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RESEARCH ARTICLE

A NEW FIBONACCI MATRIX DEFINITION AND SOME RESULTS

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ABSTRACT

A new definition of the Fibonacci Matrix is given. The elements of the matrix consist of the Fibonacci numbers. First a preliminary knowledge on Fibonacci sequences and their properties are given. Then the new definition of the matrix is given together with some properties. The difference from the common definition is also discussed. The determinant of the matrix and its properties are posed and proven. Applications to systems of algebraic equations are also outlined.

KEYWORDS

Fibonacci Numbers, Linear Algebra, Matrices, Determinants

1. Introduction

One of the most famous sequences in the history is the Fibonacci sequence. Apart from the beauty of the mathematical properties of the sequence, it has been applied extensively to understand the design in nature also (Naylor, 2002). A new Fibonacci Matrix consisting of Fibonacci numbers which is different from the existing ones in the literature is defined in this work. Based on the fundamental definition, sub-matrices are defined and their properties are investigated (Basu and Prased, 2009). The rank of the matrix, the determinants and algebraic linear equations with Fibonacci coefficients are treated as applications of the Fibonacci matrix.

2. Preliminaries

Fibonacci sequence is defined by the formula

$$b_{j+2} = b_{j+1} + b_j \,, \quad j = 0,1,2 \ldots \tag{1}$$

which produces the numbers in Table 1 for b_0=0 and b_1=1.

Table 1: The First 16 Fibonacci Numbers

b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

Equation (1) is a linear difference equation of order 2 accepting a solution of the form r^k. Substituting this solution into (1) and dividing by r^k gives the quadratic equation

$$r^2 - r - 1 = 0 \tag{2}$$

for which the solution is

$$r_{1,2} = \frac{1 \mp \sqrt{5}}{2} \tag{3}$$

Therefore, the solution of the difference equation is

$$b_k = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^k + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^k \tag{4}$$

For the initial conditions of $b_0 = 0$ and $b_1 = 1$, (second order difference equations require two initial conditions much like the case of second order differential equations) the constants are evaluated to be $c_1 = 1/\sqrt{5}$, $c_2 = -1/\sqrt{5}$. Hence any k'th term in a Fibonacci sequence can be calculated from the formula

$$b_k = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right) \tag{5}$$

The beauty of the formula is that although, it involves irrational numbers, the result is always an integer. The number

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887 \tag{6}$$

is called the Golden Ratio which found applications in spiral structures of seashells, vegetables, human body, orientation of leaves, to name a few of them (Kalman and Mona, 2003). It can be shown easily by employing (5) that the ratio of the k+1'th term to the k'th one approaches this Golden Ratio as k tends to infinity

$$\lim_{k\to\infty} \frac{b_{k+1}}{b_k} = \varphi = \frac{1+\sqrt{5}}{2} \tag{7}$$

The powers of the golden ratio can be calculated from the linear equation with Fibonacci coefficients, hence

$$\varphi^n = b_n \varphi + b_{n-1} \quad n=1,2,3,\ldots \tag{8}$$

3. Fibonacci Matrix and Properties

First the Fibonacci matrix is defined slightly different from the one existing in the literature. Then the sub matrices and their properties are investigated (Lee and Peterson, 2014).

Definition 1 (Fibonacci Matrix and Sub-Matrices)

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The *nxn* Fibonacci square matrix $\boldsymbol{F}$ is defined as

$$\boldsymbol{F}=\begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ b_2 & b_3 & b_4 & \cdots & b_{n+1} \\ b_3 & b_4 & b_5 & \cdots & b_{n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_n & b_{n+1} & b_{n+2} & \cdots & b_{2n-1} \end{bmatrix} \tag{9}$$

with each element f_{ij} calculated by the formula

$$f_{ij}=b_{i+j-1} \quad i,j=1,2,3\ldots n \tag{10}$$

where b_i are the Fibonacci numbers calculated sequentially from formula (1) or directly from (5) . The square sub-matrices of the Fibonacci matrix with m rows and m columns starting from the first term as b_k can be defined with the notation.

$$\boldsymbol{F_{mm}^k}=\begin{bmatrix} b_k & b_{k+1} & b_{k+2} & \cdots & b_{k+m-1} \\ b_{k+1} & b_{k+2} & b_{k+3} & \cdots & b_{k+m} \\ b_{k+2} & b_{k+3} & b_{k+4} & \cdots & b_{k+m+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{k+m-1} & b_{k+m} & b_{k+m+1} & \cdots & b_{k+2m-2} \end{bmatrix} \tag{11}$$

with each element f_{ij}^k calculated by the formula

$$f_{ij}^k=b_{k+i+j-2} \qquad i,j=1,2,3\ldots m \tag{12}$$

In this new definition, any path of the matrix followed partially to the right and down with broken lines represents a Fibonacci sequence. The definition is somewhat different from the triangular matrices given in the literature with the formula (Zhang and Wang, 2007; Stanimirovic et al., 2008).

$$f_{ij}=\begin{cases} b_{i-j+1} & if \quad i-j+1\geq 0 \\ 0 & if \quad i-j+1<0 \end{cases} \tag{13}$$

Note that our definition of the Fibonacci matrix is symmetric. Some theorems regarding the properties of the matrices immediately follows:

Theorem 1

The determinant of the Fibonacci matrix defined in (9) and (10) is zero if $n\geq 3$.

Proof

The determinant of a 2x2 matrix is nonzero

$$det(F_{22})=\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}=1 \tag{14}$$

but when it comes to 3x3 matrices and higher orders, the rows and columns obey the Fibonacci rule

$$f_{i\,j+2}=f_{i\,j+1}+f_{i\,j} \quad , \quad f_{i+2\,j}=f_{i+1\,j}+f_{i\,j} \tag{15}$$

which states that the third rows/columns are not independent of the previous two rows/columns. Hence the determinant vanishes and the matrix is singular in nature. The same is true for the sub-matrices (Eqs. 11 and 12) also □

Another way of expressing Theorem 1 is the following corollary.

Corollary 1.

The rank of the Fibonacci matrix is 2 □

Proof

From the proof of Theorem 1, it is stated that any 3x3 sub-matrices of the Fibonacci matrix is singular. But the 2x2 matrices are always non-singular, the determinant of the simplest first one being $det(F_{22})=1$ which makes the rank of the Fibonacci matrix 2 □

For a detailed discussion on Rank 2 matrices see (Lee and Peterson, 2014).

Lemma 2 is proven first to aid the proof of Theorem 2.

Lemma 2.

For the 2x2 sub-matrices defined in (11) and (12), $\det(F_{22}^k)+\det(F_{22}^{k+1})=0$ □

Proof

The determinants are written in open form

$$\det(F_{22}^k)+\det(F_{22}^{k+1})=\begin{vmatrix} b_k & b_{k+1} \\ b_{k+1} & b_{k+2} \end{vmatrix}+\begin{vmatrix} b_{k+1} & b_{k+2} \\ b_{k+2} & b_{k+3} \end{vmatrix} \tag{16}$$

and calculated yielding

$$\det(F_{22}^k)+\det(F_{22}^{k+1})=b_{k+2}(b_k+b_{k+1}-b_{k+2}) \tag{17}$$

But the term in parenthesis is zero from the very definition of Fibonacci sequence, i.e., Eq. 1 □

Theorem 2.

$$\det(F_{22}^k)=(-1)^{k+1} \quad □ \tag{18}$$

Proof

$det(F_{22}^1)=\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}=1$ and from lemma 1, $det(F_{22}^2)=-1$. Proceeding further $det(F_{22}^3)=1$ and so on. By induction $\det(F_{22}^k)=(-1)^{k+1}$ □

Theorem 2 in open form is known as the Cassini identity (Spivey, 2006; Kalman and Mena, 2003; Basu and Prasad, 2009)

$$b_k b_{k+2}-b_{k+1}^2=(-1)^{k+1} \tag{19}$$

An interesting relationship exists between the Q matrices and the 2x2 Fibonacci sub-matrices (Renault, 2013)

$$Q^{n+1}=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n+1}=F_{22}^n=\begin{bmatrix} b_n & b_{n+1} \\ b_{n+1} & b_{n+2} \end{bmatrix} \tag{20}$$

Therefore, if one wants to compute the m'th power of a Fibonacci 2x2 sub-matrix, it will be

$$(F_{22}^n)^m=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{m(n+1)} \tag{21}$$

The ratio of the consecutive first diagonal elements in a Fibonacci matrix converges to the golden ratio squared

$$\lim_{k\to\infty}\frac{f_{k+1k+1}}{f_{kk}}=\lim_{k\to\infty}\frac{f_{k+1k+1}}{f_{k+1k}}\frac{f_{k+1k}}{f_{kk}}=\varphi^2 \tag{22}$$

4. Systems of Linear Equations

Systems of linear equations involving Fibonacci matrices are treated in this section.

Theorem 3.

For the system of homogenous linear equations

$$\boldsymbol{F}_{nn}^k\boldsymbol{x}=\boldsymbol{0}\,, \qquad \boldsymbol{x}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{23}$$

i) There exists only trivial solution $\boldsymbol{x}=\boldsymbol{0}$ if $n\leq 2$

ii) There exists non-trivial solutions if $n>2$ □

Proof

From the theory of linear algebra, for a homogenous equation $\mathbf{A}\boldsymbol{x}=\boldsymbol{0}$, if $\det(\mathbf{A})\neq 0$, $\boldsymbol{x}=\boldsymbol{0}$ is the only solution and if $\det(\mathbf{A})=0$, non-trivial solutions exist. From the previous theorems, it is proven that $\det(\boldsymbol{F}_{nn}^k)=(-1)^{k+1}\neq 0$ for n=2 and $\det(\boldsymbol{F}_{nn}^k)=0$ for n>2. Hence cases *i* and *ii* immediately follow from the linear algebra □

Theorem 4

For the linear non-homogenous equation $\boldsymbol{F}_{22}^k\boldsymbol{x}=\boldsymbol{c}$, with $\boldsymbol{x}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\boldsymbol{c}=\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, if $\boldsymbol{c}$ consists of integers, then the components of solution $\boldsymbol{x}$ are also integers □

Proof

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From the Cramer rule

$$x_1 = \frac{\begin{vmatrix} c_1 & b_{k+1} \\ c_2 & b_{k+2} \end{vmatrix}}{|\boldsymbol{F}_{22}^k|} = \frac{c_1 b_{k+2} - c_2 b_{k+1}}{(-1)^{k+1}} \tag{24}$$

$$x_2 = \frac{\begin{vmatrix} b_k & c_1 \\ b_{k+1} & c_2 \end{vmatrix}}{|\boldsymbol{F}_{22}^k|} = \frac{c_2 b_k - c_1 b_{k+1}}{(-1)^{k+1}} \tag{25}$$

The denominators are ± 1 and the numerators are multiplications and subtractions of integers which are integers □

For instructors to design linear equations with integer solutions, by the token of the theorem, the consecutive three Fibonacci numbers may be used in the coefficient matrix with any integer numbers at the right-hand side. For the system of equations, for example $8x_1 + 13x_2 = 2$, $13x_1 + 21x_2 = 3$, the solutions are $x_1 = -3$, $x_2 = 2$.

Theorem 5

For the linear non-homogenous system of equations $\boldsymbol{F}_{33}^k \boldsymbol{x} = \boldsymbol{c}$, with $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, solutions exist only for $c_1 + c_2 = c_3$ □

Proof

Since $\det(\boldsymbol{F}_{33}^k) = 0$, from linear algebra, no unique solution exists. By the property of the Fibonacci matrix, addition of the left-hand sides of the first two equations yields the left-hand side of the last equation. $\boldsymbol{c}$ should obey this condition also, otherwise inconsistencies appear and no solution would be available □

In summary, a new Fibonacci matrix and its submatrices are defined. Some properties of the matrices are given in the theorems with proofs. Applications to systems of linear algebraic equations are also discussed.

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