ZIBELINE INTERNATIONAL™ PUBLISHING

ISSN: 2521-0831 (Print)
ISSN: 2521-084X (Online)
CODEN: MSMAD

Matrix Science Mathematic (MSMK)

DOI: http://doi.org/10.26480/msmk.02.2024.27.30

CrossMark

RESEARCH ARTICLE

A NEW COMPLEX DERIVATIVE DEFINITION WITH APPLICATIONS

Mehmet Pakdemirli*

Manisa Celal Bayar University, Yunusemre, Manisa, Turkey.
****Corresponding Author Email: pakdemirli@gmail.com***

ARTICLE DETAILS

Article History:

Received 10 March 2024
Revised 12 April 2024
Accepted 15 May 2024
Available online 19 May 2024

ABSTRACT

A new complex derivative is defined for the first time. Some theorems linked to the definitions are given first. Applications of the new derivative to calculus and dynamics are discussed. The analysis may open new horizons to undergraduate students and lecturers.

KEYWORDS

Calculus, Differentiation, Complex Analysis, Optimization, Dynamics

1. Introduction

Calculus played an extremely important role in the development of technology. Differentiation is one of the most important operations in calculus. Many physical problems are already modeled by differential equations where differentiation and its reverse, integration are the most crucial tools. Optimization problems, behavior of functions, many numerical techniques require the usage of derivative. In the way the derivative is defined, the order of differential operators are always positive integers. The derivative is not merely an algebraic operation, a geometric meaning of the derivative also exists. It is the slope of the tangent line to a function at the given point. Although known and defined a long time ago, the fractional differentiation attracted the attention of researchers in the recent decades only. The basic idea behind the fractional differential operators is that the order of derivative can be taken as a fractional number rather than an integer. Many different definitions of fractional derivatives such as the Riemann-Liouville, the Caputo, the Grunwald-Letnikov etc. appeared in the literature (Podlubny, 1999; Hilfer, 2000; Oldham and Spanier, 2006). The geometric meaning of a fractional derivative remains still undetermined although there are some complicated explanations to address the problem.

The fractional theory has been developed on the main assumption that one can take fractional numbers instead of the integers as the order of differentiation. With an inspiration from the fractional orders, another extension might be to employ complex number orders in differentiation. Instead of an *m*'th order differential operator, one may assume an *m+ni*'th order differential operator, *m* and *n* being natural numbers for the time being and $i=\sqrt{-1}$. In this work, first the definition of the complex differentiation is given. The defined derivative is new and completely different than the derivative of a complex valued function with respect to a complex variable. In finding the derivatives of complex valued functions, real integer orders of differentiation with respect to the complex variable are employed. The definition here is different because the derivative operator can be any *m+ni*'th order and *m* and *n* may be much different from each other giving additional flexibility in the calculations. The classical complex derivatives can be found in any textbook discussing complex analysis (O'Neil, 1991; Ahlfors, 1979).

Some properties of the complex derivative based on the definition are given first. Then the differential operator is applied to determining the maxima and minima of a real valued function. Using the complex derivative, the maxima and minima condition can be expressed by a single expression rather than the two successive expressions in the usual calculus. The properties of the functions associated with the derivatives such as decreasing, increasing, concave up and down can also be expressed in a more compact way using the new definition. Another potential application area of differentiation is dynamics. It is shown that the constant velocity motion, the constant acceleration motion can be expressed by a single complex differential operator. Finally, the geometric meaning of the first complex derivative is given.

This study is an introductory study which can be tracked by a freshman undergraduate or advanced junior high school student familiar with derivatives. The topic may be extended to partial differentiation, fractional complex differentiation, differential equations in further studies and higher levels. Applications in many areas of physics and engineering such as models in mathematical physics, mechanics, fluids, heat transfer etc. may be of quite interest in the future.

2. The Complex Differential Operator

The new complex differential operator is defined first.

2.1 Definition

The *m+ni*'th complex differential operator is defined as

$$\frac{d^{m+ni}}{dx^{m+ni}}=\frac{d^m}{dx^m}+i\mu(n)\frac{d^n}{dx^n},\quad \mu(n)=\begin{cases}0 & if \quad n=0\\ 1 & if \quad n=1,2,3\dots\\ -1 & if \quad n=-1,-2,-3\dots\end{cases} \tag{1}$$

where *m* is a natural number and *n* is an integer with $i=\sqrt{-1}$.

If $y=y(x)$ is a real valued function, then

$$\frac{d^{m+ni}(y)}{dx^{m+ni}}=\frac{d^m y}{dx^m}+i\mu(n)\frac{d^n y}{dx^n} \tag{2}$$

If on the other hand, $z=p(x)+iq(x)$ is a single variable complex function, then

$$\frac{d^{m+ni}(z)}{dx^{m+ni}}=\left(\frac{d^m}{dx^m}+i\mu(n)\frac{d^n}{dx^n}\right)(p+iq)=\left(\frac{d^m p}{dx^m}-\mu(n)\frac{d^n q}{dx^n}\right)+i\left(\frac{d^m q}{dx^m}+\mu(n)\frac{d^n p}{dx^n}\right) \tag{3}$$

Quick Response Code	Access this article online	
	Website: www.matrixsmathematic.com	**DOI:** 10.26480/msmk.02.2024.27.30

Cite The Article: Mehmet Pakdemirli (2024). A New Complex Derivative Definition with Applications. *Matrix Science Mathematic, 8(2): 27-30.*

2.2 Theorem 1

For the $m+ni$'th complex derivative of a single variable complex function $z = p(x) + iq(x)$

If $\frac{d^{m+ni}(z)}{dx^{m+ni}}$ is pure real, then $\frac{d^m q}{dx^m} + \mu(n)\frac{d^n p}{dx^n} = 0$ (4)

If $\frac{d^{m+ni}(z)}{dx^{m+ni}}$ is pure imaginary, then $\frac{d^m p}{dx^m} - \mu(n)\frac{d^n q}{dx^n} = 0$ (5)

If $\frac{d^{m+ni}(z)}{dx^{m+ni}} = 0$, then $\frac{d^m q}{dx^m} + \mu(n)\frac{d^n p}{dx^n} = 0, \frac{d^m p}{dx^m} - \mu(n)\frac{d^n q}{dx^n} = 0$ (6)

2.2.1 Proof

The proof is straightforward and follows directly from equation (3) □

The conditions (4)-(6) all lead to differential equations. Especially for the last condition, there are two equations with two unknowns that can be solved. The solutions of the differential equations are beyond the scope of this introductory analysis.

The $1+i$'th derivative is named as the first complex derivative, the $2+2i$'th derivative is named as the second complex derivative and the $n+ni$'th derivative is named as the n'th complex derivative for briefness in the new context of definition.

2.3 Corollary 1

For the first complex derivative ($1+i$'th derivative) of a complex function $z = p(x) + iq(x)$, if

$$\frac{d^{1+i}(z)}{dx^{1+i}} = 0 \qquad (7)$$

z is at most a complex constant number.

2.3.1 Proof

From equations (1) and (3), for m=1 and n=1

$$\frac{dp}{dx} - \frac{dq}{dx} = 0, \quad \frac{dq}{dx} + \frac{dp}{dx} = 0$$

or solving

$$\frac{dp}{dx} = 0, \quad \frac{dq}{dx} = 0$$

leading to $p = p_0$, $q = q_0$ and hence $z = p_0 + iq_0$ a constant complex number □

2.4 Theorem 2

For $m+ni$ and $k+li$ complex numbers, the operation of one complex derivative to the other leads to

$$\frac{d^{m+ni}}{dx^{m+ni}}\frac{d^{k+li}}{dx^{k+li}} = \frac{d^{m+k}}{dx^{m+k}} - \mu(n)\mu(l)\frac{d^{n+l}}{dx^{n+l}} + i\left(\mu(l)\frac{d^{m+l}}{dx^{m+l}} + \mu(n)\frac{d^{n+k}}{dx^{n+k}}\right) \qquad (8)$$

2.4.1 Proof

The proof follows immediately from the successive application of the definition (1) □

3. Application to Optimization and Functional Behaviors

By employing the complex derivatives, the ideas of basic calculus can be expressed in a more compact form. Two example theorems are given to exploit the idea

3.1 Theorem 3

For the real valued function y=$y(x)$ and constant $a_0 \neq 0$, if

$$\left.\frac{d^{2+i}(y)}{dx^{2+i}}\right|_{x=x_0} = a_0 \qquad (9)$$

then

1) x_0 is a local minimum for $a_0 > 0$,
2) x_0 is a local maximum for $a_0 < 0$,

3.1.1 Proof

From the definition of the operator

$$\left.\frac{d^{2+i}(y)}{dx^{2+i}}\right|_{x=x_0} = \left.\frac{d^2(y)}{dx^2}\right|_{x=x_0} + i\left.\frac{d(y)}{dx}\right|_{x=x_0} = a_0$$

or upon separation leads to

$$\left.\frac{d^2(y)}{dx^2}\right|_{x=x_0} = a_0, \quad \left.\frac{d(y)}{dx}\right|_{x=x_0} = 0$$

Since the first derivative is zero, the point may be a local maximum, local minimum or a saddle point depending on the second derivative. Hence from calculus, for $a_0 > 0$, x_0 is a local minimum, for $a_0 < 0$, x_0 is a local maximum □

3.2 Theorem 4

For the real valued function y=$y(x)$ and constants a_0 and b_0, if

$$\frac{d^{1+2i}(y)}{dx^{1+2i}} = a_0 + b_0 i \qquad (10)$$

- the function is increasing and concave down for $a_0 > 0, b_0 > 0$,
- the function is increasing and concave up for $a_0 > 0, b_0 < 0$,
- the function is decreasing and concave down for $a_0 < 0, b_0 > 0$,
- the function is decreasing and concave up for $a_0 < 0, b_0 < 0$.

3.2.1 Proof

From the definition of the operator

$$\frac{d^{1+2i}(y)}{dx^{1+2i}} = \frac{dy}{dx} + i\frac{d^2y}{dx^2} = a_0 + b_0 i$$

or

$$\frac{dy}{dx} = a_0, \quad \frac{d^2y}{dx^2} = b_0$$

and the classical results from calculus follows immediately □

4. Application to Dynamics

The motion of an object is represented by derivatives in the fundamental equations of kinematics. It is well known that the instantaneous velocity of an object is the derivative of the position vector and the acceleration is the derivative of the velocity vector. Using the new complex derivative notation, the rules of movement can be represented in a more compact way. Three sample theorems are given to exploit the idea.

4.1 Theorem 5

For an object with position x=$x(t)$, t representing time and a_0 a real constant, if

$$\frac{d^{2+i}(x)}{dt^{2+i}} = a_0 \qquad (11)$$

the object has zero velocity with constant acceleration. If $a_0 = 0$, the object stands still and does not start moving.

4.1.1 Proof

From the definition of the operator

$$\frac{d^{2+i}(x)}{dt^{2+i}} = \frac{d^2x}{dt^2} + i\frac{dx}{dt} = a_0$$

or upon separation leads to

$$\frac{d^2x}{dt^2} = a_0, \quad \frac{dx}{dt} = 0.$$

Since the first derivative is zero, the object has zero velocity but does have constant acceleration. If further $a_0 = 0$, the object stands still without an attempt to move □

As can be seen from the above theorem, two information can be embedded in one single derivative expression which is frequently needed in dynamics problems.

4.2 Theorem 6

For an object with position x=$x(t)$, t representing time and v_0 a real constant, if

$$\frac{d^{1+2i}(x)}{dt^{1+2i}} = v_0 \tag{12}$$

the object is moving with constant velocity.

4.2.1 Proof

From the definition of the operator

$$\frac{d^{1+2i}(x)}{dt^{1+2i}} = \frac{dx}{dt} + i\frac{d^2x}{dt^2} = v_0$$

or upon separation leads to

$$\frac{dx}{dt} = v_0, \quad \frac{d^2x}{dt^2} = 0.$$

Hence the object moves with constant velocity having no acceleration □

4.3 Theorem 7

For an object with position $x=x(t)$, t representing time and a_0 and v_0 real constants, if

$$\frac{d^{1+2i}(x)}{dt^{1+2i}} = (a_0 t + v_0) + a_0 i \tag{13}$$

the object is moving with a constant acceleration a_0 and initial velocity v_0.

4.3.1 Proof

The proof follows in a similar reasoning with the previous theorems □

5. Geometric Interpretation

The geometric interpretation of a derivative is extremely important. For fractal derivatives, this remains an open question. However, for the new complex derivative, a geometric interpretation already exists. The interpretation of the first complex derivative is given in the following theorem.

5.1 Theorem 8

For the complex single parameter function $z = r(\theta)e^{i\theta}$, the first complex derivative

$$\frac{d^{1+i}(z)}{d\theta^{1+i}} = \frac{dz}{d\theta} + i\frac{dz}{d\theta} \tag{14}$$

represents a vector in the complex plane which is always 45^0 counterclockwise to the tangent vector of the curve defined by $z = r(\theta)e^{i\theta}$ with a length of

$$\left|\frac{d^{1+i}(z)}{d\theta^{1+i}}\right| = \sqrt{2(r'^2 + r^2)}. \tag{15}$$

$\frac{dz}{d\theta}$ is the tangent vector and $i\frac{dz}{d\theta}$ is the normal vector to the curve obtained by rotating the tangent vector 90^0 counterclockwise.

5.1.1 Proof

With referral to Figure 1,

$$\frac{dz}{d\theta} = \lim_{\Delta\theta \to 0} \frac{z(\theta + \Delta\theta) - z(\theta)}{\Delta\theta} = \lim_{\Delta\theta \to 0} \frac{\Delta x + i\Delta y}{\Delta\theta}$$

or

$$\frac{dz}{d\theta} = \frac{dx}{d\theta} + i\frac{dy}{d\theta} = \frac{dx}{d\theta}\left(1 + i\frac{dy/d\theta}{dx/d\theta}\right) = \frac{dx}{d\theta}\left(1 + i\frac{dy}{dx}\right)$$

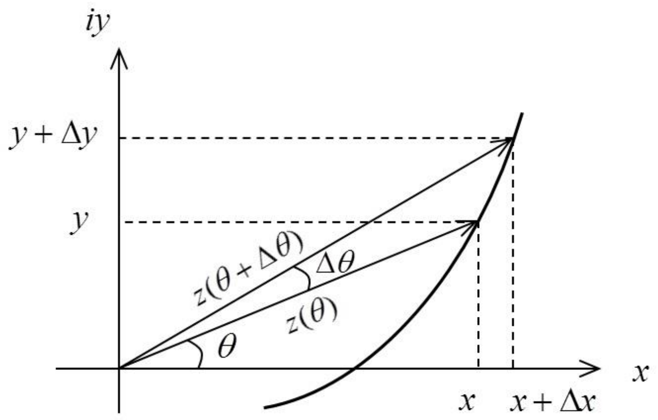

Figure 1: The parametric curve in the complex plane

Since dy/dx is the slope of the tangent line, $1+i\, dy/dx$ represents a tangent vector to the curve in the complex plane. Hence $dz/d\theta$ is a vector tangent to the curve. It is well-known and can be shown easily that multiplication of a complex number (represented by a vector in the complex plane) with i turns the complex vector 90^0 counterclockwise preserving the length. Therefore $i dz/d\theta$ is a vector with same length and normal to the tangent vector which is obtained by rotating the tangent vector 90^0 counterclockwise. The addition of the tangent and normal components leads to a resultant vector with 45^0 counterclockwise with the tangent vector (Figure 2).

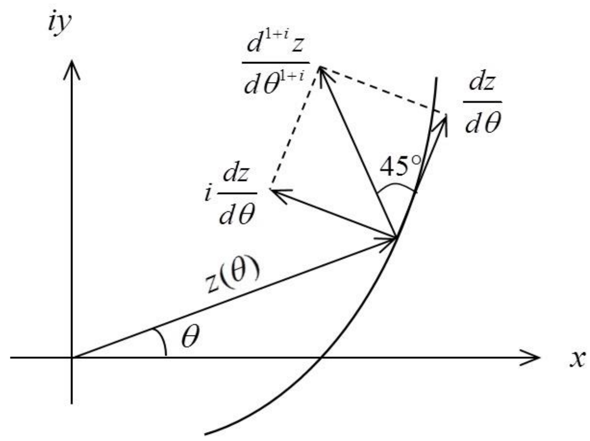

Figure 2: The geometric interpretation of the first complex derivative

For the magnitude of the vector, using (14)

$$\frac{d^{1+i}z}{d\theta^{1+i}} = \frac{dz}{d\theta} + i\frac{dz}{d\theta} = \left(r' + ir\right)(1+i)e^{i\theta}$$

and the magnitude turns out to be $\sqrt{2(r'^2 + r^2)}$□

For an object moving through a path, the unit vectors in the tangential and normal directions are of practical importance and the unit vectors can be calculated as follows

$$e_t = \frac{1}{\sqrt{r'^2+r^2}}\frac{dz}{d\theta}, \quad e_n = \frac{1}{\sqrt{r'^2+r^2}}\frac{dz}{d\theta}i \tag{16}$$

6. Concluding Remarks

A new complex derivative notation is proposed. The new notation is applied to fundamental problems in calculus and dynamics. It is shown that by employing the notation, more information can be gathered in a single expression. The geometrical interpretation of the first complex derivative is also given. The work can be extended to cover fractional derivatives, partial differentiation, more involved application problems in differential equations.

From the pedagogical point of view, a new mathematical analysis is proposed. The new analysis should start with definitions and the basic properties follow then based on the definitions. Establishing the properties, the next step may be to search for potential application areas in science and technology. The mentioned simple steps are followed in this work. I hope the present work will inspire other original definitions and theorems in related topics with new potential application fields.

References

Ahlfors, L.V., 1979. An Introduction to the Theory of Analytic Functions of One Complex Variable, McGraw-Hill, 1979.

Hilfer, R., 2000 Applications of Fractional Calculus in Physics, World Scientific, Singapore.

O'Neil., P.V., 1991. Advanced Engineering Mathematics, Wadsworth, Belmont.

Oldham, K.B., and Spanier, J., 2006. The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order, Dover.

Podlubny, I., 1999. Fractional Differential Equation, Academic Press, San Dieago.