

RESEARCH ARTICLE

CLASSIFICATION OF FUNCTIONS IN ASSOCIATION WITH DIFFERENTIAL EQUATIONS

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ABSTRACT

A new classification of functions is presented in this work in relation to ordinary differential equations. A function for which the n 'th derivative of it can be expressed in terms of linear combinations of lower derivatives and the function itself is defined as the linear differential function where n is the lowest derivative for such a relation to hold. The functions that do not obey the rule are defined as nonlinear differential functions. The properties of the differential functions are discussed through theorems and examples. One elementary application is the method of undetermined coefficients where the non-homogenous function should be a linear differential function. Variable coefficient equations are also treated within the context of linear differential functions. The approximation of nonlinear differential functions in terms of linear differential functions are discussed. An application to the perturbation solutions is given. The ideas and definitions presented in this work will add to the understanding of differential equations and their solutions as functions.

KEYWORDS

Ordinary Differential Equations, Linear Differential Functions, Non-Linear Differential Functions, Method of Undetermined Coefficients, Variable Coefficient

1. INTRODUCTION

Differential equations courses are one of the fundamental courses in mathematics and applied sciences at the undergraduate level. An understanding of the nature of the equations is critical in search of solutions. The theory of linear constant coefficient differential equations is well established with their solution procedures. This is the starting point in this work for classification of functions in association with the differential equations. A linear differential function is defined as the function whose n 'th derivative can be expressed as a linear combination of its lower derivatives and the function itself. n is the minimum order of derivative for such a relation to exist and the function is called the linear differential function of order n . Nonlinear differential functions are the functions which cannot be expressed with the mentioned relationship. The definitions immediately lead to many theorems on properties of such functions. The formalism definitely adds to the understanding of the method of undetermined coefficients where the non-homogenous function should take a special form for the method in order to be applicable. The emphasis is not on the solution procedure itself but on the properties of non-homogenous functions. In the case of variable coefficient linear differential equations, the conditions for the existence of linear differential function solutions are discussed. A sample equation treated is the Euler-Cauchy equation. For nonlinear equations, the duffing equation and the Riccati equation are analysed within the context of differential functions. For fundamental solutions of such equations, see for example (O'Neil, 1991). For detailed analysis of the method of undetermined coefficients and its extensions to variable coefficient equations, see (Oliveira, 2013; Cook and Cook, 2022; Fischer, 2022; Leon, 2010, 2015). He presented solutions of the differential equations by factorization of the operators to first order operators with variable coefficients (Euler, 2012). For a detailed treatment and solution methods of Riccati equation, see (Ndiaye, 2022).

The paper is organized as follows: First the definition of a linear differential function is given. The properties of such functions are analysed next. Nonlinear differential functions are defined also. The link to the method of undetermined coefficients is exploited. Variable coefficient second order equations are treated with special cases of Euler-Cauchy equation. The nonlinear equations are treated by analysing the Riccati equation and the Duffing equation. Perturbation type solutions of the Duffing equation are interpreted within the context of the formalism. The formalism presented here may add to the understanding of differential equations for both research and education.

2. LINEAR DIFFERENTIAL FUNCTION DEFINITIONS AND PROPERTIES

In this section, first the definition of a Linear Differential function is given. Based on the definition, the properties of such functions are determined with theorems and examples.

Definition 1

If the n 'th derivative of a given function $y(x)$ can be expressed in terms of the linear combination of its lower derivatives and the function itself

$$y^{(n)} = c_{n-1}y^{(n-1)} + c_{n-2}y^{(n-2)} + \dots + c_2y'' + c_1y' + c_0y \quad (1)$$

for some constants $c_i \in \mathbb{R}$, and $n \in \mathbb{Z}^+$ is the minimum order of derivative for which the above expression holds, then the function $y(x)$ is classified as the linear differential function of order n , denoted by $LDF(n)$ □

Although the n 'th derivative of a $LDF(n)$ function satisfies relationship (1), the same is not true for lower derivatives. A simple example is the $y(x) = \sin x$ for which

$$y' = \sqrt{1 - y^2} \quad (2)$$

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$$y'' = -y \quad (3)$$

and hence the function is LDF(2). The linear relationship appears at the second order of derivative but the first derivative has a nonlinear relationship. Two basic theorems immediately follow from the definition

Theorem 1

If an infinitely differentiable continuous function $y(x)$ is LDF(n), then all its higher derivatives can be expressed as linear relationships of their lower derivatives $\square$

Proof

Since $y(x)$ is LDF(n), equation (1) holds and differentiating once

$$y^{(n+1)} = c_{n-1}y^{(n)} + c_{n-2}y^{(n-1)} + \dots + c_2y''' + c_1y'' + c_0y' \quad (4)$$

which is again a linear relationship. Obviously, all successive derivatives will be linear relationships also $\square$

Theorem 1 explains the reason of defining n as the minimum order in the definition, because, after the minimum order, all derivatives will hold a linear relationship. Another basic theorem from the definition is

Theorem 2

If an infinitely differentiable continuous function $y(x)$ is LDF(n), then it is always possible to find a linear constant coefficient differential operator which annihilates $y(x)$, namely

$$\mathcal{L}^{(n)}y(x) = 0 \quad (5)$$

and the operator is

$$\mathcal{L}^{(n)} = D^n - c_{n-1}D^{n-1} - c_{n-2}D^{n-2} - \dots - c_2D^2 - c_1D - c_0 \quad (6)$$

where $D^k = \frac{d^k}{dx^k} \square$

Proof

Taking all terms in definition (1) to the left-hand side and expressing the derivatives in operator notation, it is easily seen that (5) holds $\square$

Based on Theorems 1 and 2, a LDF(n) function always satisfies a linear differential equation of order m for which $m \geq n$.

Some example functions and their classifications are given in Table 1.

Table 1: LDF function classifications

$y(x)$	Expression	Differential Operator	Classification
$\sin ax, \cos ax$	$y'' = -a^2 y$	$D^2 + a^2$	LDF(2)
e^{ax}	$y' = ay$	$D - a$	LDF(1)
α	$y' = 0$	D	LDF(1)
x^{n-1}	$y^{(n)} = 0$	D^n	LDF(n)
$e^{ax}\sin bx, e^{ax}\cos bx$	$y'' = 2ay' - (a^2 + b^2)y$	$D^2 - 2aD + a^2 + b^2$	LDF(2)
$x\sin ax, x\cos ax$	$y^{(4)} = -2a^2y'' - a^4y$	$(D^2 + a^2)^2$	LDF(4)
a^x	$y' = \ln a y$	$D - \ln a$	LDF(1)
$x^{m-1}e^{ax}\sin bx, x^{m-1}e^{ax}\cos bx$	$y^{(2m)} = \dots$	$(D^2 - 2aD + a^2 + b^2)^m$	LDF($2m$)

Usually, in standard textbooks on differential equations, similar tables are given without the LDF classification.

Example 1

Determine the classification of $\sin^2 x$ and $\cos^3 x$.

Solution

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ and since the operator annihilating the function from Table 1 is $\mathcal{L}^{(3)} = D(D^2 + 4)$, with the expression $y''' = -4y'$, and hence $\sin^2 x$ is LDF(3).

$\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$ and since the operator annihilating the function is $\mathcal{L}^{(4)} = (D^2 + 1)(D^2 + 9)$, with the expression $y^{(4)} = -10y'' - 9y$, and hence $\cos^3 x$ is LDF(4).

Example 1 is generalized in the following theorem:

Theorem 3

The functions $\cos^n x$ and $\sin^n x$ are LDF($n+1$) $\square$

Proof

If n is odd, $n = 2k + 1, k = 0, 1, 2, \dots$, then from the trigonometric identities

$$\cos^n x = A_1 \cos x + A_3 \cos 3x + \dots + A_{2k+1} \cos (2k + 1)x \quad (7)$$

for some constants $A_{2k+1}, k = 0, 1, 2, \dots, \frac{n-1}{2}$. The operator annihilating all these functions is

$(D^2 + 1)(D^2 + 3^2)(D^2 + 5^2) \dots (D^2 + (2k + 1)^2)$. Since there are $k + 1$ terms each with second order operators, then the highest order term is $D^{2(k+1)} = D^{n+1}$.

Similarly, if n is even, then $n = 2k, k = 0, 1, 2, \dots, \frac{n}{2}$, then from trigonometric identities

$$\cos^n x = B_0 + B_2 \cos 2x + B_4 \cos 4x + \dots + B_{2k} \cos 2kx \quad (8)$$

for some constants $B_{2k}, k = 0, 1, 2, \dots, \frac{n}{2}$. The operator annihilating all these functions is

$D(D^2 + 2^2)(D^2 + 4^2) \dots (D^2 + (2k)^2)$. Since there are k terms each with second order operators and a first order operator, then the order of the

operator is $D^{2k+1} = D^{n+1}$. Both cases imply that the function is LDF($n+1$). A similar proof can be given for $\sin^n x$ also $\square$

Example 2

Give the classification of an arbitrary polynomial

$$p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \quad (9)$$

Solution

Since $D^{n+1}p_n = 0$, or $p_n^{(n+1)} = 0$, the polynomial is LDF($n+1$). The following theorem is useful in determining the classification when there are linear combinations of several functions.

Theorem 4

If the function u is LDF(n) and the function v is LDF(m), then their linear combination $w = c_1 u + c_2 v$ is LDF(p) where $\max(n, m) \leq p \leq n + m$ $\square$

Proof

Apply the operator $\mathcal{L}^{(n)}$ and $\mathcal{L}^{(m)}$ to the function, use the linearity property and commutative property

$$\mathcal{L}^{(n)}\mathcal{L}^{(m)}w = \mathcal{L}^{(n)}\mathcal{L}^{(m)}(c_1 u + c_2 v) = c_1 \mathcal{L}^{(n)}\mathcal{L}^{(m)}(u) + c_2 \mathcal{L}^{(n)}\mathcal{L}^{(m)}(v) \quad (10)$$

$$= c_1 \mathcal{L}^{(m)}(\mathcal{L}^{(n)}(u)) + c_2 \mathcal{L}^{(n)}(\mathcal{L}^{(m)}(v)) = c_1 \mathcal{L}^{(m)}(0) + c_2 \mathcal{L}^{(n)}(0) = 0. \quad (11)$$

Hence $\mathcal{L}^{(n+m)}$ always annihilates the function w . It may happen that $\mathcal{L}^{(n)}$ and $\mathcal{L}^{(m)}$ may have common factors of differential operators which reduces the necessary order of the operator to annihilate both functions. In that case, the minimum order is the greatest of n and m , namely $\max(n, m)$ $\square$

Example 3

Give the classification of $w = c_1 x^2 + c_2 \sin x$

Solution

The function x^2 is LDF(3) and $\sin x$ is LDF(2). From Table 1, the operator $D^3(D^2 + 1)$ annihilates w and hence w is LDF(5). This is an example of the highest order $p = n + m = 3 + 2 = 5$.

Example 4

Determine the classification of $w = c_1 x \cos x + c_2 \sin x$

Solution

The function $x\cos x$ is LDF(4) and $\sin x$ is LDF(2). However w is not LDF(6) since the operators share common factors. In fact the operator $(D^2 + 1)^2$ annihilates w and hence w is LDF(4) which is the maximum order of the two operators $D^2 + 1$ and $(D^2 + 1)^2$. This is an example of the lowest order $p = \max(n, m) = \max(2, 4) = 4$.

For the multiplication of functions, the following theorem is posed

Theorem 5

If the function u is LDF(n) and the function v is LDF(m), then their multiplication

$w = uv$ is LDF(p) where $\max(n, m) \leq p \leq nm$ □

Proof

The proof is left to the reader.

Example 5

Determine the classification of $w = uv$ where $u = x^2$ and $v = \cos x$

Solution

$u = x^2$ is LDF(3) and $v = \cos x$ is LDF(2). To annihilate the multiplication of the functions, one needs $\mathcal{L}^{(6)} = (D^2 + 1)^3 = D^6 + 3D^4 + 3D^2 + 1$ hence w is LDF(6). This is an example for the highest limit since $n=3$ and $m=2$, $p = nm = 6$.

Example 6

Determine the classification of $w = uv$ where $u = e^x$ and $v = \sin x$

Solution

$u = e^x$ is LDF(1) and $v = \sin x$ is LDF(2). To annihilate the multiplication of the functions, one needs $\mathcal{L}^{(2)} = D^2 - 2D + 2$ hence w is LDF(2). This is an example for the lowest limit, since $n=1$ and $m=2$, $p = \max(n, m) = 2$.

The next two theorems outline what happens when LDF(n) function is differentiated or integrated.

Theorem 6

If the function y is LDF(n), then its derivative $u = y'$ is either LDF(n) or LDF($n - 1$) □

Proof

y satisfies equation (1). Since $u = y'$, $y = \int u dx$. Substitute into (1)

$$u^{(n-1)} = c_{n-1}u^{(n-2)} + c_{n-2}u^{(n-3)} + \dots + c_1u + c_0 \int u dx \quad (12)$$

Differentiate once

$$u^{(n)} = c_{n-1}u^{(n-1)} + c_{n-2}u^{(n-2)} + \dots + c_1u' + c_0u \quad (13)$$

which shows that $u = y'$ is LDF(n). However, if $c_0 = 0$ in (4), then

$$u^{(n-1)} = c_{n-1}u^{(n-2)} + c_{n-2}u^{(n-3)} + \dots + c_1u \quad (14)$$

and for this special case, it is evident that u is LDF($n-1$) □

Theorem 7

If the function y is LDF(n), then its integral $u = \int y dx$ is either LDF(n) or LDF($n + 1$) □

Proof

The proof is similar to the proof of Theorem 6 and skipped for brevity.

Example 7

The function $y = e^{ax}$ is LDF(1) and differentiation and integration does not change its order. Similarly the function $y = \cos ax$ is LDF(2) and differentiation and integration does not change the order. However, $y = x^n$ is LDF($n+1$) but differentiation results in a function of LDF(n) and integration results in a function of LDF($n+2$), one lower and one higher order.

A final theorem for this section is given for LDF(2) functions.

Theorem 8

If the function y is LDF(2), then $u = y^n$ ($n \in \mathbb{Z}^+$) is LDF($n + 1$) □

Proof

The proof is given for $n = 2$. The idea can be generalized to an arbitrary n . Since y is LDF(2)

$$y'' = c_1y' + c_0y \quad (15)$$

Try first if u is also LDF(2) or not. If u is LDF(2) then

$$u'' = b_1u' + b_0u \quad (16)$$

Calculate u and its derivatives in terms of y and use (15) for y''

$$u = y^2, u' = 2yy', u'' = 2y'^2 + 2c_1yy' + 2c_0y^2 \quad (17)$$

and substitute into (16) leading to the equation

$$2y'^2 + 2c_1yy' + 2c_0y^2 = b_12yy' + b_0y^2. \quad (18)$$

As can be seen, the first term $2y'^2$ cannot be balanced by adjusting the coefficients. So $u = y^2$ is not LDF(2). If u is assumed to be LDF(3), then

$$u''' = b_2u'' + b_1u' + b_0u \quad (19)$$

If $u = y^2$ is substituted in terms of y into (19) with employment of (15),

$$6c_1y'^2 + (8c_0 + 2c_1^2)yy' + 2c_0c_1y^2 = b_2(2y'^2 + 2c_1yy' + 2c_0y^2) + 2b_1yy' + b_0y^2. \quad (20)$$

Terms can be balanced now

$$b_0 = -4c_0c_1, \quad b_1 = 4c_0 - 2c_1^2, \quad b_2 = 3c_1$$

Hence (19) holds and $u = y^2$ is LDF(3). The idea can be generalized to any arbitrary power n in a similar way □

Note that Theorem 3 is a special case of Theorem 8 where u are harmonic functions.

3. NON-LINEAR DIFFERENTIAL FUNCTION DEFINITION

In this section, the definition of a Non-linear Differential Function is given.

Definition 2

If all the finite derivatives of a continuously differentiable function are expressed as nonlinear relations of its lower order derivatives and the function itself, then the function is a non-linear differential function NDF.

Another way of expressing NDF is that, for such functions, no relation of the type

$$y^{(n)} = c_{n-1}y^{(n-1)} + c_{n-2}y^{(n-2)} + \dots + c_2y'' + c_1y' + c_0y \quad (21)$$

exists for finite n .

Example 8

The function $y = \tan x$ is NDF. To see this

$$y' = 1 + y^2, \quad y'' = 2yy', \quad y''' = 2y'^2 + 2yy'', \quad y^{(4)} = 6y'y'' + 2yy''' \quad (22)$$

all being nonlinear relationships in terms of the lower orders of derivatives. The expressions cannot be linearized by successive derivations.

Example 9

The function $y = \ln x$ is NDF. To see this

$$y' = \frac{1}{x} = e^{-y}, \quad y'' = -y'e^{-y} = -e^{-2y}, \quad y''' = 2e^{-3y} \quad (23)$$

with the n 'th derivative $y^{(n)} = (n-1)!(-1)^{n+1}e^{-ny}$

being a nonlinear relationship.

Example 10

The function $y = \frac{1}{1+x}$ is NDF. The derivatives are all expressed as nonlinear relationships

$$y' = -\frac{1}{(1+x)^2} = -y^2, \quad y'' = 2y^3, \quad y''' = -6y^4, \quad \dots, y^{(n)} = n!(-1)^n y^{n+1} \quad (24)$$

Example 11

The function $y = e^{x^2}$ is NDF. The derivatives are all expressed as nonlinear relationships

$$y' = 2y\sqrt{\ln y}, \quad y'' = 2y + 4y\sqrt{\ln y}, \quad y''' = 8y\sqrt{\ln y} + 4y\ln y \quad (25)$$

4. THEOREMS FOR DIFFERENTIAL EQUATIONS

The linear and nonlinear differential function definitions have direct impact on the solutions of ordinary differential equations. Two theorems on the nature of solutions will be given in this section. The last theorem is a systemization of the method of undetermined coefficients.

Theorem 9

If a nonlinear differential equation cannot be cast into a linear constant coefficient homogenous equation by repetitive differentiation, then the solution of the original equation is an NDF□

Proof

By definition, a LDF satisfies the relationship (1). The given differential equation has an order m which is lower than n . It cannot be higher because then it would be already a linear equation since successive differentiation does not spoil the linearity of the equation. Then from the order m up to order n , one may differentiate the whole equation. If one cannot obtain a linear constant coefficient differential equation, then the solution cannot be an LDF. It must be NDF □

Example 12

The duffing equation $u'' + u + \varepsilon u^3 = 0$ possesses a NDF solution since by successive differentiations $u''' + u' + 3\varepsilon u^2 u' = 0$, $u^{(4)} + 6\varepsilon u u'^2 - (1 + \varepsilon u^2)(1 + 3\varepsilon u^2)u = 0$ a linear differential equation cannot be obtained.

The next theorem is vital for the well-known method of undetermined coefficients for which the solutions are generally determined by intuitive guesses.

Theorem 10 (The Method of Undetermined Coefficients)

For the n 'th order non-homogenous linear differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \dots + a_2y'' + a_1y' + a_0y = f(x) \quad (26)$$

i) if $f(x)$ is LDF(m), then the particular solution can be found by the method of undetermined coefficients which is LDF(p), $m \leq p \leq n + m$.

ii) if $f(x)$ is NDF, then the particular solution is NDF and cannot be found by the method of undetermined coefficients □

Proof

For case (i) the original equation possesses a linear differential operator $\mathcal{L}^{(n)}$ and the equation in compact form is $\mathcal{L}^{(n)}y = f(x)$. Since $f(x)$ is LDF(m), it can be annihilated by an operator $\mathcal{L}^{(m)}$. Applying this operator to both sides of the equation result in $\mathcal{L}^{(n+m)}y = 0$. Hence in the most general case, the solution y of this equation is LDF($n+m$). In the method of undetermined coefficients, the solution of $\mathcal{L}^{(n+m)}y = 0$ is back substituted into the original equation and the arbitrary coefficients are determined. However, if the operators do not have any common factors, then a LDF(m) solution satisfies the equation as a particular solution and hence $m \leq p \leq n + m$.

For case (ii) since $f(x)$ is NDF, it cannot be annihilated by a linear differential operator $\mathcal{L}^{(m)}$ and $\mathcal{L}^{(n+m)}y \neq 0$ for any m . Hence, the solution y cannot be a LDF and the method of undetermined coefficients fails. For this case, one may resort to other techniques such as variation of parameters to determine the NDF particular solution □

Example 13

For the equation $u'' + u = e^x$, $n=2$, $m=1$ and the particular solution $y = \frac{1}{2}e^x$ is LDF(1) ($p=m$). However, for the equation $u'' + u = \sin x$, $n=2$, $m=2$ and the particular solution is $y = -\frac{1}{2}x \cos x$, which is LDF(4) ($p=m+n$).

5. VARIABLE COEFFICIENT LINEAR DIFFERENTIAL EQUATIONS

In general, a variable coefficient linear differential equation possesses a solution of NDF. If there exists a LDF solution, then the linear constant coefficient equation should be retractable by successive differentiation.

5.1 First Order Equations

A sample problem and its generalization are given in the next examples.

Example 14

For the first order variable coefficient linear differential equation

$$y' - \frac{\cos x}{\sin x} y = 0 \quad (27)$$

the solution is $y = c \sin x$ which is LDF(2). If one differentiates the above equation once

$y'' + \frac{1-\cos^2 x}{\sin^2 x} y = 0$, which is $y'' + y = 0$ a linear relationship at the second order derivative which verifies the LDF(2) solution.

Example 15

For the variable coefficient first order linear differential equation

$$y' - a(x)y = 0 \quad (28)$$

what is the condition to obtain a LDF(2) solution?

Solution

Differentiate the equation once

$$y'' - (a' + a^2)y = 0. \quad (29)$$

The condition to obtain a LDF(2) solution is $a' + a^2 = c$ for some constant c .

$c = -1$ case was already treated in example 14 yielding $a = \cot x$.

$c = 0$ case yields $a = \frac{1}{x+a_0}$ for some constant a_0 . The solution is $y = c_1(x + a_0)$ a LDF(2) solution.

For a general constant c , however, $a = \sqrt{c} \frac{c_1 e^{2\sqrt{c}x} - 1}{c_1 e^{2\sqrt{c}x} + 1}$ for a LDF(2) solution.

5.2 Second Order Equations

Consider the second order variable coefficient linear homogenous differential equation

$$y'' + p(x)y' + q(x)y = 0 \quad (30)$$

The equation definitely possesses a LDF(2) solution for $p(x) = p_0$ and $q(x) = q_0$ for some constants p_0 and q_0 . The following theorem gives the conditions for (30) to possess LDF(3) and LDF(4) solutions.

Theorem 11

For the conditions

$$p' - p^2 + q = p_0, \quad q' - pq = q_0 \quad (31)$$

equation (30) possesses LDF(3) solutions, and for the conditions

$$p'' - 3pp' - 2pq + p^3 + 2q' = p_0, \quad q'' - pq' - 2qp' + qp^2 - q^2 = q_0 \quad (32)$$

equation (30) possesses LDF(4) solutions for some constants p_0 and q_0 .

Proof

Differentiate (30) two times and use $y'' = -p(x)y' - q(x)y$ when necessary

$$y''' + (p' - p^2 + q)y' + (q' - pq)y = 0 \quad (33)$$

$$y^{(4)} + (p'' - 3pp' - 2pq + p^3 + 2q')y' + (q'' - pq' - 2qp' + qp^2 - q^2)y = 0 \quad (34)$$

Equating the coefficients to constants, conditions (31) and (32) are obtained.

Note that Theorem 11 does not give all LDF(3) and LDF(4) solutions. For example, for LDF(3) solutions, if the solution of

$$(p' - p^2 + q)y' + (q' - pq)y = 0 \quad (35)$$

turns out to be a solution of

$$y''' = 0 \quad (36)$$

then again one obtains a LDF(3) solution.

Example 16

Find two special LDF(3) solutions of (30).

Solution

i) If one takes $p_0 = q_0 = 0$, from (31),

$$p' - p^2 + q = 0, \quad q' - pq = 0. \quad (37)$$

The second equation is solved first

$$q = c_1 e^{\int p dx} \quad (38)$$

and substituted into the first one

$$p' - p^2 + c_1 e^{\int p dx} = 0 \quad (39)$$

which is highly nonlinear. Try a solution $p = \frac{c_2}{x}$ leading to

$$-\frac{c_2}{x^2} - \frac{c_2^2}{x^2} + c_1 x^{c_2} = 0 \quad (40)$$

which dictates $c_2 = -2$ for the powers to be compatible. $c_1 = 2$ is a consequence of this choice. Then $p = -\frac{2}{x}$, $q = \frac{2}{x^2}$ and the equation is of Euler-Cauchy type

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0 \quad (41)$$

The third order equation is $y''' = 0$ with a solution $y = c_1 x^2 + c_2 x + c_3$. Substituting this to the above second order equation, $c_3 = 0$ and the solution is

$$y = c_1 x^2 + c_2 x \quad (42)$$

ii) If one takes $p_0 = 1, q_0 = 0$, from (31),

$$p' - p^2 + q = 1, \quad q' - pq = 0. \quad (43)$$

with

$$q = c_1 e^{\int p dx} \quad (44)$$

and

$$p' - p^2 + c_1 e^{\int p dx} = 1 \quad (45)$$

which is highly nonlinear. Select $c_1 = 0$, and hence $q = 0$, leading to

$$p' - p^2 = 1 \quad (46)$$

with

$$p = \tan x \quad (47)$$

leading to the second order equation

$$y'' + \tan x y' = 0 \quad (48)$$

The third order equation to be solved is $y''' + y' = 0$ with a solution $y = c_1 + c_2 \sin x + c_3 \cos x$. Substituting this to the above second order equation, $c_3 = 0$ and the solution is

$$y = c_1 + c_2 \sin x \quad (49)$$

Example 17

Find a LDF(4) solution of (30) for the special case of $q = 0, p_0 = q_0 = 0$.

Solution

From (32) for this special case, the second condition is already satisfied for $q = 0$ and the first condition reduces to

$$p'' - 3pp' + p^3 = 0 \quad (50)$$

A trial solution $p = \frac{A}{x}$ works and gives the condition

$$A(A+1)(A+2) = 0 \quad (51)$$

i) $A = 0$ leads to $y'' = 0$, a LDF(2) solution hence discarded.

ii) $A = -1$ leads to $y''' = 0$, a LDF(3) solution hence discarded.

iii) $A = -2$ leads to $y^{(4)} = 0$, a LDF(4) solution which is the solution searched. Substituting $y = c_1 x^3 + c_2 x^2 + c_3 x + c_4$ into the original equation with $p = -\frac{2}{x}$, $q = 0$, the coefficients $c_2 = c_3 = 0$, and the LDF(4) solution is $y = c_1 x^3 + c_4$.

Example 18 Euler-Cauchy Differential Equation

Consider the Euler-Cauchy differential equation

$$y'' + \frac{A}{x}y' + \frac{B}{x^2}y = 0 \quad (52)$$

Find conditions for A and B so that the equation accepts a LDF(3) solution.

Solution

Differentiate the equation once and use the equation to eliminate the second derivative

$$y''' + \frac{1}{x^2}(B - A - A^2)y' - \frac{1}{x^3}B(2 + A)y = 0 \quad (53)$$

An obvious solution is $A = -2$ and $B = 2$. However, for a more general solution take

$$\frac{1}{x^2}(B - A - A^2)y' - \frac{1}{x^3}B(2 + A)y = 0 \quad (54)$$

which yields

$$y = x^{B(2+A)/(B-A-A^2)} \quad (55)$$

and the reduced equation is $y''' = 0$. For y to satisfy the reduced equation, $B(2 + A)/(B - A - A^2) = 2$

or

$$B = -2(1 + A) \quad (56)$$

which is the condition for a LDF(3) solution.

6. NONLINEAR DIFFERENTIAL EQUATIONS

A nonlinear differential equation usually possesses a NDF solution. It may however happen that for some special cases, the solutions turn out to be LDF solutions. Sometimes those LDF solutions are exact and sometimes, the exact NDF is approximated by a LDF solution. Two sample problems, namely the Riccati and the Duffing equation are analysed in this section.

Example 19 Riccati Equation

For the specific Riccati equation

$$y' = y^2 + Axy + Bx^2 + C \quad (57)$$

find A, B, C so that the equation has a LDF(2) solution.

Solution

Differentiating the equation once

$$y'' = y(2y' + A) + x(Ay' + 2B) \quad (58)$$

The parenthesis should vanish

$$2y' + A = 0, \quad Ay' + 2B = 0 \quad (59)$$

leading to $A = 2, B = 1$. Both equations are identical now with a solution $y = -x + c_1$

In order this solution to satisfy the original equation $-1 = c_1^2 + C$. Hence $C = -1 - c_1^2$.

In summary, the LDF(2) solution for the Riccati equation

$$y' = y^2 + 2xy + x^2 - 1 - c_1^2 \quad (60)$$

is

$$y = -x + c_1 \quad (61)$$

Example 20 The Duffing Equation

In Example 12, it is shown that the Duffing solutions are NDF. However, those NDF solutions may be approximated by LDF solutions. For the Duffing equation,

$$u'' + u + \varepsilon u^3 = 0 \quad (62)$$

if ε is small, a perturbative solution can be constructed (Nayfeh, 1981) by various methods such as the Lindstedt-Poincare method, the Method of Multiple Scales, Averaging method etc. which is

$$u = a \cos \left[\left(1 + \varepsilon \frac{3}{8} a^2 \right) t + \beta \right] + \varepsilon \left\{ \frac{1}{32} a^3 \cos \left[3 \left(1 + \varepsilon \frac{3}{8} a^2 \right) t + 3\beta \right] + b \cos \left[\left(1 + \varepsilon \frac{3}{8} a^2 \right) t + \gamma \right] \right\} \quad (63)$$

This solution can be annihilated by the operator

$$\left(D^2 + \left(1 + \varepsilon \frac{3}{8} a^2 \right)^2 \right) \left(D^2 + 9 \left(1 + \varepsilon \frac{3}{8} a^2 \right)^2 \right) \quad (64)$$

hence the solution is LDF(4). Carrying out the calculations to higher orders of approximation, one may obtain the higher harmonics also. To summarize, the approximate solution of the Duffing equation is

i) LDF(4) for first correction term included

ii) LDF(6) for second correction term included

iii) LDF(2n+2) for n'th correction term included

7. CONCLUDING REMARKS

The functions are classified for the first time with reference to differential

equations. The linear differential functions and the non-linear differential functions are defined. Properties of such functions are exploited via theorems and worked examples. Applications to differential equations and their solutions are shown. With the aid of the given definitions, the method of undetermined coefficients can be understood better. Also the difference of the method of undetermined coefficients and the method of variations of parameters are systematically discussed. The material presented here might improve the understanding of the differential equations and their solutions. The formalism given in the text may be useful for designing new differential equations possessing linear differential function solutions.

COMPETING INTERESTS

Author declares no competing interests

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