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RESEARCH ARTICLE

ON LOGISTIC GROWTH MODELS BY USING THE FRACTIONAL CAPUTO-FABRIZIO DERIVATIVE

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ABSTRACT

This paper considers the conventional logistic model and how to obtain the solutions of fractional differential equations. It examines using a hybrid of Sumudu transform method, which is an approximate analytical method for solving fractional differential equations that are associated with time delay. Furthermore, the paper introduces the fractional Caputo-Fabrizio derivative and proportional time delay into the conventional logistic model to propose a general and more logistic model for the population growth. The paper considers different cases of the newly introduced general logistic model and using a hybrid of Sumudu transform method, their solutions were obtained. Using MATLAB, it displays and compares the behaviour of different cases of the general logistic model with fractional Caputo-Fabrizio derivative and proportional time delay.

KEYWORDS

Sumudu transform, Caputo-Fabrizio, Population, Logistic model, Time delay

1. INTRODUCTION

Mathematical models play substantial roles in the description of the evolution of several real-life phenomena (Crescenzo and Paraggio, 2019). An example is the Malthusian model (Malthus, 1978),

$$y'(t) = ry(t) \tag{1}$$

where y(t) represents the population size at a time t and r > 0 denotes the growth rate, which is the difference between the fertility and the mortality rates that can be taken to be constant. The Malthusian model (1) is a mathematical model that is based on differential equation, and it is solved by

$$y(t) = ce^{rt} \tag{2}$$

where c represents the population size at time t = 0. Observe that (2) is an exponential function and that renders the Malthusian growth model to be unrealistic. The model function conjectures exponential growth if r > 0 and forecasts the traditional decay if r < 0. The model is not suitable for the description of a long-term growth. There is need for the consideration of other factors such as availability of nutritional resources, adverse weather condition, emergence of virus and diseases, which are capable to alter the population growth rate. A postulate that generalizes the Malthusian model is:

$$y'(t) = ry(t)\left(1 - \frac{y(t)}{K}\right) \tag{3}$$

which is known as the logistic model and it has two parameters r and K > 0 that denote the population growth rate and the population carrying capacity, respectively (Verhulst, 1838). Using the logistic model to study the population growth analysis has been considered by some researchers that include (Karim et al., 2022; Bhowmick et al., 2014; Pal et al., 2018). The curve

$$y(t) = \frac{1}{1+(K/c-1)e^{-rt}} \tag{4}$$

solves the logistic model (3). The study of logistic model curve has ample applications in several fields including ecology, environmental sciences, economics, finance, physiology to mention just a few (Gertsev and Gertseva, 2004; Bear and Cheng, 2010; Rai et al., 2012; Jiang, 2005; Sgouralis and Layton, 2015). Delay is inseparable from several real-life phenomena. Genuine mathematical models that describe real-life phenomena must comprise of time delay. For the logistic model to be more realistic in its prediction of the population density and extinction, delay was introduced in the form.

$$y'(t) = ry(t)\left(1 - \frac{y(t-\tau)}{K}\right) \tag{5}$$

where the delay in time is the τ > 0 (Hutchinson, 1948). The population size $y(t)$ at a given time and the relative population size $y(t-\tau)$ at a preceding time determine the population growth rate $y_0(t)$ that is given by the logistic model with delay. The logistic model with a proportional delay was considered in (Aibinu et al., 2023).

A generalization of the class of classical models are the models with fractional derivatives. Replacing the first-order derivative of a given differential equation with a fractional derivative of order α, 0 < α ≤ 1, produces fractional differential equations. Fractional differential equations have been used in describing several real-life phenomena including continuum mechanics, optimal control, hydrologic modeling, variational problems, fluid mechanics, finance, viscoelasticity, economics and decay process (Carpinteri and Mainardi, 1997; Bhrawy and Ezz-Eldien, 2016; Benson et al., 2013; Ezz-Eldien, 2016; Kulish and Lage, 2002; Jiang et al., 2012; Koeller, 1984; Aibinu and Moyo, 2023; Aibinu and Moyo, 2023; Aibinu and Momoniat, 2023; Aibinu, 2023). Studies on solutions of

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fractional differential equations attracted great attention in the last decades and some of the numerical and approximate analytic methods that have been considered include spectral tau method, radial basis functions method, Haar wavelet method, fractional finite volume method, Adomian decomposition method, operational method and Sumudu variational iteration method to mention just a few (Ezz-Eldien, 2018; Hosseini et al., 2014; Chen et al., 2014; Liu et al., 2014; Khodabakhshi et al., 2014; Ezz-Eldien et al., 2017; Aibinu and Moyo, 2023).

There is no precise analytical method that yields exact solutions for the delay differential equations. Numerical and approximate analytic methods are often used to obtain their solutions. Sumudu Transform (ST) is a reform of the well-known Laplace transform. This paper considers using an approximate analytical method, which is a hybrid of ST method for finding the solutions of fractional differential equations that are associated with Caputo-Fabrizio derivative and time delay. Furthermore, the paper introduces Caputo-Fabrizio derivative and proportional time delay into (1.3), to propose a general and more logistic model for the population growth. It considers different cases and applies the hybrid of ST method to obtain their solutions. Using MATLAB, it displays and compares the behaviour of different cases of the general logistic model with Caputo-Fabrizio derivative and proportional time delay.

2. PRELIMINARIES

We give in this section some definitions and propositions that are applicable to the main results of this paper. We shall denote the sets of natural and real numbers by ℕ and ℝ, respectively.

Definition 2.1. Differential equations with a proportional delay have the form

$$y'(t) = \Omega y(t) + y(\lambda t) \tag{6}$$

where λ > 0 while Ω and Φ are real constants (see, e.g., Ockendon and Tayler, 1971). The equation (6) is called a pantograph equation when λ > 0 (λ 6= 1). When 0 < λ < 1, it is a differential equation with varying delay τ = τ(t), where $\tau\ (t) = (1-\lambda)t$ with τ(t) > 0.

Definition 2.2. For all real t ≥ 0 and y(t) ∈ Ǣ, the ST is defined as:

$$\mathrm{S}[y(t)] := \int_0^\infty y(tu)e^{-t}\,dt, u \in (-\tau_1, \tau_2) \tag{7}$$

where,

$$\bar{Æ} := \mathbb{A} = \{y(t) : \exists Q, \tau_1\tau_2 > 0, |y(t)| < Q\mathrm{e}^{|t|\tau_j}, \qquad if\ t \in (-1)^j{}_{\mathrm{x}}\,[0,\infty)\}.$$

is a set of functions (see, e.g., Belgacem and Karaballi, 2006). Denoting S[y(t)] by Y (u), the inverse ST of Y (u) is y(t), and the relation is given by y(t) = S^{-1}[Y (u)]. Recall that

$$\mathcal{L}[y(t)] = \int_0^\infty y(t)e^{-st}dt,\ s > 0 \tag{8}$$

is the Laplace transform of y(t). In a similar manner to ST, denoting $\mathcal{L}[y(t)]$ by $L(u)$, the alliance between the Sumudu and Laplace transforms can be denoted as

$$Y(1/s) = sL(s), \quad L(1/u) = uY(u).$$

The Sumudu transforms of some selected commonly functions are given in [21–24]. The ST is a linear function, that is,

$$\mathrm{S}[\phi y(t) + \psi z(t)] = \phi\mathcal{S}[y(t)] + \psi\mathcal{S}[z(t)]$$

for x(t), y(t) ∈ Ǣ, and for arbitrary constants φ and ψ (see, e.g., Watugala, 1993; Belgacem et al., 2003; Moltot and Deresse, 2022). Given a first order derivative, its ST can be expressed as

$$\mathcal{S}[y'(t)] = \tfrac{1}{u}\,[Y(u) - y(0)] \tag{9}$$

and

$$\mathrm{S}[y^n(t)] = \tfrac{1}{u^n}\,[Y(u) - \textstyle\sum_{k=0}^{n-1} u^k y^{(k)}\,(t)|_{t=0}] \tag{10}$$

denotes the ST of n$^{\mathrm{th}}$-order derivative with $y^{(k)}(t) = \frac{d^k y(t)}{dt^k}$.

Definition 2.3. Let a > 0, b > 0 be positive real numbers. Then

$${}^{CFC}_{\ \ a}D^\alpha\omega(t) = \frac{M(\alpha)}{(1\ -\ \alpha)}\int_a^t y'(\tau)e^{\psi(t-\tau)}\mathrm{d}\tau$$

and

$${}^{CFC}_{\ \ 0}D^\alpha y(t) = -\frac{M(\alpha)}{(1\ -\ \alpha)}\int_a^t y'(\tau)e^{\psi(\tau-t)}\mathrm{d}\tau,$$

denote respectively the left and right sided Caputo-Fabrizio fractional derivatives in the Caputo sense (CFC) of order α, where 0 < α < 1, M(α) is normalization function and $\psi = -\frac{\alpha}{1-\alpha}$ ST for the CFC of order α is (see, e.g., Baleanu et al., 2020),

$$\mathrm{S}[{}^{CFC}_{\ \ 0}D^\alpha y(t)] = \frac{M(\alpha)}{(1-\alpha)(1+(\alpha/(1-\alpha))u)}\big(\mathrm{S}[y(t)] - y(0)\big) \tag{11}$$

Proposition 2.4. Given the classical convolution product

$$(\varphi * \zeta)(t) = \int_0^t \varphi(t-x)\zeta(x)dx,$$

where φ, ζ: [0, ∞) → ℝ, its ST is given by

$$\mathrm{S}[(\varphi * \zeta)(t)] = \mathrm{S}[\varphi(t)]\mathrm{S}[\zeta(t)] = u\varphi(u)\zeta(u).$$

Definition 2.5. The Mittag-Leffler functions

$$E_\alpha(t) = \sum_{n=0}^\infty \frac{t^n}{\Gamma\ (\mathrm{n}\alpha+1)}, \alpha > 0,$$

and

$$E_{\alpha,\beta}(t) = \sum_{n=0}^\infty \frac{t^n}{\Gamma\ (\mathrm{n}\alpha+\beta)}, \alpha, \beta \in \mathbb{C}, \qquad Re(\alpha) > 0,$$

denote the functions of one parameter and two parameters, respectively. We shall refer to the following results about Mittag-Leffler functions and ST (see, e.g., Nanware et al., 2022):

$$\mathrm{S}[E_\alpha(-at^\alpha)\ ] = \tfrac{1}{1\,+\,au^\alpha}.$$

$$\mathrm{S}[1 - E_\alpha(-at^\alpha)\ ] = \tfrac{au^\alpha}{1\,+\,au^\alpha}.$$

3. MAIN RESULTS

In this section, Sumudu Iterative Method (SIM) is presented for solving nonlinear problems with Caputo-Fabrizio fractional derivatives in the Caputo sense (CFC) of order α.

3.1 Series Solution

Consider a nonlinear functional equation

$$y = g + N(y) \tag{12}$$

with a known function g: = g(t) and a nonlinear operator N. Suppose that (12) has a series solution of the form (see, e.g., Daftardar-Gejji and Jafari, 2006)

$$y = \textstyle\sum_{n=0}^\infty y_n \tag{13}$$

Decompose the nonlinear operator N in (12) as

$$N(\textstyle\sum_{n=0}^\infty y_n) = N(y_0) + \sum_{n=0}^\infty\{N\big(\sum_{n=0}^j y_n\big) - N\big(\sum_{n=0}^{j-1} y_n\big)\}, j = 1,2,3 \tag{14}$$

This indicates that

$$N(y) = N(y_0) + [N(y_0 + y_1) - N(y_0)] + N(y_0) + [N(y_0 + y_1 + y\ \ 2) - N(y_0 + y_1)] \tag{15}$$

Substitute (13) and (14) into (12) to obtain

$$\textstyle\sum_{n=0}^\infty y_n = g + N(y_0) + \sum_{n=0}^\infty\{N\big(\sum_{n=0}^j y_n\big) - N\big(\sum_{n=0}^{j-1} y_n\big)\} \tag{16}$$

Therefore, one can deduce from (16) that the recurrence relation is given by

$$\begin{cases} y_0 = g \\ y_1 = N(y_0) \\ y_2 = N(y_0 + y_1) - N(y_0) \\ y_3 = N(y_0 + y_1 + y_2) - N(y_0 + y_1) \\ \vdots \\ y_n = N(y_0 + y_1 + y_2 + \cdots y_n) - N(y_0 + y_1 + y_2 \ldots y_{n-1}). \end{cases} \tag{17}$$

Therefore, the conclusion is that (12) and

$$\textstyle\sum_{n=0}^\infty y_n = g + N(\sum_{n=0}^\infty y_n) \tag{18}$$

3.2 Approximate analytical solution

Consider a nonlinear differential equation with a proportional delay and CFC fractional derivative of order α,

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$$ {}^{CFC}_{\ 0}D^{\alpha} y(t) + \mathrm{R}[y(t)] + \mathrm{N}[y(\lambda t)] = \mathrm{g(t)} \tag{19}$$

with the initial condition

$$y(0) = c \tag{20}$$

a linear operator R, a nonlinear operator N, and a given continuous function g(t), where λ > .0 The ST of (19) is taken as

$$\mathrm{S}[{}^{CFC}_{\ 0}D^{\alpha} y(t)\ ] = \mathrm{S}\big[\,\mathrm{g(t)} - \mathrm{R}[y(t)] - \mathrm{N}[y(\lambda t)]\big].$$

Applying (2.6) leads to

$$\frac{M(\alpha)}{(1-\alpha)\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}\big(\mathrm{S}[y(t)] - y(0)\big) = \mathrm{S}\big[\,\mathrm{g(t)} - \mathrm{R}[y(t)] - \mathrm{N}[y(\lambda t)]\big]$$

where we have taken a ≡ 0. Using the given initial condition that y (0) = c in (20), leads to

$$\frac{M(\alpha)}{(1-\alpha)\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}(Y(u) - c) = \mathrm{S}\big[\,\mathrm{g(t)} - \mathrm{R}[y(t)] - \mathrm{N}[y(\lambda t)]\big],$$

and consequently

$$Y(u) = c + \tfrac{(1-\alpha)\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}{M(\alpha)}\mathrm{S}\big[\,\mathrm{g(t)} - \mathrm{R}[y(t)] - \mathrm{N}[y(\lambda t)]\big] \tag{21}$$

The inverse ST is taken on both sides of (21) to obtain

$$y(t) = c + \mathrm{S}^{-1}\left[\begin{array}{c}\tfrac{(1-\alpha)\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}{M(\alpha)} * \\ \mathrm{S}\big[g(t) - \mathrm{R}[y(t)] - \mathrm{N}[y(\lambda t)]\big]\end{array}\right] \tag{22}$$

Suppose that (19) has a series solution of the form (12). Substitute (12) into (22) to obtain

$$\textstyle\sum_{n=0}^{\infty} y_n\,(t) = c + \mathrm{S}^{-1}\left[\begin{array}{c}\tfrac{(1-\alpha)\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}{M(\alpha)} * \\ \mathrm{S}\big[g(t) - \mathrm{R}[y_n(t)] - \mathrm{N}[\sum_{n=0}^{\infty} y_n\,(\lambda t)]\big]\end{array}\right] \tag{23}$$

Decompose the nonlinear term "$\mathrm{N}[\sum_{n=0}^{\infty} y_n\,(\lambda t)]$" in (23) by using (13) to obtain

$$N\left(\textstyle\sum_{n=0}^{\infty} y_n(\lambda t)\right) = N(y_0) + \sum_{n=0}^{\infty}\left\{N\left(\sum_{n=0}^{j} y_n\right) - N\left(\sum_{n=0}^{j-1} y_n\right)\right\} \tag{24}$$

Substitute (24) into (23) to obtain

$$\sum_{n=0}^{\infty} y_n\,(t) = c + \mathrm{S}^{-1}\left[\frac{(1-\alpha)\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}{M(\alpha)}\mathrm{S}\big[g(t) - \mathrm{R}[y_n(t)]\big]\right]$$

$$+\mathrm{S}^{-1}\left[\tfrac{(1-\alpha)\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}{M(\alpha)}\mathrm{S}\left[\textstyle\sum_{n=0}^{\infty}\left\{N\left(\sum_{n=0}^{j} y_n\right) - N\left(\sum_{n=0}^{j-1} y_n\right)\right\}\right]\right] \tag{25}$$

Then, the recurrence relation in (25) is given by:

$$\left\{\begin{aligned} y_0(t) &= c + \mathrm{S}^{-1}\left[\tfrac{(1-\alpha)\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}{M(\alpha)}\mathrm{S}[g(t)]\right] \\ y_1 &= -\mathrm{S}^{-1}\left[\tfrac{(1-\alpha)\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}{M(\alpha)}\big(\mathrm{S}[R[y_0]] + \mathrm{S}[N[y_0]]\big)\right] \\ &\ \ \vdots \\ y_n &= -\mathrm{S}^{-1}\left[\tfrac{(1-\alpha)\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}{M(\alpha)}\mathrm{S}\left[\textstyle\sum_{n=0}^{\infty}\left\{N\left(\sum_{n=0}^{j} y_n\right) - N\left(\sum_{n=0}^{j-1} y_n\right)\right\}\right]\right]\end{aligned}\right. \tag{26}$$

for $n \geqslant 1$. Therefore, the approximate analytical solution of (19) in the truncated series form is given by

$$y(t) = \lim_{N\to\infty} \textstyle\sum_{n=0}^{N} y_n(t) = y_0 + y_1 + y_3 + \cdots \tag{27}$$

4. APPLICATIONS

The mathematical models for the population growth will be considered as an application of the computational method that is presented Section 3.

4.1 Logistic growth model

Verhulst proposed a logistic growth model that is given by (1.3) (see [3]). The model is solved by (1.4). In this section, we shall apply the computational method in Section 3 to (1.3) to verify how accurate and efficient is the method. The graph of the solution produced by the computational method in Section 3 will be compared with the graph of (1.4), which is the exact solution of (1.3). The ST of (1.3) is taken as

$$\mathrm{S}[y'(t)] = r\mathrm{S}\left[y(t) - \frac{y^2(t)}{K}\right],$$

and it gives

$$u^{-1}(Y(u) - y(0)) = r\mathrm{S}\left[y(t) - \frac{y^2(t)}{K}\right]$$

simplified to obtain

$$Y(u) = c + ru\mathrm{S}\left[y(t) - \tfrac{y^2(t)}{K}\right] \tag{28}$$

since it has been given that $y(0) = c$. Taking the inverse ST of (28) yields

$$y(t) = c + r\mathrm{S}^{-1}\left[u\mathrm{S}\left[y(t) - \tfrac{y^2(t)}{K}\right]\right] \tag{29}$$

Applying the SIM to (29) and taking (19) into account gives

$$y_0(t) = c,$$

$$y_1(t) = r\mathrm{S}^{-1}\left[u\mathrm{S}\left[c - \tfrac{c^2}{K}\right]\right] = r\mathrm{S}^{-1}\left[u\left(c - \tfrac{c^2}{K}\right)\right]$$

$$= r\left(c - \tfrac{c^2}{K}\right)\mathrm{S}^{-1}[u] = r\left(c - \tfrac{c^2}{K}\right)t.$$

Observe that

$$y_0(t) + y_1(t) = c + r\left(c - \frac{c^2}{K}\right)t.$$

Then

$$y_2(t) = N[y_0(t) + y_1(t)] - N[y_0(t)]$$

$$= r\mathrm{S}^{-1}\left[u\mathrm{S}\left[c + r\left(c - \tfrac{c^2}{K}\right)t - \tfrac{\left(c + r\left(c - \frac{c^2}{K}\right)t\right)^2}{K} - \left(c - \tfrac{c^2}{K}\right)\right]\right]$$

$$= r\mathrm{S}^{-1}\left\lfloor u\mathrm{S}\left\lfloor r\left(c - \frac{c^2}{K}\right)t - \frac{2rc\left(c - \frac{c^2}{K}\right)t + r^2\left(c - \frac{c^2}{K}\right)^2 t^2}{K} - \left(c - \frac{c^2}{K}\right)\right\rfloor\right\rfloor$$

$$= r^2\left(c - \tfrac{c^2}{K}\right)\mathrm{S}^{-1}\left[u\mathrm{S}\left[t - \tfrac{2rct + r^2\left(c - \frac{c^2}{K}\right)t^2}{K}\right]\right]$$

$$= r^2\left(c - \tfrac{c^2}{K}\right)\mathrm{S}^{-1}\left[u\mathrm{S}\left[\left(1 - \tfrac{2c}{K}\right)t - \tfrac{r}{K}\left(c - \tfrac{c^2}{K}\right)t^2\right]\right]$$

$$= r^2\left(c - \tfrac{c^2}{K}\right)\mathrm{S}^{-1}\left[u\left(\left(1 - \tfrac{2c}{K}\right)u - \tfrac{r}{K}\left(c - \tfrac{c^2}{K}\right)2u^2\right)\right]$$

$$= r^2\left(c - \frac{c^2}{K}\right)\mathrm{S}^{-1}\left[\left(1 - \frac{2c}{K}\right)u^2 - \frac{r}{K}\left(c - \frac{c^2}{K}\right)2u^3\right]$$

$$= r^2\left(c - \tfrac{c^2}{K}\right)\left(\left(1 - \tfrac{2c}{K}\right)\tfrac{t^2}{2} - \tfrac{r}{K}\left(c - \tfrac{c^2}{K}\right)\tfrac{t^3}{3}\right)$$

$$= r^2\left(1 - \tfrac{2c}{K}\right)\left(c - \tfrac{c^2}{K}\right)\tfrac{t^2}{2} - \tfrac{r^3}{K}\left(c - \tfrac{c^2}{K}\right)^2\tfrac{t^3}{3}.$$

Therefore, according to (3.16), the solution of (1.3) is

$$y(t) = \lim_{N\to\infty} \textstyle\sum_{n=0}^{N} y_n(t) = y_0 + y_1 + y_3 + \cdots$$

$$= c + r\left(c - \tfrac{c^2}{K}\right)t + r^2\left(1 - \tfrac{2c}{K}\right)\left(c - \tfrac{c^2}{K}\right)\tfrac{t^2}{2} - \tfrac{r^3}{K}\left(c - \tfrac{c^2}{K}\right)^2\tfrac{t^3}{3} + \cdots$$

The given exact solution is compared with the obtained approximate analytical solution in Figure 1 for the logistic model (1.3). The exact solution is (1.4) while the approximate analytical solution is (4.3), which is obtained by using the SIM, which is presented in Section 3. The Figure shows that the obtained approximate analytical solution agrees well with the given exact solution. This shows the efficiency of the SIM in yielding reliable solutions for the differential equations.

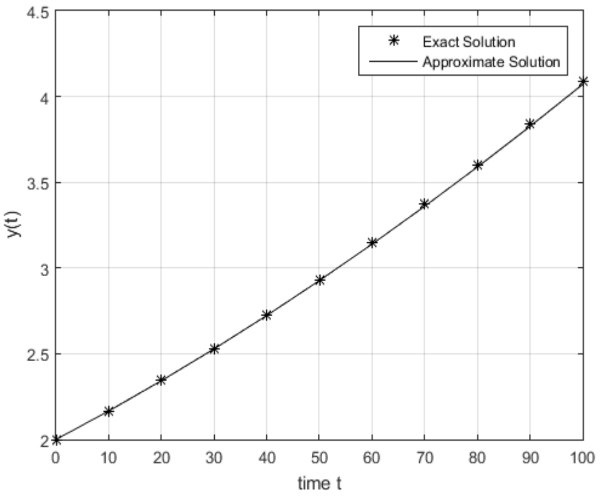

Figure 1: Exact and approximate analytical solutions of logistic model (1.3).

4.2 Logistic model with fractional derivative and proportional delay

This study considers an important modification to Hutchinson's model (1948). The results of the integer-order calculus sometime show discrepancy when compared with the experimental results. This study assigns CFC of order α to transform Hutchinson's model. Several dynamical systems such as population dynamics are naturally inseparable from delays. Consider a logistic growth model with delay and which is associated with CFC of order α that is given by

$$ {}^{CFC}_{\ 0}D^{\alpha}y(t) = ry(t)\left(1-\frac{y(\lambda t)}{K}\right), 0<\lambda<1 \tag{30}$$

with $y_0 = c,$ where c is the initial population, and it is a constant.

Case 1: Observe that for λ = 0 in (30), it gives

$$ {}^{CFC}_{\ 0}D^{\alpha}y(t) = r\left(1-\frac{c}{K}\right)y(t), y(0)=c \tag{31}$$

Taking the ST of both sides of (31) as

$$\frac{M(\alpha)}{(1-\alpha)\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}\left(S[y(t)]-y(0)\right) = r\left(1-\frac{c}{K}\right)S[y(t)],$$

gives

$$\frac{M(\alpha)}{(1-\alpha)}\frac{Y(u)}{\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)} - \frac{M(\alpha)}{(1-\alpha)}\frac{c}{\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}$$

Factorizing gives

$$\left(\frac{M(\alpha)}{(1-\alpha)}\frac{1}{\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)} - r\left(1-\frac{c}{K}\right)\right)Y(u) = \frac{M(\alpha)}{(1-\alpha)}\frac{c}{\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)},$$

and consequently

$$Y(u) = \frac{\frac{M(\alpha)}{(1-\alpha)}\frac{c}{\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}}{\frac{M(\alpha)}{(1-\alpha)}\frac{1}{\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}-r\left(1-\frac{c}{K}\right)} \tag{32}$$

Taking the inverse ST of (32) yields

$$y(t) = S^{-1}\left[\frac{\frac{M(\alpha)}{(1-\alpha)}\frac{c}{\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}}{\frac{M(\alpha)}{(1-\alpha)}\frac{1}{\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}-r\left(1-\frac{c}{K}\right)}\right] = \frac{M(\alpha)}{M(\alpha)-r\left(1-\frac{c}{K}\right)(1-\alpha)}exp\left(\frac{\alpha r\left(1-\frac{c}{K}\right)t}{-M(\alpha)+r\left(1-\frac{c}{K}\right)(1-\alpha)}\right)$$

(33)

The logistic model with fractional derivative and proportional delay is given by (29) and for λ = 0, the model reduces to (30). Through the introduction of t, the fractional derivative and K > 0, which is the carrying capacity of the population are present in (30) and it modifies the Malthusian model (1.1). It is observed that as K → ∞, the new model reduces to a Malthusian equation with a fractional derivative. The solution of the new model is given by (33) and Figure 2 shows its graph.

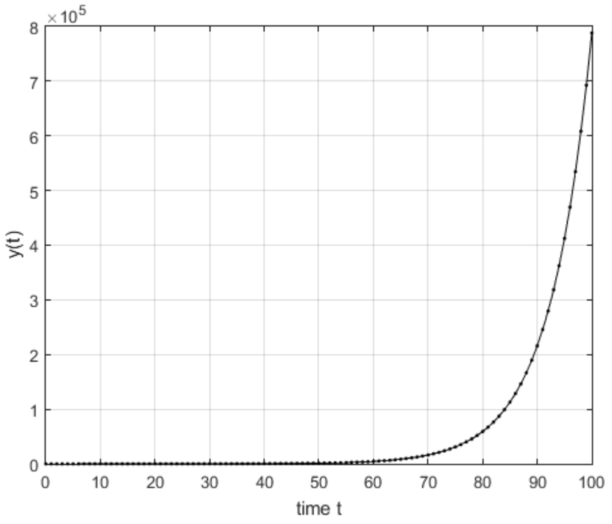

Figure 2: Modified Malthusian equation with fractional derivative and the carrying capacity of the population, K > 0.

Case 2: Notice that for λ = 1 in (28), it gives

$$ {}^{CFC}_{\ 0}D^{\alpha}y(t) = r\left(y(t)-\frac{y^{2}(t)}{K}\right)y(t),\ y(0)=c \tag{34}$$

Taking the ST of both sides of (34) as

$$\frac{M(\alpha)}{(1-\alpha)\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}\left(S[y(t)]-y(0)\right) = rS\left[y(t)-\frac{y^{2}(t)}{K}\right],$$

gives

$$Y(u) = c + \frac{(1-\alpha)\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}{M(\alpha)}rS\left[y(t)-\frac{y^{2}(t)}{K}\right] \tag{35}$$

since $y(0)=c.$ Taking the inverse ST of (35) yields

$$y(t) = c + r\frac{(1-\alpha)}{M(\alpha)}S^{-1}\left[\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)S\left[y(t)-\frac{y^{2}(t)}{K}\right]\right] \tag{36}$$

Applying the SIM to (4.10) and taking (3.15) and (3.16) into account gives

$$y_0(t) = c,$$

$$y_1(t) = r\frac{(1-\alpha)}{M(\alpha)}S^{-1}\left[\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)S\left[y_0(t)-\frac{y_0{}^{2}(t)}{K}\right]\right]$$

$$= r\frac{(1-\alpha)}{M(\alpha)}S^{-1}\left[\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)S\left[c-\frac{c^{2}}{K}\right]\right]$$

$$= r\frac{(1-\alpha)}{M(\alpha)}\left(c-\frac{c^{2}}{K}\right)S^{-1}\left[\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)\right]$$

$$= r\frac{(1-\alpha)}{M(\alpha)}\left(c-\frac{c^{2}}{K}\right)\left(1+\left(\frac{\alpha}{1-\alpha}\right)t\right).$$

Notice that

$$y_0(t)+y_1(t) = c + r\frac{(1-\alpha)}{M(\alpha)}\left(c-\frac{c^{2}}{K}\right)\left(1+\left(\frac{\alpha}{1-\alpha}\right)t\right),$$

then

$$y_2(t) = N[y_0(t)+y_1(t)] - N[y_0(t)]$$

$$= r\frac{(1-\alpha)}{M(\alpha)}S^{-1}\left[\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)S\left[\begin{array}{c} c+r\frac{(1-\alpha)}{M(\alpha)}\left(c-\frac{c^{2}}{K}\right)\left(1+\left(\frac{\alpha}{1-\alpha}\right)t\right) \\ -\frac{\left(c+r\frac{(1-\alpha)}{M(\alpha)}\left(c-\frac{c^{2}}{K}\right)\left(1+\left(\frac{\alpha}{1-\alpha}\right)t\right)\right)^{2}}{K}-\left(c-\frac{c^{2}}{K}\right)\end{array}\right]\right]$$

$$= r^{2}\left(\frac{(1-\alpha)}{M(\alpha)}\right)^{2}\left(c-\frac{c^{2}}{K}\right)S^{-1}\left[\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right) * S\left[\begin{array}{c}\left(1+\left(\frac{\alpha}{1-\alpha}\right)t\right) \\ -\frac{r}{K}\left(\frac{(1-\alpha)}{M(\alpha)}\right)\left(c-\frac{c^{2}}{K}\right)\left(1+2\left(\frac{\alpha}{1-\alpha}\right)t+\left(\frac{\alpha}{1-\alpha}\right)^{2}t^{2}\right)\end{array}\right]\right]$$

$$= r^2\left(\frac{(1-\alpha)}{M(\alpha)}\right)^2\left(c-\frac{c^2}{K}\right)\left(1-\frac{2c}{K}\right)\left(1+2\left(\frac{\alpha}{1-\alpha}\right)t+\left(\frac{\alpha}{1-\alpha}\right)^2\frac{t^2}{2}\right)$$

$$-r^3\left(\frac{(1-\alpha)}{M(\alpha)}\right)^3\left(c-\frac{c^2}{K}\right)^2\left(1+3\left(\frac{\alpha}{1-\alpha}\right)t+2\left(\frac{\alpha}{1-\alpha}\right)^2t^2+\left(\frac{\alpha}{1-\alpha}\right)^3\frac{t^3}{3}\right).$$

Therefore, according to (3.16), the solution of (4.8) is

$$y(t)=\lim_{N\to\infty}\sum_{n=0}^{N}y_n(t)=y_0+y_1+y_3+\cdots$$

$$=c+r\frac{(1-\alpha)}{M(\alpha)}\left(c-\frac{c^{\,2}}{K}\right)\left(1+\left(\frac{\alpha}{1-\alpha}\right)t\right)$$

$$+r^2\left(\frac{1-\alpha}{M(\alpha)}\right)^2\left(c-\frac{c^2}{K}\right)\left(1-\frac{2c}{K}\right)\left(1+2\left(\frac{\alpha}{1-\alpha}\right)t+\left(\frac{\alpha}{1-\alpha}\right)^2\frac{t^2}{2}\right) \qquad (37)$$

$$-r^3\left(\frac{1-\alpha}{M(\alpha)}\right)^3\left(c-\frac{c^2}{K}\right)^2\left(1+3\left(\frac{\alpha}{1-\alpha}\right)t+2\left(\frac{\alpha}{1-\alpha}\right)^2t^2+\left(\frac{\alpha}{1-\alpha}\right)^3\frac{t^3}{3}\right).$$

For λ = 1 in (4.4), which is the logistic model with fractional derivative and proportional delay, it reduces to (4.8). Through the introduction of the fractional derivative, the new logistic model modifies the conventional logistic model (1.3). The solution of the logistic model with fractional derivative is given by (4.11) and Figure 3 shows its graph.

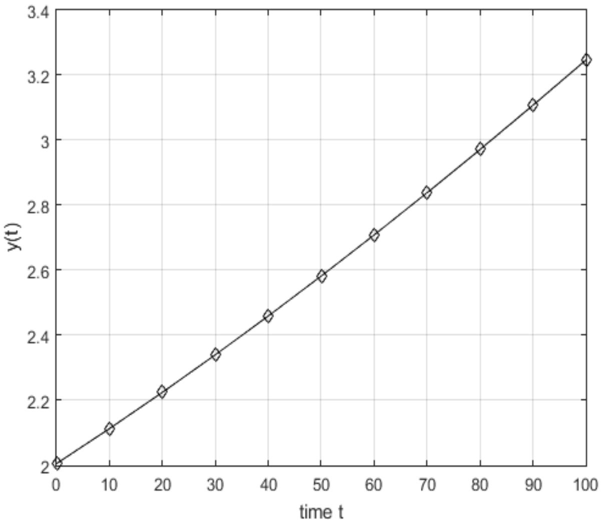

Figure 3: Modified logistic model with fractional derivative.

Case 3: Consider the case where 0 < λ < 1 in (4.4), which can be expressed as

$${}^{CFC}_{\;\;0}D^{\alpha}y(t)=r\left(y(t)\;-\frac{y(t)y(\lambda t)}{K}\right)y(t),\;\; y(0)=c. \qquad (38)$$

Taking the ST of both sides of (4.12) as

$$\frac{M(\alpha)}{(1-\alpha)\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}\left(\mathrm{S}[y(t)]-y(0)\right)=r\mathrm{S}\left[y(t)\;-\frac{y(t)y(\lambda t)}{K}\right],$$

gives

$$Y(u)=c+\frac{(1-\alpha)\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)}{M(\alpha)}r\mathrm{S}\left[y(t)\;-\frac{y(t)y(\lambda t)}{K}\right] \qquad (39)$$

since $y_0(t)=c$. Taking the inverse ST of (4.13) leads to

$$y(t)=c+r\frac{(1-\alpha)}{M(\alpha)}\mathrm{S}^{-1}\left[\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)\mathrm{S}\left[y(t)\;-\frac{y(t)y(\lambda t)}{K}\right]\right] \qquad (40)$$

Applying the SIM to (4.14) and taking (3.15) and (3.16) into account gives

$$y_0(t)=y_0(\lambda t)=c,$$

$$y_1(t)=r\frac{(1-\alpha)}{M(\alpha)}\mathrm{S}^{-1}\left[\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)\mathrm{S}\left[y_0(t)\;-\frac{{y_0}^{2}(t)}{K}\right]\right]$$

$$=r\frac{(1-\alpha)}{M(\alpha)}\mathrm{S}^{-1}\left[\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)\mathrm{S}\left[c-\frac{c^{\,2}}{K}\right]\right]$$

$$=r\frac{(1-\alpha)}{M(\alpha)}\left(c-\frac{c^{\,2}}{K}\right)\mathrm{S}^{-1}\left[\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)\right]$$

$$=r\frac{(1-\alpha)}{M(\alpha)}\left(c-\frac{c^{\,2}}{K}\right)\left(1+\left(\frac{\alpha}{1-\alpha}\right)t\right).$$

Notice that

$$y_0(t)+y_1(t)=c+r\frac{(1-\alpha)}{M(\alpha)}\left(c-\frac{c^{\,2}}{K}\right)\left(1+\left(\frac{\alpha}{1-\alpha}\right)t\right),$$

then

$$y_2(t)=N[y_0(t)+y_1(t)]-N[y_0(t)]$$

$$=r\frac{(1-\alpha)}{M(\alpha)}\mathrm{S}^{-1}\left\lfloor\left(1+\left(\frac{\alpha}{1-\alpha}\right)u\right)\mathrm{S}\left\lfloor\begin{array}{c} c+r\frac{(1-\alpha)}{M(\alpha)}\left(c-\frac{c^{\,2}}{K}\right)\left(1+\left(\frac{\alpha}{1-\alpha}\right)t\right) \\ -\left(c+r\frac{(1-\alpha)}{M(\alpha)}\left(c-\frac{c^{\,2}}{K}\right)\left(1+\left(\frac{\alpha}{1-\alpha}\right)t\right)\right)* \\ \left(c+r\frac{(1-\alpha)}{M(\alpha)}\left(c-\frac{c^{\,2}}{K}\right)\left(1+\left(\frac{\alpha}{1-\alpha}\right)t\right)\right)/K \\ -\left(c-\frac{c^{\,2}}{K}\right)\end{array}\right\rfloor\right\rfloor$$

$$=r^2\left(\frac{1-\alpha}{M(\alpha)}\right)^2\left(c-\frac{c^2}{K}\right)\left(1+2\left(\frac{\alpha}{1-\alpha}\right)t+\left(\frac{\alpha}{1-\alpha}\right)^2\frac{t^2}{2}\right)$$

$$-\frac{r^{2c}\left(\frac{1-\alpha}{M(\alpha)}\right)^2\left(c-\frac{c^2}{K}\right)}{K}\left(2+3\left(\frac{\alpha}{1-\alpha}\right)t+\left(\frac{\alpha}{1-\alpha}\right)^2(1+\lambda)\frac{t^2}{2}\right)$$

$$-\frac{r^3\left(\frac{1-\alpha}{M(\alpha)}\right)^3\left(c-\frac{c^2}{K}\right)^2}{K}\left(1+(2+\lambda)\left(\frac{\alpha}{1-\alpha}\right)t+2(1+\lambda)\left(\frac{\alpha}{1-\alpha}\right)^2t^2+\lambda\left(\frac{\alpha}{1-\alpha}\right)^3\frac{t^3}{3}\right).$$

Therefore, according to (3.16), the solution of (4.12) is

$$y(t)=\lim_{N\to\infty}\sum_{n=0}^{N}y_n(t)=y_0+y_1+y_3+\cdots$$

$$=c+r\frac{(1-\alpha)}{M(\alpha)}\left(c-\frac{c^{\,2}}{K}\right)\left(1+\left(\frac{\alpha}{1-\alpha}\right)t\right)$$

$$+r^2\left(\frac{1-\alpha}{M(\alpha)}\right)^2\left(c-\frac{c^2}{K}\right)\left(1+2\left(\frac{\alpha}{1-\alpha}\right)t+\left(\frac{\alpha}{1-\alpha}\right)^2\frac{t^2}{2}\right) \qquad (41)$$

$$-\frac{r^{2c}\left(\frac{1-\alpha}{M(\alpha)}\right)^2\left(c-\frac{c^2}{K}\right)}{K}\left(2+3\left(\frac{\alpha}{1-\alpha}\right)t+\left(\frac{\alpha}{1-\alpha}\right)^2(1+\lambda)\frac{t^2}{2}\right)$$

$$-\frac{r^3\left(\frac{1-\alpha}{M(\alpha)}\right)^3\left(c-\frac{c^2}{K}\right)^2}{K}\left(1+(2+\lambda)\left(\frac{\alpha}{1-\alpha}\right)t+2(1+\lambda)\left(\frac{\alpha}{1-\alpha}\right)^2t^2+\lambda\left(\frac{\alpha}{1-\alpha}\right)^3\frac{t^3}{3}\right).$$

A most general form of the conventional logistic model is the logistic model with fractional derivative and proportional delay (4.4). SIM is applied to obtain its solution and Figure 4 displays its graph when α = 0.65, c = 2, r = 0.01, K = 10.4, M = 1 and λ = 0.65 in (4.15). Figure 5 displays an inconspicuous effect of variation of λ when the fractional derivative α is held constant. For a constant proportional delay λ, the effects of variation of the fractional derivative, α, is displayed in Figure 6.

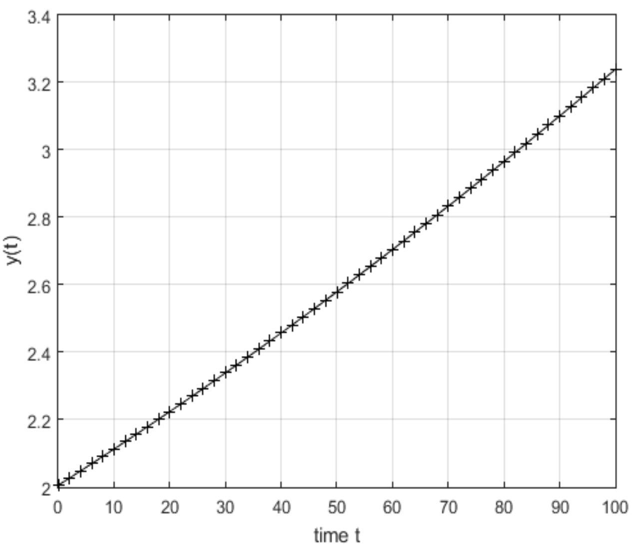

Figure 4: Logistic model with fractional derivative and proportional delay.

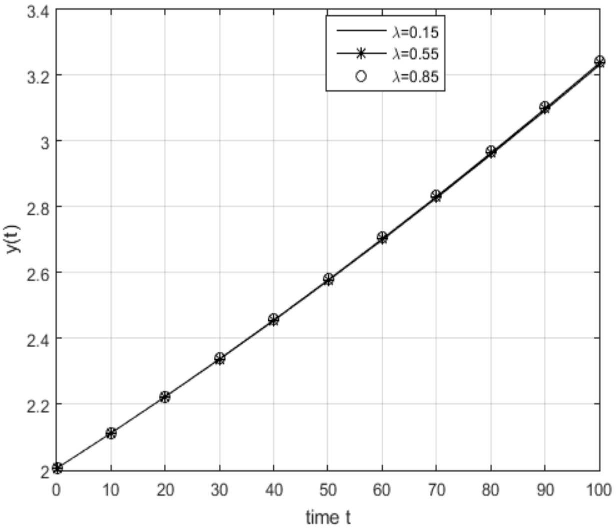

Figure 5: Logistic model with constant fractional derivative and variable proportional delay.

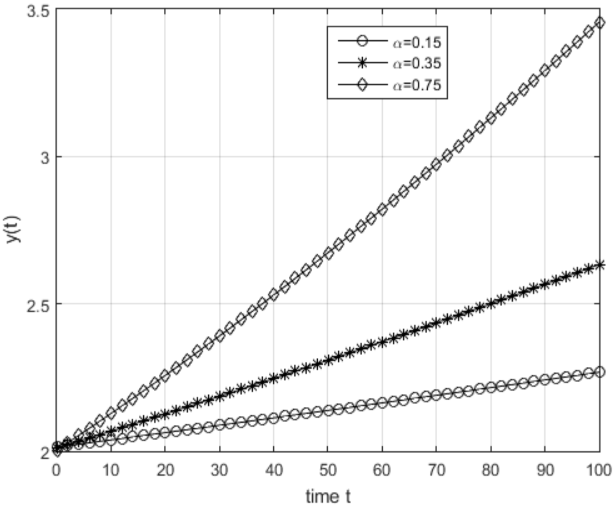

Figure 6: Logistic model with variable fractional derivative and a constant proportional delay.

5. CONCLUSION

The logistic model tops the list among the most popular growth models that are often used in population growth analysis, and it has ample applications in several fields. Recently, various mathematicians and modelers in the applied sciences have paid great attention to the study of differential equations with fractional derivatives due to their significant relevance in real data analysis. This paper considered the conventional logistic model and methods for solving differential equations with fractional derivatives, which are mathematical models that describe several real-life phenomena. This paper introduced the fractional Caputo-Fabrizio derivative and proportional time delay into the conventional logistic model to propose a general and more logistic model with non-locality property for the population growth. Using SIM, the solutions of the different cases of the newly introduced and general logistic model were obtained and their graphs were displayed with the aid of MATLAB.

ABBREVIATION

SIM: Sumudu Iteration Method

CONFLICTS OF INTEREST

The authors declare no Conflicts of Interest.

AVAILABILITY OF DATA AND MATERIALS

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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