

Proof: Let $(X,L)\underset{+}{\overset{*}{\sim}}(G,L)=(T,L)$, where $\forall\ddot{o}\in L$;

$$T(\ddot{o})=\begin{cases}X'(\ddot{o}), & \ddot{o}\in L-L=\emptyset\\ X'(\ddot{o})\cup G(\ddot{o}), & \ddot{o}\in L\cap L=L\end{cases}$$

Let $(T,L)\underset{+}{\overset{*}{\sim}}(H,L)=(M,L)$, where $\forall\ddot{o}\in L$;

$$M(\ddot{o})=\begin{cases}T'(\ddot{o}), & \ddot{o}\in L-L=\emptyset\\ T'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap L=L\end{cases}$$

Thus,

$$M(\ddot{o})=\begin{cases}T'(\ddot{o}), & \ddot{o}\in L-L=\emptyset\\ [F(\ddot{o})\cap G'(\ddot{o})]\cup H(\ddot{o}), & \ddot{o}\in L\cap L=L\end{cases}$$

Let $(G,L)\underset{+}{\overset{*}{\sim}}(H,L)=(R,L)$, where $\forall\ddot{o}\in L$;

$$R(\ddot{o})=\begin{cases}G'(\ddot{o}), & \ddot{o}\in L-L=\emptyset\\ G'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap L=L\end{cases}$$

Let $(X,L)\underset{+}{\overset{*}{\sim}}(R,L)=(N,L)$, where $\forall\ddot{o}\in L$;

$$N(\ddot{o})=\begin{cases}X'(\ddot{o}), & \ddot{o}\in L-L=\emptyset\\ X'(\ddot{o})\cup R(\ddot{o}), & \ddot{o}\in L\cap L=L\end{cases}$$

Thus,

$$N(\ddot{o})=\begin{cases}X'(\ddot{o}), & \ddot{o}\in L-L=\emptyset\\ X'(\ddot{o})\cup[G(\ddot{o})\cap H'(\ddot{o})], & \ddot{o}\in L\cap L=L\end{cases}$$

It is seen that $M\neq N$.

That is, for the soft sets whose parameter set are the same, the operation $\underset{+}{\overset{*}{\sim}}$ is not associative. Moreover, we have the following:

3) $[(X,L)\underset{+}{\overset{*}{\sim}}(G,\mho)]\underset{+}{\overset{*}{\sim}}(H,C)\neq(X,L)\underset{+}{\overset{*}{\sim}}[(G,\mho)\underset{+}{\overset{*}{\sim}}(H,C)]$

Proof: Let $(X,L)\underset{+}{\overset{*}{\sim}}(G,\mho)=(T,L)$, where $\forall\ddot{o}\in L$;

$$T(\ddot{o})=\begin{cases}X'(\ddot{o}), & \ddot{o}\in L-\mho\\ X'(\ddot{o})\cup G(\ddot{o}), & \ddot{o}\in L\cap\mho\end{cases}$$

Let $(T,L)\underset{+}{\overset{*}{\sim}}(H,C)=(M,L)$, where $\forall\ddot{o}\in L$;

$$M(\ddot{o})=\begin{cases}T'(\ddot{o}), & \ddot{o}\in L-C\\ T'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap C\end{cases}$$

Thus,

$$M(\ddot{o})=\begin{cases}X(\ddot{o}), & \ddot{o}\in(L-\mho)-C=L\cap\mho'\cap C'\\ X(\ddot{o})\cap G'(\ddot{o}), & \ddot{o}\in(L\cap\mho)-C=L\cap\mho\cap C'\\ X'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(L-\mho)\cap C=L\cap\mho'\cap C\\ [X(\ddot{o})\cap G'(\ddot{o})]\cup H(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap C=L\cap\mho\cap C\end{cases}$$

Let $(G,\mho)\underset{+}{\overset{*}{\sim}}(H,C)=(K,\mho)$, where $\forall\ddot{o}\in\mho$;

$$K(\ddot{o})=\begin{cases}G'(\ddot{o}), & \ddot{o}\in\mho-C\\ G'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in\mho\cap C\end{cases}$$

Let $(X,L)\underset{+}{\overset{*}{\sim}}(K,\mho)=(S,L)$, where $\forall\ddot{o}\in L$;

$$S(\ddot{o})=\begin{cases}X'(\ddot{o}), & \ddot{o}\in L-\mho\\ X'(\ddot{o})\cup K(\ddot{o}), & \ddot{o}\in L\cap\mho\end{cases}$$

Thus,

$$S(\ddot{o})=\begin{cases}X'(\ddot{o}), & \ddot{o}\in L-\mho\\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in L\cap(\mho-C)=L\cap\mho\cap C'\\ X'(\ddot{o})\cup[G'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in L\cap(\mho\cap C)=L\cap\mho\cap C\end{cases}$$

Here let's handle $\ddot{o}\in L-\mho$ in the second equation of the first line. Since $L-\mho=L\cap\mho'$, if $\ddot{o}\in\mho'$, then $\ddot{o}\in C-\mho$ or $\ddot{o}\in(\mho\cup C)'$. Hence, if $\ddot{o}\in L-\mho$, then $\ddot{o}\in L\cap\mho'\cap C'$ or $\ddot{o}\in L\cap\mho'\cap C$. Thus, it is seen that $M\neq S$. That is, the operation $\underset{+}{\overset{*}{\sim}}$ is not associative on the set $S_E(U)$.

4) $(X,L)\underset{+}{\overset{*}{\sim}}(G,\mho)\neq(G,\mho)\underset{+}{\overset{*}{\sim}}(X,L)$.

Proof: Let $(X,L)\underset{+}{\overset{*}{\sim}}(G,\mho)=(H,L)$. Then, $\forall\ddot{o}\in L$;

$$H(\ddot{o})=\begin{cases}X'(\ddot{o}), & \ddot{o}\in L-\mho\\ X'(\ddot{o})\cup G(\ddot{o}), & \ddot{o}\in L\cap\mho\end{cases}$$

Let $(G,Ʊ)\overset{*}{\underset{+}{\sim}}(X,L)=(T,Ʊ)$. Then, $\forall ö\in Ʊ$;

$$T(ö)=\begin{cases} G'(ö), & ö\in Ʊ-L \\ G'(ö)\cup X(ö), & ö\in Ʊ\cap L \end{cases}$$

Here, while the parameter set of the soft set of the left hand side is L; the parameter set of the soft set of the right hand side is Ʊ. Thus, by the definition of soft equality

$$(X,L)\overset{*}{\underset{+}{\sim}}(G,Ʊ)\neq(G,Ʊ)\overset{*}{\underset{+}{\sim}}(X,L).$$

Hence, the operation $\overset{*}{\underset{+}{\sim}}$ is not commutative in the set $S_E(U)$,.

5) $(X,L)\overset{*}{\underset{+}{\sim}}(X,L)=U_L$.

Proof: Let $(X,L)\overset{*}{\underset{+}{\sim}}(X,L)=(H,L)$. Then, $\forall ö\in L$;

$$H(ö)=\begin{cases} X'(ö), & ö\in L-L=\emptyset \\ X'(ö)\cup X(ö), & ö\in L\cap L=L \end{cases}$$

Here, $\forall ö\in L$, $H(ö)=X'(ö)\cup X(ö)=U$, hence $(H,L)=U_L$.

That is, the operation $\overset{*}{\underset{+}{\sim}}$ does not have idempotency property on the set $S_E(U)$.

6) $(X,L)\overset{*}{\underset{+}{\sim}}U_L=U_L$

Proof: Let $U_L=(T,L)$. Hence, $\forall ö\in L$, $T(ö)=U$. Let $(X,L)\overset{*}{\underset{+}{\sim}}(T,L)=(H,L)$. Hence, $\forall ö\in L$;

$$H(ö)=\begin{cases} X'(ö), & ö\in L-L=\emptyset \\ X'(ö)\cup T(ö), & ö\in L\cap L=L \end{cases}$$

Hence, $\forall ö\in L$ $H(ö)=X'(ö)\cup T(ö)=X'(ö)\cup U=U$, so $(H,L)=U_L$.

Note that, for the soft sets whose parameter set is L, U_L is the right-absorbing element for the operation $\overset{*}{\underset{+}{\sim}}$.

7) $U_L\overset{*}{\underset{+}{\sim}}(X,L)=(X,L)$.

Proof: Let $U_A=(T,L)$. Hence, $\forall ö\in L$, $T(ö)=U$. Let $(T,L)\overset{*}{\underset{+}{\sim}}(X,L)=(H,L)$, so $\forall ö\in L$,

$$H(ö)=\begin{cases} T'(ö), & ö\in L-L=\emptyset \\ T'(ö)\cup X(ö), & ö\in L\cap L=L \end{cases}$$

Hence $\forall ö\in L$, $H(ö)=T'(ö)\cup X(ö)=\emptyset\cup X(ö)=X(ö)$, so $(H,L)=(X,L)$.

Note that, for the soft sets whose parameter set is L, U_L is the left-identity element for the operation $\overset{*}{\underset{+}{\sim}}$.

8) $(X,L)\overset{*}{\underset{+}{\sim}}U_E=U_L$.

Proof: Let $U_E=(T,E)$. Hence, $\forall ö\in E$, $T(ö)=U$. Let $(X,L)\overset{*}{\underset{+}{\sim}}(T,E)=(H,L)$, then $\forall ö\in L$;

$$H(ö)=\begin{cases} X'(ö), & ö\in L-E=\emptyset \\ X'(ö)\cup T(ö), & ö\in L\cap E=L \end{cases}$$

Since, $X'(ö)\cup T(ö)=X'(ö)\cup U=U$,

$$H(ö)=\begin{cases} X'(ö), & ö\in L-E=\emptyset \\ U, & ö\in L\cap E=L \end{cases}$$

$H(ö)=U$ for all $ö\in L$. Thus, $(H,L)=U_L$.

9) $U_E\overset{*}{\underset{+}{\sim}}(X,L)=(X,L)$.

Proof: Let $U_E=(T,E)$ .Thus, $\forall ö\in E$, $T(ö)=U$. Let $(T,E)\overset{*}{\underset{+}{\sim}}(X,L)=(H,E)$, so $\forall ö\in E$,

$$H(ö)=\begin{cases} T'(ö), & ö\in E-L \\ T'(ö)\cup X(ö), & ö\in E\cap L \end{cases}$$

Hence,

$$H(ö)=\begin{cases} \emptyset, & ö\in E-L=L' \\ X(ö), & ö\in E\cap L \end{cases}$$

Thus, $\forall ö\in L$, $H(ö)=X(ö)$, so $(H,L)=(X,L)$.

Note that, U_E is the left-identity element for the operation $\overset{*}{\underset{+}{\sim}}$ in the set $S_E(U)$.

10) $(X,L)\overset{*}{\underset{+}{\sim}}\emptyset_L=(X,L)^r$.

Proof: Let $\emptyset_L=(S,L)$. Hence, $\forall ö\in L$; $S(ö)=\emptyset$. Let $(X,L)\overset{*}{\underset{+}{\sim}}(S,L)=(H,L)$. Then, $\forall ö\in L$,

$$H(ö)=\begin{cases} X'(ö), & ö\in L-L=\emptyset \\ X'(ö)\cup S(ö), & ö\in L\cap L=L \end{cases}$$

Thus, $\forall ö\in L$, $H(ö)=X'(ö)\cup S(ö)=X'(ö)\cup\emptyset=X'(ö)$. Hence $(H,L)=(X,L)^r$.

11) $\emptyset_L\overset{*}{\underset{+}{\sim}}(X,L)=U_L$

Proof: Let $\emptyset_L=(S,L)$. Hence, $\forall ö\in L$, $S(ö)=\emptyset$. Let $(S,L)\overset{*}{\underset{+}{\sim}}(X,L)=(H,L)$. Then, $\forall ö\in L$,

$$H(ö)=\begin{cases} S'(ö), & ö\in L-L=\emptyset \\ S'(ö)\cup X(ö), & ö\in L\cap L=L \end{cases}$$

Hence, ∀ö∈L $H(ö)= S'(ö) \cup X(ö)= U\cup X(ö)=U$, so $(H,L)= U_L$.

12) $(X,L) \underset{+}{\overset{*}{\sim}} \emptyset_C=(X,L)^r$.

Proof: Let $\emptyset_C=(S,C)$. Hence, ∀ö∈C, $S(ö)=\emptyset$. Let $(X,L) \underset{+}{\overset{*}{\sim}} (S,C)=(H,L)$. Thus, ∀ö∈L

$$H(ö)=\begin{cases} X'(ö), & ö\in L-C \\ X'(ö)\cup S(ö), & ö\in L\cap C \end{cases}$$

Hence,

$$H(ö)=\begin{cases} X'(ö), & ö\in L-C \\ X'(ö), & ö\in L\cap C \end{cases}$$

Thus, ∀ö∈L-C, $H(ö)= X'(ö)$ and thus $(H,L)= (X,L)^r$.

13) $\emptyset_C \underset{+}{\overset{*}{\sim}} (X,L)=U_C$.

Proof: Let $\emptyset_C=(S,C)$. Hence, ∀ö∈C, $S(ö)=\emptyset$. Let $(S,C) \underset{+}{\overset{*}{\sim}} (X,L)=(H,C)$. Thus, ∀ö∈C

$$H(ö)=\begin{cases} S'(ö), & ö\in C-L \\ S'(ö)\cup X(ö), & ö\in C\cap L \end{cases}$$

Hence,

$$H(ö)=\begin{cases} U, & ö\in C-L \\ U, & ö\in C\cap L \end{cases}$$

Thus, ∀ö∈C, $H(ö)= U$ and thus $(H,L)= U_C$.

14) $(X,L) \underset{+}{\overset{*}{\sim}} \emptyset_E=(X,L)^r$.

Proof: Let $\emptyset_E=(S,E)$. Hence ∀ö∈E; $S(ö)=\emptyset$. Let $(X,L) \underset{+}{\overset{*}{\sim}} (S,E)=(H,L)$. Thus, ∀ö∈L,

$$H(ö)=\begin{cases} X'(ö), & ö\in L-E=\emptyset \\ X'(ö)\cup S(ö), & ö\in L\cap E=L \end{cases}$$

Hence, ∀ö∈L $H(ö)= X'(ö) \cup S(ö)= X'(ö) \cup \emptyset= X'(ö)$, so $(H,L)= (X,L)^r$.

15) $\emptyset_E \underset{+}{\overset{*}{\sim}} (X,L) = U_E$

Proof: Let $\emptyset_E=(S,E)$. Hence ∀ö∈E; $S(ö)=\emptyset$. Let $(S,E) \underset{+}{\overset{*}{\sim}} (X,L)=(H,E)$. Thus, ∀ö∈E,

$$H(ö)=\begin{cases} S'(ö), & ö\in E-L=L' \\ S'(ö)\cup X(ö), & ö\in L\cap E=L \end{cases}$$

Hence, ∀ö∈E, $H(ö)= S'(ö) \cup X(ö)= U\cup X(ö)=U$, so $(H,L)= U_E$.

16) $(X,L) \underset{+}{\overset{*}{\sim}} (X,L)^r=(X,L)^r$.

Proof: Let $(X,L)^r=(H,L)$, so ∀ö∈L, $H(ö)=X'(ö)$. Let $(X,L) \underset{+}{\overset{*}{\sim}} (H,L)=(T,L)$, so ∀ö∈L,

$$T(ö)=\begin{cases} X'(ö), & ö\in L-L=\emptyset \\ X'(ö)\cup H(ö), & ö\in L\cap L=L \end{cases}$$

Hence, ∀ö∈L, $T(ö)= X'(ö) \cup H(ö)= X'(ö) \cup X'(ö)=X'(ö)$, so $(T,L)=(X,L)^r$

Note that, relative complement of a soft set is the right-absorbing element of its own soft set for the operation $\underset{+}{\overset{*}{\sim}}$ in the set $S_E(U)$. Moreover, every soft set is the left identity for its own complement under the operation $\underset{+}{\overset{*}{\sim}}$ in the set $S_E(U)$.

17) $(X,L)^r \underset{+}{\overset{*}{\sim}} (X,L)=(X,L)$.

Proof: Let $(X,L)^r=(H,L)$. Hence ∀ö∈L, $H(ö)=X'(ö)$. Let $(H,L) \underset{+}{\overset{*}{\sim}} (X,L)=(T,L)$, so ∀ö∈L,

$$T(ö)=\begin{cases} H'(ö), & ö\in L-L=\emptyset \\ H'(ö)\cup X(ö), & ö\in L\cap L=L \end{cases}$$

Hence, ∀ö∈L, $T(ö)= H'(ö) \cup X(ö)= X(ö) \cup X(ö)=X(ö)$, so $(T,L)=(X,L)$.

Note that, relative complement of a soft set is the left identity element of its own soft set for the operation $\underset{+}{\overset{*}{\sim}}$ in the set $S_E(U)$.

18) $[(X,L) \underset{+}{\overset{*}{\sim}} (G,Ʊ)]^r=(X,L) \tilde{\setminus} (G,Ʊ)$.

Proof: Let $(X,L) \underset{+}{\overset{*}{\sim}} (G,Ʊ)=(H,L)$. Then, ∀ö∈L,

$$H(ö)=\begin{cases} X'(ö), & ö\in L-Ʊ \\ X'(ö)\cup G(ö), & ö\in L\cap Ʊ \end{cases}$$

Let $(H,L)^r=(T,L)$, so ∀ö∈L,

$$T(ö)=\begin{cases} X(ö), & ö\in L-Ʊ \\ X(ö)\cap G'(ö), & ö\in L\cap Ʊ \end{cases}$$

Thus, $(T,L) =(X,L) \tilde{\setminus} (G,Ʊ)$.

In classical theory, $A \cup B = \emptyset \Leftrightarrow A = \emptyset$ and $B = \emptyset$. Now, we have the following:

19) $(X, L) \overset{*}{\underset{+}{\sim}} (G, L) = \emptyset_L \Leftrightarrow (X, A) = U_L$ and $(G, L) = \emptyset_L$.

Proof: Let $(X, L) \overset{*}{\underset{+}{\sim}} (G, L) = (T,L)$. Hence, $\forall ö \in L$,

$$T(ö) = \begin{cases} X'(ö), & ö \in L-L=\emptyset \\ X'(ö) \cup G(ö), & ö \in L\cap L = L \end{cases}$$

Since $(T,L)=\emptyset_L$, $\forall ö \in L$, $T(ö)= \emptyset$. Hence, $\forall ö \in L$, $T(ö)= X'(ö) \cup G(ö)=\emptyset \Leftrightarrow \forall ö \in L$, $X'(ö)= \emptyset$ and $G(ö)= \emptyset \Leftrightarrow \forall ö \in L$, $X(ö)=U$ and $G(ö)= \emptyset \Leftrightarrow (X, L) = U_L$ and $(G, L) = \emptyset_L$.

20) $\emptyset_L \tilde{\subseteq} (X, L) \overset{*}{\underset{+}{\sim}} (G,\mho)$ and $\emptyset_P \tilde{\subseteq} (G,\mho) \overset{*}{\underset{+}{\sim}} (X,L)$.

21) $(X, L) \overset{*}{\underset{+}{\sim}} (G,\mho) \tilde{\subseteq} U_L$ and $(G,\mho) \overset{*}{\underset{+}{\sim}} (X, L) \tilde{\subseteq} U_\mho$

22) $(X,L)^r \tilde{\subseteq} (X, L) \overset{*}{\underset{+}{\sim}} (G,\mho)$ however neither $(G,\mho)^r$ nor $(G,\mho)$ needs not to be a soft subset of $(X, L) \overset{*}{\underset{+}{\sim}} (G,\mho)$.

Proof: Let $(X, L) \overset{*}{\underset{+}{\sim}} (G,\mho)=(H,L)$. First of all, $L\subseteq L$. Moreover, $\forall ö \in L$,

$$H(ö) = \begin{cases} X'(ö), & ö \in L-\mho \\ X'(ö) \cup G(ö), & ö \in L\cap\mho \end{cases}$$

Since $\forall ö \in L$, $X'(ö) \subseteq X'(ö)$ and $\forall ö \in L\cap\mho$, $X'(ö) \subseteq X'(ö) \cup G(ö)$, hence $\forall ö \in L$, $X'(ö) \subseteq H(ö)$. Therefore, $(X,L)^r \tilde{\subseteq} (X, L) \overset{*}{\underset{+}{\sim}} (G,\mho) =(H,L)$.

23) $(X,L)^r \tilde{\subseteq} (X, L) \overset{*}{\underset{+}{\sim}} (G, L)$, moreover $(G,L) \tilde{\subseteq} (X, L) \overset{*}{\underset{+}{\sim}} (G,L)$

Proof: Let $(X, L) \overset{*}{\underset{+}{\sim}} (G,L)=(H,L)$. First of all, $L\subseteq L$. Moreover, $\forall ö \in L$,

$$H(ö) = \begin{cases} X'(ö), & ö \in L-L=\emptyset \\ X'(ö) \cup G(ö), & ö \in L\cap L=L \end{cases}$$

Since $\forall ö \in L$, $X'(ö) \subseteq X'(ö) \cup G(ö)= H(ö)$, hence $(X,L)^r \tilde{\subseteq} (X, L) \overset{*}{\underset{+}{\sim}} (G,L)$.

Moreover, since $\forall ö \in L$, $G(ö) \subseteq X'(ö) \cup G(ö)$, hence $(G,L) \tilde{\subseteq} (X, L) \overset{*}{\underset{+}{\sim}} (G,L)$.

4. Distribution Rules

In this section, distribution of complemetary soft binary piecewise plus (+) operation with complement over other soft set operations such as extended and soft binary piecewise intersection and union, complementary extended and complementary soft binary piecewise theta and star and restricted intersection, union, theta and star are examined in detail and many interesting results are obtained.

4.1 Distribution of Soft Binary Piecewise Plus (+) Operation With Complement Over Extended Soft Set Operations:

1) $(X,L) \overset{*}{\underset{+}{\sim}} [(G,\mho)\cap_\varepsilon(H,C)] = [(X,L) \overset{*}{\underset{+}{\sim}} (G,\mho)] \cup_\varepsilon [(X,L) \overset{*}{\underset{+}{\sim}} (H,C)]$, where $L\cap\mho\cap C=\emptyset$

Proof: Let first handle the left hand side of the equality and let $(G,\mho)\cap_\varepsilon(H,C)=(M,\mho\cup C)$, so $\forall ö \in \mho\cup C$,

$$M(ö) = \begin{cases} G(ö), & ö \in \mho-C \\ H(ö), & ö \in C-\mho \\ G(ö)\cap H(ö), & ö \in \mho\cap C \end{cases}$$

Let $(X,L) \overset{*}{\underset{+}{\sim}} (M,\mho\cup C)=(N,L)$, $\forall ö \in L$,

$$N(ö) = \begin{cases} X'(ö), & ö \in L-(\mho\cup C) \\ X'(ö) \cup M(ö), & ö \in L\cap(\mho\cup C) \end{cases}$$

$$N(ö) = \begin{cases} X'(ö) \cap [(G(ö)\cup H(ö)]', & ö \in \Gamma\cup\Omega\cup C= \Gamma\cup\Omega\cup C \\ X'(ö) \cap H(ö)', & ö \in \Gamma\cup(C-\Omega)= \Gamma\cup\Omega'\cup C \\ X'(ö) \cap G(ö)', & ö \in \Gamma\cup(\Omega-C)= \Gamma\cup\Omega\cup C' \\ X'(ö)', & ö \in \Gamma-(\Omega\cap C) =\Gamma\cup\Omega'\cup C' \end{cases}$$

Now let's handle the right hand side of the equality: $[(X,L) \overset{*}{\underset{+}{\sim}} (G,\mho)] \cup_\varepsilon [(X,L) \overset{*}{\underset{+}{\sim}} (H,C)]$.

Assume that $(X,L) \overset{*}{\underset{+}{\sim}} (G,\mho)=(V,L)$, then for $\forall ö \in L$,

$$V(ö) = \begin{cases} X'(ö), & ö \in L-\mho \\ X'(ö) \cup G(ö), & ö \in L\cap\mho \end{cases}$$

Now let $(F,A) \overset{*}{\underset{+}{\sim}} (H,C)=(W,L)$. Then, $\forall ö \in L$,

$$W(ö) = \begin{cases} X'(ö), & ö \in L-C \\ X'(ö) \cup H(ö), & ö \in L\cap C \end{cases}$$

Assume that $(V,L) \cup_\varepsilon (W,L)=(T,L)$, then $\forall ö \in L$,

$$T(ö) = \begin{cases} V(ö), & ö \in L-L=\emptyset \\ W(ö), & ö \in L-L=\emptyset \\ V(ö) \cup W(ö), & ö \in L\cap L=L \end{cases}$$

$$T(ö) = \begin{cases} X'(ö) \cup X'(ö), & ö \in (L-\mho)\cap(L-C) \\ X'(ö) \cup [X'(ö) \cup H(ö)], & ö \in (L-\mho)\cap(L\cap C) \\ [X'(ö) \cup G(ö)] \cup X'(ö), & ö \in (L\cap\mho)\cap(L-C) \\ [X'(ö) \cup G(ö)] \cup [X'(ö) \cup H(ö)], & ö \in (L\cap\mho)\cap(L\cap C) \end{cases}$$

Thus,

$$T(ö) = \begin{cases} X'(ö), & ö \in L\cap\mho'\cap C' \\ X'(ö) \cup H(ö), & ö \in L\cap\mho'\cap C \\ X'(ö) \cup G(ö), & ö \in L\cap\mho\cap C' \\ [X'(ö) \cup G(ö)] \cap [X'(ö) \cup H(ö)], & ö \in L\cap\mho\cap C \end{cases}$$

It is seen that N=T.

2) $[(X,L)\cap_{\varepsilon}(G,\mho)]\overset{*}{\underset{+}{\sim}}(H,C)=[(X,L)\overset{*}{\underset{+}{\sim}}(H,C)]\cup_{\varepsilon}[(G,\mho)\overset{*}{\underset{+}{\sim}}(H,C)]$

Proof: Let first handle the left hand side of the equality. Let $(X,L)\cap_{\varepsilon}(G,\mho)=(M,L\cup\mho)$, then $\forall\ddot{o}\in L\cup\mho$,

$$M(\ddot{o})=\begin{cases} X(\ddot{o}), & \ddot{o}\in L-\mho \\ G(\ddot{o}), & \ddot{o}\in \mho-L \\ X(\ddot{o})\cap G(\ddot{o}), & \ddot{o}\in L\cap\mho \end{cases}$$

Let $(M,L\cup\mho)\overset{*}{\underset{+}{\sim}}(H,C)=(N,L\cup\mho)$, so $\forall\ddot{o}\in L\cup\mho$,

$$N(\ddot{o})=\begin{cases} M'(\ddot{o}), & \ddot{o}\in(L\cup\mho)-C \\ M'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(L\cup\mho)\cap C \end{cases}$$

Thus,

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in(L-\mho)-C=L\cap\mho'\cap C' \\ G'(\ddot{o}), & \ddot{o}\in(\mho-L)-C=L'\cap\mho\cap C' \\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in(L\cap\mho)-C=L\cap\mho\cap C' \\ X'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(L-\mho)\cap C=L\cap\mho'\cap C \\ G'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(\mho-L)\cap C=L'\cap\mho\cap C \\ [X'(\ddot{o})\cup G'(\ddot{o})]\cup H(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap C=L\cap\mho\cap C \end{cases}$$

Now let's handle the right hand side of the equality: $(X,L)\overset{*}{\underset{+}{\sim}}(H,C)]\cup_{\varepsilon}[(G,\mho)\overset{*}{\underset{+}{\sim}}(H,C)$. Assume that $(X,L)\overset{*}{\underset{+}{\sim}}(H,C)=(V,L)$, so $\forall\ddot{o}\in L$,

$$V(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-C \\ X'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Let $(G,\mho)\overset{*}{\underset{+}{\sim}}(H,C)=(W,\mho)$, then $\forall\ddot{o}\in\mho$,

$$W(\ddot{o})=\begin{cases} G'(\ddot{o}), & \ddot{o}\in\mho-C \\ G'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in\mho\cap C \end{cases}$$

Assume that $(V,A)\cup_{\varepsilon}(W,\mho)=(T,L\cup\mho)$, then $\forall\ddot{o}\in L\cup\mho$,

$$T(\ddot{o})=\begin{cases} V(\ddot{o}), & \ddot{o}\in L-\mho \\ W(\ddot{o}), & \ddot{o}\in\mho-L \\ V(\ddot{o})\cup W(\ddot{o}), & \ddot{o}\in L\cap\mho \end{cases}$$

Hence,

$$T(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in(L-C)-\mho=L\cap\mho'\cap C' \\ X'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(L\cap C)-\mho=L\cap\mho'\cap C \\ G'(\ddot{o}), & \ddot{o}\in(\mho-C)-L=L'\cap\mho\cap C' \\ G'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(\mho\cap C)-L=L'\cap\mho\cap C \\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in(L-C)\cap(\mho-C)=L\cap\mho\cap C' \\ X'(\ddot{o})\cup[G'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L-C)\cap(\mho\cap C)=\emptyset \\ [X'(\ddot{o})\cup H(\ddot{o})]\cup G'(\ddot{o}), & \ddot{o}\in(L\cap C)\cap(\mho-C)=\emptyset \\ [X'(\ddot{o})\cup H(\ddot{o})]\cup[G'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L\cap C)\cap(\mho\cap C)=L\cap\mho\cap C \end{cases}$$

It is seen that N=T..

3) $(X,L)\overset{*}{\underset{+}{\sim}}[(G,\mho)\cup_{\varepsilon}(H,C)]=[(X,L)\overset{*}{\underset{+}{\sim}}(G,\mho)]\cup_{\varepsilon}[(X,L)\overset{*}{\underset{+}{\sim}}(H,C)]$, where $L\cap\mho\cap C=\emptyset$.

Proof: Let first handle the left hand side of the equality. Let $(G,\mho)\cup_{\varepsilon}(H,C)=(M,\mho\cup C)$, so $\forall\ddot{o}\in\mho\cup C$,

$$M(\ddot{o})=\begin{cases} G(\ddot{o}), & \ddot{o}\in\mho-C \\ H(\ddot{o}), & \ddot{o}\in C-\mho \\ G(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in\mho\cap C \end{cases}$$

Assume that $(X,L)\overset{*}{\underset{+}{\sim}}(M,\mho\cup C)=(N,L)$, so $\forall\ddot{o}\in L$,

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-(\mho\cup C) \\ X'(\ddot{o})\cup M(\ddot{o}), & \ddot{o}\in L\cap(\mho\cup C) \end{cases}$$

Thus,

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-(\mho\cup C)=L\cap\mho'\cap C' \\ X'(\ddot{o})\cup G(\ddot{o}), & \ddot{o}\in L\cap(\mho-C)=L\cap\mho\cap C' \\ X'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap(C-\mho)=L\cap\mho'\cap C \\ X'(\ddot{o})\cup[(G(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in L\cap\mho\cap C=L\cap\mho\cap C \end{cases}$$

Now let's handle the right hand side of the equality, that is, $[(X,L)\overset{*}{\underset{+}{\sim}}(G,\mho)]\cap_{\varepsilon}[(X,L)\overset{*}{\underset{+}{\sim}}(H,C)]$. Let $(X,L)\overset{*}{\underset{+}{\sim}}(G,\mho)=(V,L)$, so $\forall\ddot{o}\in L$,

$$V(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-\mho \\ X'(\ddot{o})\cup G(\ddot{o}), & \ddot{o}\in L\cap\mho \end{cases}$$

Let $(X,L)\overset{*}{\underset{+}{\sim}}(H,C)=(W,L)$, so $\forall\ddot{o}\in L$,

$$W(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-C \\ X'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Assume that $(V,L)\cup_{\varepsilon}(W,L)=(T,L)$. Hence, $\forall\ddot{o}\in L$,

$$T(\ddot{o})=\begin{cases} V(\ddot{o}), & \ddot{o}\in L-L=\emptyset \\ W(\ddot{o}), & \ddot{o}\in L-L=\emptyset \\ V(\ddot{o})\cup W(\ddot{o}), & \ddot{o}\in L\cap L=L \end{cases}$$

So,

$$T(\ddot{o})=\begin{cases} X'(\ddot{o})\cup X'(\ddot{o}), & \ddot{o}\in(L-\mho)\cap(L-C) \\ X'(\ddot{o})\cup[X'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L-\mho)\cap(L\cap\mho) \\ [X'(\ddot{o})\cup G(\ddot{o})]\cup X'(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap(L-C) \\ [X'(\ddot{o})\cup G(\ddot{o})]\cup[X'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L\cap\mho)\cap(L\cap C) \end{cases}$$

Hence

$$T(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L\cap\mho'\cap C' \\ X'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap\mho'\cap C \\ X'(\ddot{o})\cup G(\ddot{o}), & \ddot{o}\in L\cap\mho\cap C' \\ [X'(\ddot{o})\cup G(\ddot{o})]\cup[X'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in L\cap\mho\cap C \end{cases}$$

It is seen that N=T.

4) $[(X,L)\cup_{\varepsilon}(G,\mho)]\overset{*}{\underset{+}{\sim}}(H,C)=[(X,L)\overset{*}{\underset{+}{\sim}}(H,C)]\cap_{\varepsilon}[(G,\mho)\overset{*}{\underset{+}{\sim}}(H,C)]$

Proof: Let first handle the left hand side of the equality and let $(X,L)\cap_{\varepsilon}(G,\mho)=(M,L\cup\mho)$, so $\forall\ddot{o}\in L\cup\mho$,

$$M(\ddot{o})=\begin{cases}X(\ddot{o}), & \ddot{o}\in L-\mho\\ G(\ddot{o}), & \ddot{o}\in\mho-L\\ X(\ddot{o})\cup G(\ddot{o}), & \ddot{o}\in L\cap\mho\end{cases}$$

Let $(M,L\cup\mho)\overset{*}{\underset{+}{\sim}}(H,C)=(N,L\cup\mho)$, so $\forall\ddot{o}\in L\cup\mho$,

$$N(\ddot{o})=\begin{cases}M'(\ddot{o}), & \ddot{o}\in(L\cup\mho)-C\\ M'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(L\cup\mho)\cap C\end{cases}$$

$$N(\ddot{o})=\begin{cases}X'(\ddot{o}), & \ddot{o}\in(L-\mho)-C=L\cap\mho'\cap C'\\ G'(\ddot{o}), & \ddot{o}\in(\mho-L)-C=L'\cap\mho\cap C'\\ X'(\ddot{o})\cap G'(\ddot{o}), & \ddot{o}\in(L\cap\mho)-C=L\cap\mho\cap C'\\ X'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(L-\mho)\cap C=L\cap\mho'\cap C\\ G'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(\mho-L)\cap C=L'\cap\mho\cap C\\ [X'(\ddot{o})\cap G'(\ddot{o})]\cup H(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap C=L\cap\mho\cap C\end{cases}$$

Now let's handle the right hand side of the equality: $[(X,L)\overset{*}{\underset{+}{\sim}}(H,C)]\cap_{\varepsilon}[(G,\mho)\overset{*}{\underset{+}{\sim}}(H,C)]$. Let $(X,L)\overset{*}{\underset{+}{\sim}}(H,C)=(V,L)$, so $\forall\ddot{o}\in L$,

$$V(\ddot{o})=\begin{cases}X'(\ddot{o}), & \ddot{o}\in L-C\\ X'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap C\end{cases}$$

Let $(G,\mho)\overset{*}{\underset{+}{\sim}}(H,C)=(W,\mho)$, so $\forall\ddot{o}\in\mho$,

$$W(\ddot{o})=\begin{cases}G'(\ddot{o}), & \ddot{o}\in\mho-C\\ G'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in\mho\cap C\end{cases}$$

Assume that $(V,L)\cap_{\varepsilon}(W,\mho)=(T,L\cup\mho)$, so $\forall\ddot{o}\in L\cup\mho$,

$$T(\ddot{o})=\begin{cases}V(\ddot{o}), & \ddot{o}\in L-\mho\\ W(\ddot{o}), & \ddot{o}\in\mho-L\\ V(\ddot{o})\cap W(\ddot{o}), & \ddot{o}\in L\cap\mho\end{cases}$$

$$T(\ddot{o})=\begin{cases}X'(\ddot{o}), & \ddot{o}\in(L-C)-\mho=L\cap\mho'\cap C'\\ X'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(L\cap C)-\mho=L\cap\mho'\cap C\\ G'(\ddot{o}), & \ddot{o}\in(\mho-C)-L=L'\cap\mho\cap C'\\ G'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(\mho\cap C)-L=L'\cap\mho\cap C\\ X'(\ddot{o})\cap G'(\ddot{o}), & \ddot{o}\in(L-C)\cap(\mho-C)=L\cap\mho\cap C'\\ X'(\ddot{o})\cap[G'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L-C)\cap(\mho\cap C)=\emptyset\\ [X'(\ddot{o})\cup H(\ddot{o})]\cap G'(\ddot{o}), & \ddot{o}\in(L\cap C)\cap(\mho-C)=\emptyset\\ [X'(\ddot{o})\cup H(\ddot{o})]\cap[G'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L\cap C)\cap(\mho\cap C)=L\cap\mho\cap C\end{cases}$$

It is seen that N=T.

4.2 Distribution of Soft Binary Piecewise Plus (+) Operation With Complement Over Extended Soft Set Operations With Complement:

1) $(X,L)\overset{*}{\underset{+}{\sim}}[(G,\mho)\overset{*}{\theta}_{\varepsilon}(H,C)]=[(X,L)\overset{*}{\underset{*}{\sim}}(G,\mho)]\cup_{\varepsilon}[(X,L)\overset{*}{\underset{*}{\sim}}(H,C)]$, where $L\cap\mho\cap C=\emptyset$

Proof: Let first handle the left hand side of the equality. Assume $(G,\mho)\overset{*}{\theta}_{\varepsilon}(H,C)=(M,\mho\cup C)$, so $\forall\ddot{o}\in\mho\cup C$,

$$M(\ddot{o})=\begin{cases}G'(\ddot{o}), & \ddot{o}\in\mho-C\\ H'(\ddot{o}), & \ddot{o}\in C-\mho\\ G'(\ddot{o})\cap H'(\ddot{o}), & \ddot{o}\in\mho\cap C\end{cases}$$

Let $(X,L)\overset{*}{\underset{+}{\sim}}(M,\mho\cup C)=(N,L)$, then $\forall\ddot{o}\in L$,

$$N(\ddot{o})=\begin{cases}X'(\ddot{o}), & \ddot{o}\in L-(\mho\cup C)\\ X'(\ddot{o})\cup M(\ddot{o}), & \ddot{o}\in L\cap(\mho\cup C)\end{cases}$$

Hence,

$$N(\ddot{o})=\begin{cases}X'(\ddot{o}), & \ddot{o}\in L-(\mho\cup C)=L\cap\mho'\cap C'\\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in L\cap(\mho-C)=L\cap\mho\cap C'\\ X'(\ddot{o})\cup H'(\ddot{o}), & \ddot{o}\in L\cap(C-\mho)=L\cap\mho'\cap C\\ X'(\ddot{o})\cup[(G'(\ddot{o})\cap H'(\ddot{o})], & \ddot{o}\in L\cap\mho\cap C=L\cap\mho\cap C\end{cases}$$

Now let's handle the right hand side of the equality: $[(X,L)\overset{*}{\underset{*}{\sim}}(G,\mho)]\cup_{\varepsilon}[(X,L)\overset{*}{\underset{*}{\sim}}(H,C)]$. Let $(X,L)\overset{*}{\underset{*}{\sim}}(G,\mho)=(V,L)$, so $\forall\ddot{o}\in L$,

$$V(\ddot{o})=\begin{cases}X'(\ddot{o}), & \ddot{o}\in L-\mho\\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in L\cap\mho\end{cases}$$

Let $(X,L)\overset{*}{\underset{*}{\sim}}(H,C)=(W,L)$, hence $\forall\ddot{o}\in L$,

$$W(\ddot{o})=\begin{cases}X'(\ddot{o}), & \ddot{o}\in L-C\\ X'(\ddot{o})\cup H'(\ddot{o}),, & \ddot{o}\in L\cap C\end{cases}$$

Assume that $(V,L)\cup_{\varepsilon}(W,L)=(T,L)$, hence $\forall\ddot{o}\in L$,

$$T(\ddot{o})=\begin{cases}V(\ddot{o}), & \ddot{o}\in L-L=\emptyset\\ W(\ddot{o}), & \ddot{o}\in L-L=\emptyset\\ V(\ddot{o})\cup W(\ddot{o}), & \ddot{o}\in L\cap L=L\end{cases}$$

Hence,

$$T(\ddot{o})=\begin{cases}X'(\ddot{o})\cup X'(\ddot{o}), & \ddot{o}\in(L-\mho)\cap(L-C)\\ X'(\ddot{o})\cup[X'(\ddot{o})\cup H'(\ddot{o})], & \ddot{o}\in(L-\mho)\cap(L\cap\mho)\\ [X'(\ddot{o})\cup G'(\ddot{o})]\cup X'(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap(L-C)\\ [X'(\ddot{o})\cup G'(\ddot{o})]\cup[X'(\ddot{o})\cup H'(\ddot{o})], & \ddot{o}\in(L\cap\mho)\cap(L\cap C)\end{cases}$$

Thus,

$$T(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L\cap\mho'\cap C' \\ X'(\ddot{o})\cup H'(\ddot{o}), & \ddot{o}\in L\cap\mho'\cap C \\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in L\cap\mho\cap C' \\ [X'(\ddot{o})\cup G'(\ddot{o})]\cup[X'(\ddot{o})\cup H'(\ddot{o})], & \ddot{o}\in L\cap\mho\cap C \end{cases}$$

It is seen that N=T.

2) $[(X,L)\ \overset{*}{\theta}_{\varepsilon}\ (G,\mho)]\ \overset{*}{\underset{+}{\sim}}\ (H,C)=[(X,L)\ \tilde{\cup}\ (H,C)]\ \cup_{\varepsilon}\ [(G,\mho)\ \tilde{\cup}\ (H,C)]$

Proof: Let first handle the left hand side of the equality, suppose $(X,L)\ \overset{*}{\theta}_{\varepsilon}\ (G,\mho)=(M,L\cup\mho)$, so $\forall\ddot{o}\in L\cup\mho$,

$$M(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-\mho \\ G'(\ddot{o}), & \ddot{o}\in\mho-L \\ X'(\ddot{o})\cap G'(\ddot{o}), & \ddot{o}\in L\cap\mho \end{cases}$$

Let $(M,L\cup\mho)\ \overset{*}{\underset{+}{\sim}}\ (H,C)=(N,L\cup\mho)$, then $\forall\ddot{o}\in L\cup\mho$,

$$N(\ddot{o})=\begin{cases} M'(\ddot{o}), & \ddot{o}\in(L\cup\mho)-C \\ M'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(L\cup\mho)\cap C \end{cases}$$

Thus,

$$N(\ddot{o})=\begin{cases} X(\ddot{o}), & \ddot{o}\in(L-\mho)-C=L\cap\mho'\cap C' \\ G(\ddot{o}), & \ddot{o}\in(\mho-L)-C=L'\cap\mho\cap C' \\ X(\ddot{o})\cup G(\ddot{o}), & \ddot{o}\in(L\cap\mho)-C=L\cap\mho\cap C' \\ X(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(L-\mho)\cap C=L\cap\mho'\cap C \\ G(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(\mho-L)\cap C=L'\cap\mho\cap C \\ [X(\ddot{o})\cup G(\ddot{o})]\cup H(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap C=L\cap\mho\cap C \end{cases}$$

Now let's handle the right hand side of the equality $[(X,L)\ \tilde{\cup}\ (H,C)]\ \cup_{\varepsilon}\ [(G,\mho)\ \tilde{\cup}\ (H,C)]$. Let $(X,L)\tilde{\cup}(H,C)=(V,L)$ so $\ddot{o}\in L$,

$$V(\ddot{o})=\begin{cases} X(\ddot{o}), & \ddot{o}\in L-C \\ X(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Let $(G,\mho)\tilde{\cup}(H,C)=(W,\mho)$, so $\forall\ddot{o}\in L$,

$$W(\ddot{o})=\begin{cases} G(\ddot{o}), & \ddot{o}\in\mho-C \\ G(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in\mho\cap C \end{cases}$$

Assume that $(V,L)\cup_{\varepsilon}(W,\mho)=(T,L\cup\mho)$ and $\forall\ddot{o}\in L\cup\mho$,

$$T(\ddot{o})=\begin{cases} V(\ddot{o}), & \ddot{o}\in L-\mho \\ W(\ddot{o}), & \ddot{o}\in\mho-L \\ V(\ddot{o})\cup W(\ddot{o}), & \ddot{o}\in L\cap\mho \end{cases}$$

Thus,

$$T(\ddot{o})=\begin{cases} X(\ddot{o}), & \ddot{o}\in(L-C)-\mho=L\cap\mho'\cap C' \\ X(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(L\cap C)-\mho=L\cap\mho'\cap C \\ G(\ddot{o}), & \ddot{o}\in(\mho-C)-L=L'\cap\mho\cap C' \\ G(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(\mho\cap C)-L=L'\cap\mho\cap C \\ X(\ddot{o})\cup G(\ddot{o}), & \ddot{o}\in(L-C)\cap(\mho-C)=L\cap\mho\cap C' \\ X(\ddot{o})\cup[G(\ddot{o})\cup H(\ddot{o})] & \ddot{o}\in(L-C)\cap(\mho\cap C)=\emptyset \\ X(\ddot{o})\cup H(\ddot{o})]\cup G(\ddot{o}), & \ddot{o}\in(L\cap C)\cap(\mho-C)=\emptyset \\ [X(\ddot{o})\cup H(\ddot{o})]\cup[G(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L\cap C)\cap(\mho\cap C)=L\cap\mho\cap C \end{cases}$$

It is seen that N=T.

3) $(X,L)\ \overset{*}{\underset{+}{\sim}}\ [(G,\mho)\ \overset{*}{*}_{\varepsilon}\ (H,C)]=[(X,L)\ \overset{*}{\underset{*}{\sim}}\ (G,\mho)]\ \cup_{\varepsilon}\ [(X,L)\ \overset{*}{\underset{*}{\sim}}\ (H,C)]$.

Proof: Let first handle the left hand side of the equality, assume $(G,\mho)\ \overset{*}{*}_{\varepsilon}\ (H,C)=(M,\mho\cup C)$

So $\forall\ddot{o}\in\mho\cup C$,

$$M(\ddot{o})=\begin{cases} G'(\ddot{o}), & \ddot{o}\in\mho-C \\ H'(\ddot{o}), & \ddot{o}\in C-\mho \\ G'(\ddot{o})\cup H'(\ddot{o}), & \ddot{o}\in\mho\cap C \end{cases}$$

Let $(X,L)\ \overset{*}{\underset{+}{\sim}}\ (M,\mho\cup C)=(N,L)$ and $\forall\ddot{o}\in L$,

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-(\mho\cup C) \\ X'(\ddot{o})\cup M(\ddot{o}), & \ddot{o}\in L\cap(\mho\cup C) \end{cases}$$

Thus,

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-(\mho\cup C)=L\cap\mho'\cap C' \\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in L\cap(\mho-C)=L\cap\mho\cap C' \\ X'(\ddot{o})\cup H'(\ddot{o}), & \ddot{o}\in L\cap(C-\mho)=L\cap\mho'\cap C \\ X'(\ddot{o})\cup[G'(\ddot{o})\cup H'(\ddot{o})], & \ddot{o}\in L\cap\mho\cap C=L\cap\mho\cap C \end{cases}$$

Now let's handle the right hand side of the equality: $[(X,L)\ \overset{*}{\underset{*}{\sim}}\ (G,\mho)]\ \cup_{\varepsilon}\ [(X,L)\ \overset{*}{\underset{*}{\sim}}\ (H,C)]$. Let $(X,L)\ \overset{*}{\underset{*}{\sim}}\ (G,\mho)=(V,L)$, so $\forall\ddot{o}\in L$,

$$V(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-\mho \\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in L\cap\mho \end{cases}$$

Let $(X,L)\ \overset{*}{\underset{*}{\sim}}\ (H,C)=(W,L)$, so $\forall\ddot{o}\in L$;

$$W(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-C \\ X'(\ddot{o})\cup H'(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Assume that $(V,L)\cup_{\varepsilon}(W,L)=(T,L)$ and $\forall\ddot{o}\in L$,

$$T(\ddot{o})=\begin{cases} V(\ddot{o}) & \ddot{o}\in L-L=\emptyset \\ W(\ddot{o}) & \ddot{o}\in L-L=\emptyset \\ V(\ddot{o})\cup W(\ddot{o}) & \ddot{o}\in L\cap L=L \end{cases}$$

Hence,

$$T(\ddot{o})=\begin{cases} X'(\ddot{o})\cup X'(\ddot{o}), & \ddot{o}\in(L-\mho)\cap(L-C) \\ X'(\ddot{o})\cup[X'(\ddot{o})\cup H'(\ddot{o})], & \ddot{o}\in(L-\mho)\cap(L\cap\mho) \\ [X'(\ddot{o})\cup G'(\ddot{o})]\cup X'(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap(L-C) \\ [X'(\ddot{o})\cup G'(\ddot{o})]\cup[X'(\ddot{o})\cup H'(\ddot{o})], & \ddot{o}\in(L\cap\mho)\cap(L\cap C) \end{cases}$$

Thus,

$$T(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L\cap\mho'\cap C' \\ X'(\ddot{o})\cup H'(\ddot{o}), & \ddot{o}\in L\cap\mho'\cap C \\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in L\cap\mho\cap C' \\ [X'(\ddot{o})\cup G'(\ddot{o})]\cup[X'(\ddot{o})\cup H'(\ddot{o})], & \ddot{o}\in L\cap\mho\cap C \end{cases}$$

It is een that N=T.

4) $[(X,L) \underset{*_\varepsilon}{*} (G,\mho)] \overset{*}{\underset{+}{\sim}} (H,C) = [(X,L)\, \tilde{\mho}(H,C)] \cap_\varepsilon [(G,\mho)\tilde{\mho}(H,C)]$

Proof: Let first handle the left hand side of the equality, assume $(X,L) \underset{*_\varepsilon}{*} (G,\mho)=(M,L\cup\mho)$ and $\forall \ddot{o}\in L\cup\mho$,

$$M(\ddot{o}) = \begin{cases} X'(\ddot{o}), & \ddot{o}\in L-\mho \\ G'(\ddot{o}), & \ddot{o}\in \mho-L \\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in L\cap\mho \end{cases}$$

Let $(M,L\cup\mho) \overset{*}{\underset{+}{\sim}} (H,C)=(N,L\cup\mho)$ and $\forall \ddot{o}\in L\cup\mho$,

$$N(\ddot{o}) = \begin{cases} M'(\ddot{o}), & \ddot{o}\in (L\cup\mho)-C \\ M'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in (L\cup\mho)\cap C \end{cases}$$

Thus,

$$N(\ddot{o}) = \begin{cases} X(\ddot{o}), & \ddot{o}\in (L-\mho)-C=L\cap\mho'\cap C' \\ G(\ddot{o}), & \ddot{o}\in (\mho-L)-C=L'\cap\mho\cap C' \\ X(\ddot{o})\cap G(\ddot{o}), & \ddot{o}\in (L\cap\mho)-C=L\cap\mho\cap C' \\ X(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in (L-\mho)\cap C=L\cap\mho'\cap C \\ G(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in (\mho-L)\cap C=L'\cap\mho\cap C \\ [X(\ddot{o})\cap G(\ddot{o})]\cup H(\ddot{o}), & \ddot{o}\in (L\cap\mho)\cap C=L\cap\mho\cap C \end{cases}$$

Now let's handle the right hand side of the equality: $[(X,L)\ \tilde{\mho}\ (H,C)] \cap_\varepsilon [(G,\mho)\ \tilde{\mho}\ (H,C)]$.

Let $(X,L)\ \tilde{\mho}\ (H,C)=(V,L)$ and $\forall \ddot{o}\in L$,

$$V(\ddot{o}) = \begin{cases} X(\ddot{o}), & \ddot{o}\in L-C \\ X(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Let $(G,\mho)\ \tilde{\mho}\ (H,C)=(W,\mho)$ and $\forall \ddot{o}\in\mho$,

$$W(\ddot{o}) = \begin{cases} G(\ddot{o}), & \ddot{o}\in \mho-C \\ G(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in \mho\cap C \end{cases}$$

Assume that $(V,L)\cap_\varepsilon (W,\mho)=(T,L\cup\mho)$ and $\forall \ddot{o}\in L\cup\mho$,

$$T(\ddot{o}) = \begin{cases} V(\ddot{o}), & \ddot{o}\in L-\mho \\ W(\ddot{o}), & \ddot{o}\in \mho-L \\ V(\ddot{o})\cap W(\ddot{o}), & \ddot{o}\in L\cap\mho \end{cases}$$

Hence,

$$T(\ddot{o}) = \begin{cases} X(\ddot{o}), & \ddot{o}\in (L-C)-\mho=L\cap\mho'\cap C' \\ X(\ddot{o})\cup H(\ddot{o}) & \ddot{o}\in (L\cap C)-\mho=L\cap\mho'\cap C \\ G(\ddot{o}), & \ddot{o}\in (\mho-C)-L=L'\cap\mho\cap C' \\ G(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in (\mho\cap C)-L=L'\cap\mho\cap C \\ X(\ddot{o})\cap G(\ddot{o}), & \ddot{o}\in (L-C)\cap(\mho-C)=L\cap\mho\cap C' \\ X(\ddot{o})\cap [G(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in (L-C)\cap(\mho\cap C)=\emptyset \\ [X(\ddot{o})\cup H(\ddot{o})]\cap G(\ddot{o}), & \ddot{o}\in (L\cap C)\cap(\mho-C)=\emptyset \end{cases}$$

It is seen that N=T.

4.3 Distribution of Soft Binary Piecewise Plus (+) Operation With Complement Over Soft Binary Piecewise Operations:

1) $(X,L) \overset{*}{\underset{+}{\sim}} [(G,\mho)\ \tilde{\cap}\ (H,C)] = [(X,L) \overset{*}{\underset{+}{\sim}} (G,\mho)]\ \tilde{\cap}\ [(X,L) \overset{*}{\underset{+}{\sim}} (H,C)]$ where $L\cap\mho\cap C'=\emptyset$

Proof: Let first handle the left hand side of the equality and let $(G,\mho)\ \tilde{\cap}\ (H,C)=(M,\mho)$, so $\forall \ddot{o}\in\mho\cup C$,

$$M(\ddot{o}) = \begin{cases} G(\ddot{o}), & \ddot{o}\in \mho-C \\ G(\ddot{o})\cap H(\ddot{o}), & \ddot{o}\in \mho\cap C \end{cases}$$

$(X,L) \overset{*}{\underset{+}{\sim}} (M,\mho)=(N,L)$, where $\forall a\in A$;

$$N(\ddot{o}) = \begin{cases} X'(\ddot{o}), & \ddot{o}\in L-\mho \\ X'(\ddot{o})\cup M(\ddot{o}), & \ddot{o}\in L\cap\mho \end{cases}$$

$$N(\ddot{o}) = \begin{cases} X'(\ddot{o}), & \ddot{o}\in L-\mho \\ X'(\ddot{o})\cup G(\ddot{o}), & \ddot{o}\in L\cap(\mho-C)=L\cap\mho\cap C' \\ X'(\ddot{o})\cup [G(\ddot{o})\cap H(\ddot{o})], & \ddot{o}\in L\cap\mho\cap C=L\cap\mho\cap C \end{cases}$$

Now let's handle the right hand side of the equality: $[(X,L) \overset{*}{\underset{+}{\sim}} (G,\mho)]\ \tilde{\cap}\ [(X,L) \overset{*}{\underset{+}{\sim}} (H,C)]$. Assume that $(X,L) \overset{*}{\underset{+}{\sim}} (G,\mho)=(V,L)$, then for $\forall \ddot{o}\in L$,

$$V(\ddot{o}) = \begin{cases} X'(\ddot{o}), & a\in L-\mho \\ X'(\ddot{o})\cup G(\ddot{o}), & a\in L\cap\mho \end{cases}$$

Now let $(F,A) \overset{*}{\underset{+}{\sim}} (H,C)=(W,L)$. Then, $\forall \ddot{o}\in L$,

$$W(\ddot{o}) = \begin{cases} X'(\ddot{o}), & \ddot{o}\in L-C \\ X'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Assume that $(V,L)\ \tilde{\cap}(W,L)=(T,L)$, then $\forall \ddot{o}\in L$,

$$T(\ddot{o}) = \begin{cases} V(\ddot{o}) , & \ddot{o}\in L-L=\emptyset \\ V(\ddot{o})\cap W(\ddot{o}), & \ddot{o}\in L\cap L=L \end{cases}$$

Thus,

$$T(\ddot{o}) = \begin{cases} X'(\ddot{o})\cap X'(\ddot{o}), & \ddot{o}\in (L-\mho)\cap(L-C) \\ X'(\ddot{o})\cap [X'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in (L-\mho)\cap(L\cap C) \\ [X'(\ddot{o})\cup G(\ddot{o})]\cap X'(\ddot{o}), & \ddot{o}\in (L\cap\mho)\cap(L-C) \\ [X'(\ddot{o})\cup G(\ddot{o})]\cap [X'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in (L\cap\mho)\cap(L\cap C) \end{cases}$$

$$T(\ddot{o}) = \begin{cases} X'(\ddot{o}), & \ddot{o}\in L\cap\mho'\cap C' \\ X'(\ddot{o}), & \ddot{o}\in L\cap\mho'\cap C \\ X'(\ddot{o}) & \ddot{o}\in L\cap\mho\cap C' \\ [X'(\ddot{o})\cup G(\ddot{o})]\cap [X'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in L\cap\mho\cap C \end{cases}$$

Here let's handle $\ddot{o}\in L-\mho$ in the first equation. Since $L-\mho= L\cap\mho'$, if $\ddot{o}\in\mho'$, then $\ddot{o}\in C-\mho$ or $\ddot{o}\in(\mho\cup C)'$. Hence, if $\ddot{o}\in L-\mho$, $\ddot{o}\in L\cap\mho'\cap C'$ or $\ddot{o}\in L\cap\mho'\cap C$. Thus, it is seen that N=T.

2) $[(X,L)\ \tilde{\cap}\ (G,Ʊ)] \overset{*}{\underset{+}{\sim}} (H,C) = [(X,L) \overset{*}{\underset{+}{\sim}} (H,C)]\ \tilde{\cup}\ [(G,Ʊ) \overset{*}{\underset{+}{\sim}} (H,C)]$

Proof: Let first handle the left hand side of the equality. Suppose $(X,L)\ \tilde{\cap}\ (G,Ʊ)=(M,L)$, so $\forall ö\in L$ için,

$$M(ö)=\begin{cases} X(ö), & ö\in L-Ʊ \\ X(ö)\cap G(ö), & ö\in L\cap Ʊ \end{cases}$$

Let $(M,L) \overset{*}{\underset{+}{\sim}} (H,C)=(N,L)$, so $\forall ö\in L$,

$$N(ö)=\begin{cases} M'(ö), & ö\in L-C \\ M'(ö)\cup H(ö), & ö\in L\cap C \end{cases}$$

Thus,

$$N(ö)=\begin{cases} X'(ö), & ö\in(L-Ʊ)-C=L\cap Ʊ'\cap C' \\ X'(ö)\cup G'(ö), & ö\in(L\cap Ʊ)-C=L\cap Ʊ\cap C' \\ X'(ö)\cup H(ö), & ö\in(L-Ʊ)\cap C=L\cap Ʊ'\cap C \\ [X'(ö)\cup G'(ö)]\cup H(ö), & ö\in(L\cap Ʊ)\cap C=L\cap Ʊ\cap C \end{cases}$$

Now let's handle the right hand side of the equality: $[(X,L) \overset{*}{\underset{+}{\sim}} (H,C)]\ \tilde{\cup}\ [(G,Ʊ) \overset{*}{\underset{+}{\sim}} (H,C)]$. Let $(X,L) \overset{*}{\underset{+}{\sim}} (H,C)=(V,L)$, so $\forall ö\in L$,

$$V(ö)=\begin{cases} X'(ö), & ö\in L-C \\ X'(ö)\cup H(ö), & ö\in L\cap C \end{cases}$$

Let $(G,Ʊ) \overset{*}{\underset{+}{\sim}} (H,C)=(W,Ʊ)$, so $\forall ö\in Ʊ$,

$$W(ö)=\begin{cases} G'(ö), & ö\in Ʊ-C \\ G'(ö)\cup H(ö), & ö\in Ʊ\cap C \end{cases}$$

Assume that $(V,L)\tilde{\cup}(W,Ʊ)=(T,L)$, so $\forall ö\in L$,

$$T(ö)=\begin{cases} V(ö), & ö\in L-Ʊ \\ V(ö)\cup W(ö), & ö\in L\cap Ʊ \end{cases}$$

Hence,

$$T(ö)=\begin{cases} X'(ö), & ö\in(L-C)-Ʊ=L\cap Ʊ'\cap C' \\ X'(ö)\cup H(ö), & ö\in(L\cap C)-Ʊ=L\cap Ʊ'\cap C \\ X'(ö)\cup G'(ö), & ö\in(L-C)\cap(Ʊ-C)=L\cap Ʊ\cap C' \\ X'(ö)\cup[G'(ö)\cup H(ö)], & ö\in(L-C)\cap(Ʊ\cap C)=\emptyset \\ [X'(ö)\cup H(ö)]\cup G'(ö), & ö\in(L\cap C)\cap(Ʊ-C)=\emptyset \\ [X'(ö)\cup H(ö)]\cup[G'(ö)\cup H(ö)] & ö\in(L\cap C)\cap(Ʊ\cap C)=L\cap Ʊ\cap C \end{cases} \quad (2)$$

It is seen that N=T.

3) $(X,L) \overset{*}{\underset{+}{\sim}} [(G,Ʊ)\ \tilde{\cup}\ (H,C)] = [(X,L) \overset{*}{\underset{+}{\sim}} (G,Ʊ)]\ \tilde{\cap}\ [(X,L) \overset{*}{\underset{+}{\sim}} (H,C)]$, where $L\cap Ʊ\cap C'=\emptyset$

Proof: Let first handle the left hand side of the equality and let $(G,Ʊ)\tilde{\cup}(H,C)=(M,Ʊ)$, so $\forall ö\in Ʊ\cup C$,

$$M(ö)=\begin{cases} G(ö), & ö\in Ʊ-C \\ G(ö)\cup H(ö), & ö\in Ʊ\cap C \end{cases}$$

$(X,L) \overset{*}{\underset{+}{\sim}} (M,Ʊ)=(N,L)$, where $\forall a\in A$;

$$N(ö)=\begin{cases} X'(ö), & ö\in L-Ʊ \\ X'(ö)\cap M(ö), & ö\in L\cap Ʊ \end{cases}$$

$$N(ö)=\begin{cases} X'(ö), & ö\in L-Ʊ \\ X'(ö)\cup G(ö), & ö\in L\cap(Ʊ-C)=L\cap Ʊ\cap C' \\ X'(ö)\cup[G(ö)\cup H(ö)], & ö\in L\cap Ʊ\cap C=L\cap Ʊ\cap C \end{cases}$$

Now let's handle the right hand side of the equality: $[(X,L) \overset{*}{\underset{+}{\sim}} (G,Ʊ)]\ \tilde{\cap}\ [(X,L) \overset{*}{\underset{+}{\sim}} (H,C)]$. Assume that $(X,L) \overset{*}{\underset{+}{\sim}} (G,Ʊ)=(V,L)$, then for $\forall ö\in L$,

$$V(ö)=\begin{cases} X'(ö), & a\in L-Ʊ \\ X'(ö)\cup G(ö), & a\in L\cap Ʊ \end{cases}$$

Now let $(F,A) \overset{*}{\underset{+}{\sim}} (H,C)=(W,L)$. Then, $\forall ö\in L$,

$$W(ö)=\begin{cases} X'(ö), & ö\in L-C \\ X'(ö)\cup H(ö), & ö\in L\cap C \end{cases}$$

Assume that $(V,L)\ \tilde{\cap}(W,L)=(T,L)$, then $\forall ö\in L$,

$$T(ö)=\begin{cases} V(ö), & ö\in L-L=\emptyset \\ V(ö)\cap W(ö), & ö\in L\cap L=L \end{cases}$$

Thus,

$$T(ö)=\begin{cases} X'(ö)\cap X'(ö), & ö\in(L-Ʊ)\cap(L-C) \\ X'(ö)\cap[X'(ö)\cup H(ö)], & ö\in(L-Ʊ)\cap(L\cap C) \\ [X'(ö)\cup G(ö)]\cap X'(ö), & ö\in(L\cap Ʊ)\cap(L-C) \\ [X'(ö)\cup G(ö)]\cap[X'(ö)\cup H(ö)], & ö\in(L\cap Ʊ)\cap(L\cap C) \end{cases}$$

Thus,

$$T(ö)=\begin{cases} X'(ö), & ö\in L\cap Ʊ'\cap C' \\ X'(ö), & ö\in L\cap Ʊ'\cap C \\ X'(ö), & ö\in L\cap Ʊ\cap C' \\ [X'(ö)\cup G(ö)]\cap[X'(ö)\cup H(ö)], & ö\in L\cap Ʊ\cap C \end{cases}$$

Here let's handle $ö\in L-Ʊ$ in the first equation. Since $L-Ʊ=L\cap Ʊ'$, if $ö\in Ʊ'$, then $ö\in C-Ʊ$ or $ö\in(Ʊ\cup C)'$. Hence, if $ö\in L-Ʊ$, $ö\in L\cap Ʊ'\cap C'$ or $ö\in L\cap Ʊ'\cap C$. Thus, it is seen that N=T.

4) $[(X,L)\ \tilde{\cup}\ (G,Ʊ)] \overset{*}{\underset{+}{\sim}} (H,C) = [(X,L) \overset{*}{\underset{+}{\sim}} (H,C)]\ \tilde{\cap}\ [(G,Ʊ) \overset{*}{\underset{+}{\sim}} (H,C)]$

Proof: Let first handle the left hand side of the equality. Suppose $(X,L)\ \tilde{\cup}\ (G,Ʊ)=(M,L)$, so $\forall ö\in L$,

$$M(ö)=\begin{cases} X(ö), & ö\in L-Ʊ \\ X(ö)\cup G(ö), & ö\in L\cap Ʊ \end{cases}$$

Let $(M,L) \underset{+}{\overset{*}{\sim}} (H,C)=(N,L)$, so $\forall \ddot{o}\in L$,

$$N(\ddot{o})=\begin{cases} M'(\ddot{o}), & \ddot{o}\in L-C \\ M'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Thus,

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in(L-\mho)-C=L\cap\mho'\cap C' \\ X'(\ddot{o})\cap G'(\ddot{o}), & \ddot{o}\in(L\cap\mho)-C=L\cap\mho\cap C' \\ X'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(L-\mho)\cap C=L\cap\mho'\cap C \\ [X'(\ddot{o})\cap G'(\ddot{o})]\cup H(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap C=L\cap\mho\cap C \end{cases}$$

Now let's handle the right hand side of the equality: $[(X,L) \underset{+}{\overset{*}{\sim}} (H,C)]\ \tilde{\cap}\ [(G,\mho) \underset{+}{\overset{*}{\sim}} (H,C)]$.

Let $(X,L) \underset{+}{\overset{*}{\sim}} (H,C)=(V,L)$, so $\forall \ddot{o}\in L$,

$$V(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-C \\ X'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Let $(G,\mho) \underset{+}{\overset{*}{\sim}} (H,C)=(W,\mho)$, so $\forall \ddot{o}\in\mho$,

$$W(\ddot{o}=\begin{cases} G'(\ddot{o}), & \ddot{o}\in\mho-C \\ G'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in\mho\cap C \end{cases}$$

Assume that $(V,L)\ \tilde{\cap}\ (W,\mho)=(T,L)$, so $\forall \ddot{o}\in L$

$$T(\ddot{o})=\begin{cases} V(\ddot{o}), & \ddot{o}\in L-\mho \\ V(\ddot{o})\cap W(\ddot{o}), & \ddot{o}\in L\cap\mho \end{cases}$$

Thus,

$$T(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in(L-C)-\mho=L\cap\mho'\cap C' \\ X'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(L\cap C)-\mho=L\cap\mho'\cap C \\ X'(\ddot{o})\cap G'(\ddot{o}), & \ddot{o}\in(L-C)\cap(\mho-C)=L\cap\mho\cap C' \\ X'(\ddot{o})\cap[G'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L-C)\cap(\mho\cap C)=\emptyset \\ [X'(\ddot{o})\cup H(\ddot{o})]\cap G'(\ddot{o}), & \ddot{o}\in(L\cap C)\cap(\mho-C)=\emptyset \\ [X'(\ddot{o})\cup H(\ddot{o})]\cap[G'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L\cap C)\cap(\mho\cap C)=L\cap\mho\cap C \end{cases}$$

It is seen that N=T.

4.4 Distribution of Soft Binary Piecewise Plus (+) Operation With Complement Over Soft Binary Piecewise Operations With Complement:

1) $(X,L) \underset{+}{\overset{*}{\sim}} [(G,\mho) \underset{\theta}{\overset{*}{\sim}} (H,C)] = [(X,L) \underset{*}{\overset{*}{\sim}} (G,\mho)]\ \tilde{\cup}\ [(H,C) \underset{*}{\overset{*}{\sim}} (X,L)]$, where $L\cap\mho'\cap C=\emptyset$

Proof: Let first handle the left hand side of the equality, suppose $(G,\mho) \underset{\theta}{\overset{*}{\sim}} (H,C)=(M,\mho)$, so $\forall \ddot{o}\in\mho$,

$$M(\ddot{o})=\begin{cases} G'(\ddot{o}), & \ddot{o}\in\mho-C \\ G'(\ddot{o})\cup H'(\ddot{o}), & \ddot{o}\in\mho\cap C \end{cases}$$

Let $(X,L) \underset{+}{\overset{*}{\sim}} (M,\mho)=(N,L)$, so $\forall \ddot{o}\in L$,

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-B \\ X'(\ddot{o})\cup M(\ddot{o}), & \ddot{o}\in L\cap\mho \end{cases}$$

Thus,

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-\mho \\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in L\cap(\mho-C)=L\cap\mho\cap C' \\ X'(\ddot{o})\cup[G'(\ddot{o})\cup H'(\ddot{o})], & \ddot{o}\in L\cap\mho\cap C=L\cap\mho\cap C \end{cases}$$

Now let's handle the right hand side of the equality: $[(X,L) \underset{*}{\overset{*}{\sim}} (G,\mho)]\ \tilde{\cup}\ [(H,C) \underset{*}{\overset{*}{\sim}} (X,L)]$.

Let $(X,L) \underset{*}{\overset{*}{\sim}} (G,\mho)=(V,L)$, so $\forall \ddot{o}\in L$,

$$V(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-\mho \\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in L\cap\mho \end{cases}$$

Let $(H,C) \underset{*}{\overset{*}{\sim}} (X,L)=(W,C)$, so $\forall \ddot{o}\in C$,

$$W(\ddot{o})=\begin{cases} H'(\ddot{o}), & \ddot{o}\in C-L \\ H'(\ddot{o})\cup X'(\ddot{o}), & \ddot{o}\in C\cap L \end{cases}$$

Assume that $(V,L)\tilde{\cup}(W,C)=(T,L)$, so $\forall \ddot{o}\in L$,

$$T(\ddot{o})=\begin{cases} V(\ddot{o}), & \ddot{o}\in L-C \\ V(\ddot{o})\cup W(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Thus,

$$T(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in(L-\mho)-C \\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in(L\cap\mho)-C \\ X'(\ddot{o})\cup H'(\ddot{o}), & \ddot{o}\in(L-\mho)\cap(C-L) \\ X'(\ddot{o})\cup[H'(\ddot{o})\cup X'(\ddot{o})], & \ddot{o}\in(L-\mho)\cap(C\cap L) \\ [X'(\ddot{o})\cup G'(\ddot{o})]\cup H'(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap(C-L) \\ [X'(\ddot{o})\cup G'(\ddot{o})]\cup[H'(\ddot{o})\cup X'(\ddot{o})], & \ddot{o}\in(L\cap\mho)\cap(C\cap L) \end{cases}$$

Hence,

$$T(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L\cap\mho'\cap C' \\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in L\cap\mho\cap C' \\ X'(\ddot{o})\cup H'(\ddot{o}), & \ddot{o}\in(L-\mho)\cap(C-L)=\emptyset \\ X'(\ddot{o})\cup H'(\ddot{o}), & \ddot{o}\in L\cap\mho'\cap C \\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap(C-L)=\emptyset \\ [X'(\ddot{o})\cup G'(\ddot{o})]\cup[H'(\ddot{o})\cup X'(\ddot{o})], & \ddot{o}\in L\cap\mho\cap C \end{cases}$$

Here let's handle $\ddot{o}\in L-Ʊ$ in the first equation. Since $L-Ʊ= L\cap Ʊ'$, if $\ddot{o}\in Ʊ'$, then $\ddot{o}\in C-Ʊ$ or $\ddot{o}\in(Ʊ\cup C)'$. Hence, if $\ddot{o}\in L-Ʊ$, $\ddot{o}\in L\cap Ʊ'\cap C'$ or $\ddot{o}\in L\cap Ʊ'\cap C$. Thus, it is seen that N=T.

2) $[(F, A) \underset{\theta}{\overset{*}{\sim}} (G,Ʊ)] \underset{+}{\overset{*}{\sim}} (H,C)= [(X, L) \tilde{\cup} (H,C)] \tilde{\cup} [(G,Ʊ) \tilde{\cup} (H,C)]$

Proof: Let first handle the left hand side of the equality. Let $(X, L) \underset{\theta}{\overset{*}{\sim}} (G,Ʊ)=(M,L)$, so $\forall\ddot{o}\in L$,

$$M(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-Ʊ \\ X'(\ddot{o})\cap G'(\ddot{o}), & \ddot{o}\in L\cap Ʊ \end{cases}$$

Let $(M,L) \underset{+}{\overset{*}{\sim}} (H,C)=(N,L)$, so $\forall\ddot{o}\in L$,

$$N(\ddot{o})=\begin{cases} M'(\ddot{o}), & \ddot{o}\in L-C \\ M'(\ddot{o}) \cup H(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Thus,

$$N(\ddot{o})=\begin{cases} X(\ddot{o}), & \ddot{o}\in(L-Ʊ)-C =L\cap Ʊ'\cap C' \\ X(\ddot{o})\cup G(\ddot{o}), & \ddot{o}\in(L\cap Ʊ)-C= L\cap Ʊ\cap C' \\ X(\ddot{o}) \cup H(\ddot{o}), & \ddot{o}\in(L-Ʊ) \cap C = L\cap Ʊ'\cap C \\ [X(\ddot{o})\cup G(\ddot{o})] \cup H(\ddot{o}), & \ddot{o}\in L\cap Ʊ\cap C= L\cap Ʊ\cap C \end{cases}$$

Now let's handle the right hand side of the equality: $[(X, L) \tilde{\cup} (H,C)] \tilde{\cup} [(G,Ʊ) \tilde{\cup} (H,C)$. Let $(X, L) \tilde{\cap} (H,C)=(V,L)$, so $\forall\ddot{o}\in L$,

$$V(\ddot{o})=\begin{cases} X(\ddot{o}), & \ddot{o}\in L-C \\ X(\ddot{o}) \cup H(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Let $(G,Ʊ) \tilde{\cap} (H,C)=(W,Ʊ)$, so $\forall\ddot{o}\in Ʊ$,

$$W€=\begin{cases} G(\ddot{o}), & \ddot{o}\in Ʊ-C \\ G(\ddot{o}) \cup H(\ddot{o}), & \ddot{o}\in Ʊ\cap C \end{cases}$$

Let $(V,L) \tilde{\cup} (W,Ʊ)=(T,L)$, so $\forall\ddot{o}\in L$,

$$T(\ddot{o})=\begin{cases} V(\ddot{o}), & \ddot{o}\in L-Ʊ \\ V(\ddot{o})\cup W(\ddot{o}), & \ddot{o}\in L\cap Ʊ \end{cases}$$

Thus,

$$T(\ddot{o})=\begin{cases} X(\ddot{o}), & \ddot{o}\in(L-C)-Ʊ=L\cap Ʊ'\cap C' \\ X(\ddot{o}) \cup H(\ddot{o}), & \ddot{o}\in(L\cap C)-Ʊ=L\cap Ʊ'\cap C \\ X(\ddot{o})\cup G(\ddot{o}), & \ddot{o}\in(L-C)\cap(Ʊ-C)=L\cap Ʊ\cap C' \\ X(\ddot{o})\cup [G(\ddot{o}) \cup H(\ddot{o})], & \ddot{o}\in(L-C)\cap(Ʊ\cap C)=\emptyset \\ [X(\ddot{o}) \cup H(\ddot{o})] \cup G(\ddot{o}), & \ddot{o}\in(L\cap C)\cap(Ʊ-C)=\emptyset \\ [X(\ddot{o}) \cup H(\ddot{o})] \cup [G(\ddot{o}) \cup H(\ddot{o})], & \ddot{o}\in(L\cap C)\cap(Ʊ\cap C)=L\cap Ʊ\cap C \end{cases}$$

It is seen that N=T.

3) $(X,L) \underset{+}{\overset{*}{\sim}} [(G,Ʊ) \underset{*}{\overset{*}{\sim}} (H,C)] = [(X, L) \underset{*}{\overset{*}{\sim}} (G,Ʊ)] \tilde{\cup} [(H,C) \underset{*}{\overset{*}{\sim}} (X,L)]$, where $L\cap Ʊ'\cap C=\emptyset$

Proof: Let first handle the left hand side of the equality, suppose $(G,Ʊ) \underset{*}{\overset{*}{\sim}} (H,C)=(M,Ʊ)$, so $\forall\ddot{o}\in Ʊ$,

$$M(\ddot{o})=\begin{cases} G'(\ddot{o}), & \ddot{o}\in Ʊ-C \\ G'(\ddot{o})\cup H'(\ddot{o}), & \ddot{o}\in Ʊ\cap C \end{cases}$$

Let $(X, L) \underset{+}{\overset{*}{\sim}} (M,Ʊ)=(N,L)$, so $\forall\ddot{o}\in L$,

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L- B \\ X'(\ddot{o}) \cup M(\ddot{o}), & \ddot{o}\in L\cap Ʊ \end{cases}$$

Thus,

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-Ʊ \\ X'(\ddot{o}) \cup G'(\ddot{o}), & \ddot{o}\in L\cap(Ʊ-C)= L\cap Ʊ\cap C' \\ X'(\ddot{o}) \cup [(G'(\ddot{o})\cup H'(\ddot{o})], & \ddot{o}\in L\cap Ʊ\cap C= L\cap Ʊ\cap C \end{cases}$$

Now let's handle the right hand side of the equality: $[(X, L) \underset{*}{\overset{*}{\sim}} (G,Ʊ)] \tilde{\cup} [(H,C) \underset{*}{\overset{*}{\sim}} (X,L)]$.

Let $(X,L) \underset{*}{\overset{*}{\sim}} (G,Ʊ)=(V,L)$, so $\forall\ddot{o}\in L$,

$$V(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-Ʊ \\ X'(\ddot{o}) \cup G'(\ddot{o}), & \ddot{o}\in L\cap Ʊ \end{cases}$$

Let $(H,C) \underset{*}{\overset{*}{\sim}} (X,L) =(W,C)$, so $\forall\ddot{o}\in C$,

$$W(\ddot{o})=\begin{cases} H'(\ddot{o}), & \ddot{o}\in C-L \\ H'(\ddot{o}) \cup X'(\ddot{o}), & \ddot{o}\in C\cap L \end{cases}$$

Assume that $(V,L)\tilde{\cup} (W,C)=(T,L)$, so $\forall\ddot{o}\in L$,

$$T(\ddot{o})=\begin{cases} V(\ddot{o}), & \ddot{o}\in L-C \\ V(\ddot{o})\cup W(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Thus,

$$T(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in(L-Ʊ)-C \\ X'(\ddot{o}) \cup G'(\ddot{o}), & \ddot{o}\in(L\cap Ʊ)-C \\ X'(\ddot{o})\cup H'(\ddot{o}), & \ddot{o}\in(L-Ʊ)\cap(C-L) \\ X'(\ddot{o})\cup [H'(\ddot{o}) \cup X'(\ddot{o})], & \ddot{o}\in(L-Ʊ)\cap(C\cap L) \\ [X'(\ddot{o}) \cup G'(\ddot{o})] \cup H'(\ddot{o}), & \ddot{o}\in(L\cap Ʊ)\cap(C-L) \\ [X'(\ddot{o}) \cup G'(\ddot{o})] \cup [H'(\ddot{o}) \cup X'(\ddot{o})], & \ddot{o}\in(L\cap Ʊ)\cap(C\cap L) \end{cases}$$

Hence,

$$T(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L\cap\mho'\cap C' \\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in L\cap\mho\cap C' \\ X'(\ddot{o})\cup H'(\ddot{o}), & \ddot{o}\in(L-\mho)\cap(C-L)=\emptyset \\ X'(\ddot{o})\,)\cup H'(\ddot{o}) & \ddot{o}\in L\cap\mho'\cap C \\ [X'(\ddot{o})\cup G'(\ddot{o})]\;\cup H'(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap(C-L)=\emptyset \\ [X'(\ddot{o})\cup G'(\ddot{o})]\;\cup\;[H'(\ddot{o})\cup X'(\ddot{o})], & \ddot{o}\in L\cap\mho\cap C \end{cases}$$

Here let's handle $\ddot{o}\in L-\mho$ in the first equation. Since $L-\mho= L\cap\mho'$, if $\ddot{o}\in\mho'$, then $\ddot{o}\in C-\mho$ or $\ddot{o}\in(\mho\cup C)'$. Hence, if $\ddot{o}\in L-\mho$, $\ddot{o}\in L\cap\mho'\cap C'$ or $\ddot{o}\in L\cap\mho'\cap C$. Thus, it is seen that N=T.

4) $(F, A) \underset{*}{\overset{*}{\sim}} (G,\mho)] \underset{+}{\overset{*}{\sim}} (H,C)= [(X, L)\ \tilde{\cup}\ (H,C)]\ \tilde{\cap}\ [(G,\mho)\ \tilde{\cup}\ (H,C)]$

Proof: Let first handle the left hand side of the equality, let $(X,L) \underset{*}{\overset{*}{\sim}} (G,\mho)=(M,L)$, so $\forall\ddot{o}\in L$,

$$M(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-\mho \\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in L\cap\mho \end{cases}$$

Let $(M,L) \underset{+}{\overset{*}{\sim}} (H,C)=(N,L)$, so $\forall\ddot{o}\in L$,

$$N(\ddot{o})=\begin{cases} M'(\ddot{o}), & \ddot{o}\in L-C \\ M'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Thus,

$$N(\ddot{o})=\begin{cases} X(\ddot{o}), & \ddot{o}\in(L-\mho)-C) =L\cap\mho'\cap C' \\ X(\ddot{o})\cap G(\ddot{o}), & \ddot{o}\in(L\cap\mho)-C= L\cap\mho\cap C' \\ X(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(L-\mho)\cap C= L\cap\mho'\cap C \\ [X(\ddot{o})\cap G(\ddot{o})]\;\cup H(\ddot{o}), & \ddot{o}\in L\cap\mho\cap C= L\cap\mho\cap C \end{cases}$$

Now let's handle the right hand side of the equality: $[(X, L)\ \tilde{\cup}\ (H,C)]\ \tilde{\cap}\ [(G,\mho)\cup(H,C)]$. Let $(X, L)\ \tilde{\cap}\ (H,C)=(V,L)$, so $\forall\ddot{o}\in L$,

$$V(\ddot{o})=\begin{cases} X(\ddot{o}), & \ddot{o}\in L-C \\ X(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Let $(G,\mho)\ \tilde{\cap}\ (H,C)=(W,\mho)$, so $\forall\ddot{o}\in\mho$,

$$W(\ddot{o})=\begin{cases} G(\ddot{o}), & \ddot{o}\in\mho-C \\ G(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in\mho\cap C \end{cases}$$

Assume that $(V,L)\ \tilde{\cap}\ (W,\mho)=(T,L)$, so $\forall\ddot{o}\in L$,

$$T(\ddot{o})=\begin{cases} V(\ddot{o}), & \ddot{o}\in L-\mho \\ V(\ddot{o})\cap W(\ddot{o}), & \ddot{o}\in L\cap\mho \end{cases}$$

$$T(\ddot{o})=\begin{cases} X(\ddot{o}), & \ddot{o}\in(L-C)-\mho=L\cap\mho'\cap C' \\ X(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(L\cap C)-\mho=L\cap\mho'\cap C \\ X(\ddot{o})\cap G(\ddot{o}), & \ddot{o}\in(L-C)\cap(\mho-C)=L\cap\mho\cap C' \\ X(\ddot{o})\cap\;[G(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L-C)\cap(\mho\cap C)=\emptyset \\ [X(\ddot{o})\cup H(\ddot{o})]\;\cap G(\ddot{o}), & \ddot{o}\in(L\cap C)\cap(\mho-C)=\emptyset \\ [X(\ddot{o})\cup H(\ddot{o})]\;\cap\;[G(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L\cap C)\cap(\mho\cap C)=L\cap\mho\cap C \end{cases}$$

Hence, it is seen that N=T.

4.5 Distribution of Soft Binary Piecewise Plus (+) Operation With Complement Over Restricted Soft Set Operations:

1) $(X, L) \underset{+}{\overset{*}{\sim}} [(G,\mho)\cap_R(H,C)] = [(X, L) \underset{+}{\overset{*}{\sim}} (G,\mho)]\ \cap_R\ [(X, L) \underset{+}{\overset{*}{\sim}} (H,C)]$.

Proof: Let first handle the left hand side of the equality, suppose $(G,\mho)\cap_R(H,C)=(M, \mho\cap C)$ and so $\forall\ddot{o}\in\mho\cap C$, $M(\ddot{o})=G(\ddot{o})\cap H(\ddot{o})$. Let $(X, L) \underset{+}{\overset{*}{\sim}} (M, \mho\cap C)=(N,L)$, so $\forall\ddot{o}\in L$,

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-(\mho\cap C) \\ X'(\ddot{o})\cup M(\ddot{o}), & \ddot{o}\in L\cap(\mho\cap C) \end{cases}$$

Thus

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-(\mho\cap C) \\ X'(\ddot{o})\cup\;[G(\ddot{o})\cap H(\ddot{o})], & \ddot{o}\in L\cap(\mho\cap C) \end{cases}$$

Now let's handle the right hand side of the equality: $[(X, L) \underset{+}{\overset{*}{\sim}} (G,\mho)]\ \cap_R\ [(X, L) \underset{+}{\overset{*}{\sim}} (H,C)]$. Let $(X, L) \underset{+}{\overset{*}{\sim}} (G,\mho)=(V,L)$, so $\forall\ddot{o}\in L$,

$$V(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-\mho \\ X'(\ddot{o})\cup G(\ddot{o}), & \ddot{o}\in L\cap\mho \end{cases}$$

Let $(X, L) \underset{+}{\overset{*}{\sim}} (H,C)=(W,L)$, so $\forall\ddot{o}\in L$,

$$W(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-C \\ X'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Assume that $(V,L)\cap_R(W,L)=(T,L)$, and so $\forall\ddot{o}\in L$, $T(\ddot{o}) =V(\ddot{o})\cap W(\ddot{o})$,

$$T(\ddot{o})=\begin{cases} X'(\ddot{o})\cap X'(\ddot{o}), & \ddot{o}\in(L-\mho)\cap(L-C) \\ X'(\ddot{o})\cap\;[X'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L-\mho)\cap(L\cap C) \\ [X'(\ddot{o})\cup G(\ddot{o})]\;\cap X'(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap(L-C) \\ [X'(\ddot{o})\cup G(\ddot{o})]\;\cap\;[X'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L\cap\mho)\cap(L\cap C) \end{cases}$$

Hence,

$$T(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L\cap\mho'\cap C' \\ X'(\ddot{o}), & \ddot{o}\in L\cap\mho'\cap C \\ X'(\ddot{o}), & \ddot{o}\in L\cap\mho\cap C' \\ [X'(\ddot{o})\cup G(\ddot{o})]\;\cap\;[X'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in L\cap\mho\cap C \end{cases}$$

Considering the parameter set of the first equation of the first row, that is, $L-(\mho\cap C)$; since $L-(\mho\cap C) =L\cap(\mho\cap C)'$, an element in $(\mho\cap C)'$ may be in $\mho-C$, in $C-\mho$ or $(\mho\cup C)$. Then, $L-(\mho\cap C)$ is equivalent to the following 3 states: $L\cap(\mho\cap C')$, $L\cap(\mho'\cap C)$ and $L\cap(\mho'\cap C')$.Hence, N=T.

2) $[(X, L)\cap_R(G,\mho)] \underset{+}{\overset{*}{\sim}} (H,C)= [(X,L) \underset{+}{\overset{*}{\sim}} (H,C)]\ \cup_R\ [(G,\mho) \underset{+}{\overset{*}{\sim}} (H,C)]$.

Proof: Let first handle the left hand side of the equality. Let $(X, L)\ \cap_R(G,\mho)=(M,L\cap\mho)$, so $\forall\ddot{o}\in L\cap\mho$, $M(\ddot{o})=X(\ddot{o})\cap G(\ddot{o})$. Let $(M, L\cap\mho) \underset{+}{\overset{*}{\sim}}$

$(H,C)=(N,L\cap\mho)$, so $\forall\ddot{o}\in L\cap\mho$,

$$N(\ddot{o})=\begin{cases} M'(\ddot{o}), & \ddot{o}\in(L\cap\mho)-C \\ M'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap C \end{cases}$$

Thus,

$$N(\ddot{o})=\begin{cases} X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in(L\cap\mho)-C=L\cap\mho\cap C' \\ [X'(\ddot{o})\cup G'(\ddot{o})]\cup H(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap C \end{cases}$$

Now let's handle the right hand side of the equality: $[(X,L)\overset{*}{\underset{+}{\sim}}(H,C)]\cup_R[(G,\mho)\overset{*}{\underset{+}{\sim}}(H,C)]$. Let $(X,L)\overset{*}{\underset{+}{\sim}}(H,C)=(V,L)$, so $\forall\ddot{o}\in L$,

$$V(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-C \\ X'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Let $(G,\mho)\overset{*}{\underset{+}{\sim}}(H,C)=(W,\mho)$, so $\forall\ddot{o}\in\mho$,

$$W(\ddot{o})=\begin{cases} G'(\ddot{o}), & \ddot{o}\in\mho-C \\ G'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in\mho\cap C \end{cases}$$

Assume that$(V,L)\cup_R(W,\mho)=(T,L\cap\mho)$, so $\forall\ddot{o}\in L\cap\mho$, $T(\ddot{o})=V(\ddot{o})\cup W(\ddot{o})$,

$$T(\ddot{o})=\begin{cases} X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in(L-C)\cap(\mho-C)=L\cap\mho\cap C' \\ X'(\ddot{o})\cup[G'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L-C)\cap(\mho\cap C)=\emptyset \\ [X'(\ddot{o})\cup H(\ddot{o})]\cup G'(\ddot{o}), & \ddot{o}\in(L\cap C)\cap(\mho-C)=\emptyset \\ [X'(\ddot{o})\cup H(\ddot{o})]\cup[G'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L\cap C)\cap(\mho\cap C)=L\cap\mho\cap C \end{cases}$$

It is seen that N=T.

3) $(X,L)\overset{*}{\underset{+}{\sim}}[(G,\mho)\cup_R(H,C)]=[(X,L)\overset{*}{\underset{\gamma}{\sim}}(G,\mho)]\cup_R[(X,L)\overset{*}{\underset{\gamma}{\sim}}(H,C)]$, where$L\cap\mho\cap C=\emptyset$

Proof: Let first handle the left hand side of the equality, suppose $(G,\mho)\cup_R(H,C)=(M,\mho\cap C)$ and so $\forall\ddot{o}\in\mho\cap C$, $M(\ddot{o})=G(\ddot{o})\cup H(\ddot{o})$. Let $(X,L)\overset{*}{\underset{+}{\sim}}(M,\mho\cap C)=(N,L)$, so $\forall\ddot{o}\in L$,

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-(\mho\cap C) \\ X'(\ddot{o})\cup M(\ddot{o}), & \ddot{o}\in L\cap(\mho\cap C \end{cases}$$

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-(B\cap C) \\ X'(\ddot{o})\cup[G(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in L\cap(\mho\cap C) \end{cases}$$

Now let's handle the right hand side of the equality $[(X,L)\overset{*}{\underset{\gamma}{\sim}}(G,\mho)]\cup_R[(X,L)\overset{*}{\underset{\gamma}{\sim}}(H,C)]$. Let $(X,L)\overset{*}{\underset{\gamma}{\sim}}(G,\mho)=(V,L)$, so $\forall\ddot{o}\in L$,

$$V(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-\mho \\ X'(\ddot{o})\cap G(\ddot{o}), & \ddot{o}\in L\cap\mho \end{cases}$$

Let $(X,L)\overset{*}{\underset{\gamma}{\sim}}(H,C)=(W,L)$, so $\forall\ddot{o}\in L$,

$$W(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-C \\ X'(\ddot{o})\cap H(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Assume that $(V,L)\cup_R(W,L)=(T,L)$, and so $\forall\ddot{o}\in L$, $T(\ddot{o})=V(\ddot{o})\cup W(\ddot{o})$,

$$T(\ddot{o})=\begin{cases} X'(\ddot{o})\cup X'(\ddot{o}), & \ddot{o}\in(L-\mho)\cap(L-C) \\ X'(\ddot{o})\cup[X'(\ddot{o})\cap H(\ddot{o})], & \ddot{o}\in(L-\mho)\cap(L\cap C) \\ [X'(\ddot{o})\cap G(\ddot{o})]\cup X'(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap(L-C) \\ [X'(\ddot{o})\cap G(\ddot{o})]\cup[X'(\ddot{o})\cap H(\ddot{o})], & \ddot{o}\in(L\cap\mho)\cap(L\cap C) \end{cases}$$

Hence,

$$T(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L\cap\mho'\cap C' \\ X'(\ddot{o}), & \ddot{o}\in L\cap\mho'\cap C \\ X'(\ddot{o}), & \ddot{o}\in L\cap\mho\cap C' \\ [X'(\ddot{o})\cap G(\ddot{o})]\cup[X'(\ddot{o})\cap H(\ddot{o})], & \ddot{o}\in L\cap\mho\cap C \end{cases}$$

Since $L-(\mho\cap C)$ is equal to the following 3 cases: $L\cap(\mho\cap C')$, $L\cap(\mho'\cap C)$ and $L\cap(\mho'\cap C')$, it is seen that N=T.

4) $[(X,L)\cup_R(G,\mho)]\overset{*}{\underset{+}{\sim}}(H,C)=[(X,L)\overset{*}{\underset{+}{\sim}}(H,C)]\cap_R[(G,\mho)\overset{*}{\underset{+}{\sim}}(H,C)]$.

Proof: Let first handle the left hand side of the equality, suppose $(X,L)\cup_R(G,\mho)=(M,L\cap\mho)$ so, $\forall\ddot{o}\in L\cap\mho$, $M(\ddot{o})=X(\ddot{o})\cup G(\ddot{o})$. Let $(M,L\cap\mho)\overset{*}{\underset{+}{\sim}}(H,C))=(N,L\cap\mho)$, so $\forall\ddot{o}\in L\cap\mho$,

$$N(\ddot{o})=\begin{cases} M'(\ddot{o}), & \ddot{o}\in(L\cap\mho)-C \\ M'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap C \end{cases}$$

Hence,

$$N(\ddot{o})=\begin{cases} X'(\ddot{o})\cap G'(\ddot{o}), & \ddot{o}\in(L\cap\mho)-C=L\cap\mho\cap C' \\ [X'(\ddot{o})\cap G'(\ddot{o})]\cup H(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap C \end{cases}$$

Now let's handle the right hand side of the equality, $[(X,L)\overset{*}{\underset{+}{\sim}}(H,C)]\cap_R[(G,\mho)\overset{*}{\underset{+}{\sim}}(H,C)]$. Let $(X,L)\overset{*}{\underset{+}{\sim}}(H,C)=(V,L)$, so $\forall\ddot{o}\in L$,

$$V(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-C \\ X'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Let $(G,\mho)\overset{*}{\underset{+}{\sim}}(H,C)=(W,\mho)$, so $\forall\ddot{o}\in\mho$,

$$W(\ddot{o})=\begin{cases} G'(\ddot{o}), & \ddot{o}\in\mho-C \\ G'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in\mho\cap C \end{cases}$$

Suppose that $(V,L)\cap_R (W,\mho)=(T,L\cap\mho)$, so $\forall\ddot{o}\in L\cap\mho$, $T(\ddot{o})=V(\ddot{o})\cap W(\ddot{o})$,

$$T(\ddot{o})=\begin{cases} X'(\ddot{o})\cap G'(\ddot{o}), & \ddot{o}\in(L-C)\cap(\mho-C)=L\cap\mho\cap C' \\ X'(\ddot{o})\cap[G'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L-C)\cap(\mho\cap C)=\emptyset \\ [X'(\ddot{o})\cup H(\ddot{o})]\cap G'(\ddot{o}), & \ddot{o}\in(L\cap C)\cap(\mho-C)=\emptyset \\ [X'(\ddot{o})\cup H(\ddot{o})]\cap[G'(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L\cap C)\cap(\mho\cap C)=L\cap\mho\cap C \end{cases}$$

It is seen that N=T.

5) $(X,L)\underset{+}{\overset{*}{\sim}}[(G,\mho)\,\theta_R\,(H,C)] = [(X,L)\underset{*}{\overset{*}{\sim}}(G,\mho)]\cap_R[(X,L)\underset{*}{\overset{*}{\sim}}(H,C)]$.

Proof: Let first handle the left hand side of the equality, suppose $(G,\mho)\,\theta_R\,(H,C)=(M,\mho\cap C)$ and so $\forall\ddot{o}\in\mho\cap C$, $M(\ddot{o})=G'(\ddot{o})\cap H'(\ddot{o})$. Let $(X,L)\underset{+}{\overset{*}{\sim}}(M,\mho\cap C)=(N,L)$, so $\forall\ddot{o}\in L$,

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-(\mho\cap C) \\ X'(\ddot{o})\cup M(\ddot{o}), & \ddot{o}\in L\cap(\mho\cap C) \end{cases}$$

Thus,

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-(\mho\cap C) \\ X'(\ddot{o})\cup[G'(\ddot{o})\cap H'(\ddot{o})], & \ddot{o}\in L\cap(\mho\cap C) \end{cases}$$

Now let's handle the right hand side of the equality, $[(X,L)\underset{*}{\overset{*}{\sim}}(G,\mho)]\cap_R[(X,L)\underset{*}{\overset{*}{\sim}}(H,C)]$. Let $(X,L)\underset{*}{\overset{*}{\sim}}(G,\mho)=(V,L)$, so $\forall\ddot{o}\in L$,

$$V(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-\mho \\ X'(\ddot{o})\cup G'(\ddot{o}), & \ddot{o}\in L\cap\mho \end{cases}$$

Let $(X,L)\underset{*}{\overset{*}{\sim}}(H,C)=(W,L)$, so $\forall\ddot{o}\in L$,

$$W(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-C \\ X'(\ddot{o})\cup H'(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

Assume that $(V,L)\cap_R(W,L)=(T,L)$, and so $\forall\ddot{o}\in L$, $T(\ddot{o})=V(\ddot{o})\cap W(\ddot{o})$,

$$T(\ddot{o})=\begin{cases} X'(\ddot{o})\cap X'(\ddot{o}), & \ddot{o}\in(L-\mho)\cap(L-C) \\ X'(\ddot{o})\cap[X'(\ddot{o})\cup H'(\ddot{o})], & \ddot{o}\in(L-\mho)\cap(L\cap C) \\ [X'(\ddot{o})\cup G'(\ddot{o})]\cap X'(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap(L-C) \\ [X'(\ddot{o})\cup G'(\ddot{o})]\cap[X'(\ddot{o})\cup H'(\ddot{o})], & \ddot{o}\in(L\cap\mho)\cap(L\cap C) \end{cases}$$

Hence

$$T(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L\cap\mho'\cap C' \\ X'(\ddot{o}), & \ddot{o}\in L\cap\mho'\cap C \\ X'(\ddot{o}), & \ddot{o}\in L\cap\mho\cap C' \\ [X'(\ddot{o})\cup G'(\ddot{o})]\cap[X'(\ddot{o})\cup H'(\ddot{o})], & \ddot{o}\in L\cap\mho\cap C \end{cases}$$

Since the case L-(B∩C) is equal to the following 3 cases: L∩(Ʊ∩C'), L∩(P'∩C) and L∩(P'∩C'), it is seen that N=T.

6) $(X,L)\,\theta_R\,(G,\mho)]\underset{+}{\overset{*}{\sim}}(H,C)=[(X,L)\,\tilde{\mho}\,(H,C)]\cup_R[(G,\mho)\,\tilde{\mho}\,(H,C)]$.

Proof: Let first handle the left hand side of the equality. Let $(X,L)\,\theta_R(G,\mho)=(M,L\cap\mho)$, so $\forall\ddot{o}\in L\cap\mho$, $M(\ddot{o})=X'(\ddot{o})\cap G'(\ddot{o})$. Let $(M,A\cap\mho)\underset{+}{\overset{*}{\sim}}(H,C)=(N,L\cap\mho)$, so $\forall\ddot{o}\in L\cap\mho$,

$$N(\ddot{o})=\begin{cases} M'(\ddot{o}), & \ddot{o}\in(L\cap\mho)-C \\ M'(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap C \end{cases}$$

Hence,

$$N(\ddot{o})=\begin{cases} X(\ddot{o})\cup G(\ddot{o}), & \ddot{o}\in(L\cap\mho)-C=L\cap\mho\cap C' \\ [X(\ddot{o})\cup G(\ddot{o})]\cup H(\ddot{o}), & \ddot{o}\in(L\cap\mho)\cap C \end{cases}$$

Now let's handle the right hand side of the equality: $(X,L)\,\tilde{\mho}\,(H,C)]\cup_R[(G,\mho)\,\tilde{\mho}\,(H,C)]$.

Let $(X,L)\,\tilde{\mho}\,(H,C)=(V,L)$, so $\forall\ddot{o}\in L$,

$$V(\ddot{o})=\begin{cases} X(\ddot{o}), & \ddot{o}\in L-C \\ X(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in L\cap C \end{cases}$$

LeSt $(G,\mho)\,\tilde{\mho}\,(H,C)=(W,\mho)$, so $\forall\ddot{o}\in\mho$,

$$W(\ddot{o})=\begin{cases} G(\ddot{o}), & \ddot{o}\in\mho-C \\ G(\ddot{o})\cup H(\ddot{o}), & \ddot{o}\in\mho\cap C \end{cases}$$

Assume that $(V,L)\cup_R(W,\mho)=(T,L\cap\mho)$, so $\forall\ddot{o}\in L\cap\mho$, $T(\ddot{o})=V(\ddot{o})\cup W(\ddot{o})$. Hence,

$$T(\ddot{o})=\begin{cases} X(\ddot{o})\cup G(\ddot{o}), & \ddot{o}\in(L-C)\cap(\mho-C)=L\cap\mho\cap C' \\ X(\ddot{o})\cup[G(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L-C)\cap(\mho\cap C)=\emptyset \\ [X(\ddot{o})\cup H(\ddot{o})]\cup G(\ddot{o}), & \ddot{o}\in(L\cap C)\cap(\mho-C)=\emptyset \\ [X(\ddot{o})\cup H(\ddot{o})]\cup[G(\ddot{o})\cup H(\ddot{o})], & \ddot{o}\in(L\cap C)\cap(\mho\cap C)=L\cap\mho\cap C \end{cases}$$

It is seen that N=T.

7) $(X,L)\underset{+}{\overset{*}{\sim}}[(G,\mho)\ *_R(H,C)]=[(X,L)\underset{\theta}{\overset{*}{\sim}}(G,\mho)]\cup_R[(X,L)\underset{\theta}{\overset{*}{\sim}}(H,C)]$, where L∩Ʊ∩C=∅.

Proof: Let first handle the left hand side of the equality, suppose $(G,\mho)\ *_R(H,C)=(M,\mho\cap C)$ and $\forall\ddot{o}\in\mho\cap C$, $M(\ddot{o})=G'(\ddot{o})\cup H'(\ddot{o})$. Let $(X,L)\underset{+}{\overset{*}{\sim}}(M,\mho\cap C)=(N,L)$ and $\forall\ddot{o}\in L$,

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-(\mho\cap C) \\ X'(\ddot{o})\cup M(\ddot{o}), & \ddot{o}\in L\cap(\mho\cap C) \end{cases}$$

$$N(\ddot{o})=\begin{cases} X'(\ddot{o}), & \ddot{o}\in L-(\mho\cap C) \\ X'(\ddot{o})\cup[G'(\ddot{o})\cup H'(\ddot{o})], & \ddot{o}\in L\cap(\mho\cap C) \end{cases}$$

Now let's handle the right hand side of the equality: $[(X,L)\underset{\theta}{\overset{*}{\sim}}(G,\mho)]\cup_R[(X,L)\underset{\theta}{\overset{*}{\sim}}(H,C)]$. Let $(X,L)\underset{\theta}{\overset{*}{\sim}}(G,\mho)=(V,L)$, and $\forall\ddot{o}\in L$,

$$V(ö)=\begin{cases} X'(ö), & ö\in L-\mho \\ X'(ö)\cap G'(ö), & ö\in L\cap\mho \end{cases}$$

Let $(X,L)\ \underset{\theta}{\overset{*}{\sim}}\ (H,C)=(W,L)$ and $\forall ö\in L$,

$$W(ö)=\begin{cases} X'(ö), & ö\in L-C \\ X'(ö)\cap H'(ö), & ö\in L\cap C \end{cases}$$

Assume that $(V,L)\cup_R (W,L)=(T,L)$, so $\forall ö\in T(ö)=V(ö)\cup W(ö)$,

$$T(ö)=\begin{cases} X'(ö)\cup X'(ö), & ö\in(L-\mho)\cap(L-C) \\ X'(ö)\cup[X'(ö)\cap H'(ö)], & ö\in(L-\mho)\cap(L\cap C) \\ [X'(ö)\cap G'(ö)]\cup X'(ö), & ö\in(L\cap\mho)\cap(L-C) \\ [X'(ö)\cap G'(ö)]\cup[X'(ö)\cap H'(ö)], & ö\in(L\cap\mho)\cap(L\cap C) \end{cases}$$

Thus,

$$T(ö)=\begin{cases} X'(ö), & ö\in L\cap\mho'\cap C' \\ X'(ö), & ö\in L\cap\mho'\cap C \\ X'(ö), & ö\in L\cap\mho\cap C' \\ [X'(ö)\cap G'(ö)]\cup[X'(ö)\cap H'(ö)], & ö\in L\cap\mho\cap C \end{cases}$$

Since the case L-(B∩C) is equal to the following 3 cases: L∩(Ʊ∩C'), L∩(P'∩C) and L∩(P'∩C'), it is seen that N=T.

8) $(X,L)\ *_R\ (G,\mho)]\ \underset{+}{\overset{*}{\sim}}\ (H,C)=[(X,L)\ \tilde{\mho}\ (H,C)]\ \cap_R\ [(G,\mho)\ \tilde{\mho}\ (H,C)]$.

Proof: Let first handle the left hand side of the equality, suppose $(X,L)\ *_R(G,\mho)=(M,L\cap\mho)$, so $\forall ö\in L\cap\mho$, $M(ö)=X'(ö)\cup G'(ö)$. Let $(M,L\cap\mho)\ \underset{+}{\overset{*}{\sim}}\ (H,C)=(N,L\cap\mho)$ and $\forall ö\in L\cap\mho$,

$$N(ö)=\begin{cases} M'(ö), & ö\in(L\cap\mho)-C \\ M'(ö)\cup H(ö), & ö\in(L\cap\mho)\cap C \end{cases}$$

$$N(ö)=\begin{cases} X(ö)\cap G(ö), & ö\in(L\cap\mho)-C=L\cap\mho\cap C' \\ [X(ö)\cap G(ö)]\cup H(ö), & ö\in(L\cap\mho)\cap C \end{cases}$$

Now let's handle the right hand side of the equality $[(X,L)\ \tilde{\mho}\ (H,C)]\ \cap_R\ [(G,\mho)\ \tilde{\mho}\ (H,C)]$.

Let $(X,L)\ \tilde{\mho}\ (H,C)=(V,L)$, so $\forall ö\in L$,

$$V(ö)=\begin{cases} X(ö), & ö\in L-C \\ X(ö)\cup H(ö), & ö\in L\cap C \end{cases}$$

Let $(G,\mho)\ \tilde{\mho}\ (H,C)=(W,\mho)$, so $\forall ö\in\mho$,

$$W(ö)=\begin{cases} G(ö), & ö\in\mho-C \\ G(ö)\cup H(ö), & ö\in\mho\cap C \end{cases}$$

Assume that $(V,L)\cap_R (W,\mho)=(T,L\cap\mho)$, and $\forall ö\in L\cap\mho$, $T(ö)=V(ö)\cap W(ö)$,

$$T(ö)=\begin{cases} X(ö)\cap G(ö), & ö\in(L-C)\cap(\mho-C)=L\cap\mho\cap C' \\ X(ö)\cap[G(ö)\cup H(ö)], & ö\in(L-C)\cap(\mho\cap C)=\emptyset \\ [X(ö)\cup H(ö)]\cap G(ö), & ö\in(L\cap C)\cap(\mho-C)=\emptyset \\ [X(ö)\cup H(ö)]\cap[G(ö)\cup H(ö)], & ö\in(L\cap C)\cap(\mho\cap C)=L\cap\mho\cap C \end{cases}$$

It is seen that N=T.

5) Conclusion

In this paper, we aim to contribute to the soft set literature by defining a new kind of soft set operation which we call complementary soft binary piecewise plus operation. The basic algebraic properties of the operations are investigated. Moreover by examing the distribution rules, we obtain the relationships between this new soft set operation and other types of soft set operations such as extended and soft binary piecewise intersection and union, complementary extended and complementary soft binary piecewise theta and star and restricted intersection, union, theta and star.This paper can be regarded as a theoretical study for soft sets and some future studies may continue by examining the distribution of other soft set operations over complementary soft binary piecewise plus operation and some new types of soft set operations can be defined in the following studies. Also, since soft sets are a strong mathematical tool for decision making, researchers may be able to propose novel soft set-based cryptography or decision making procedures using this new soft set operation.

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