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REVIEW ARTICLE

A HIGHER ORDER A-STABLE DIAGONALLY IMPLICIT 2-POINT SUPER CLASS OF BLOCK EXTENDED BACKWARD DIFFERENTIATION FORMULA FOR SOLVING STIFF INITIAL VALUE PROBLEMS

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ABSTRACT

This paper presents the formulation of higher order diagonally implicit 2-point super class of block extended backward differentiation formula (2DSBEBDF) for solving first order stiff initial value problems. The order of the 2DSBEBDF method is derived and found to be four. The Stability analysis of the method shows that the method is zero-stable and its absolute stability region shows that the method is A-stable within the stiff stability interval $-1 \leq \rho < 1$. The numerical experiments demonstrate the effectiveness of the 2DSBEBDF method in solving stiff initial value and oscillatory problems over the existing stiff solver found in the literature.

KEYWORDS

Stiff Initial Value Problems; Diagonally Implicit Method; Stability; and Block Extended Backward Differentiation Formula.

1. INTRODUCTION

Stiff initial value problems (IVPs) arise in various scientific and engineering applications, particularly in chemical kinetics, control theory, electrical circuits, and fluid dynamics (Yusuf et al., 2024). These problems exhibit rapid transients and require specialized numerical techniques to ensure stability and efficiency (Suleiman et al., 2014). The term "stiffness" in differential equations was first formally introduced, who observed that standard explicit methods fail to handle problems with vastly different timescales efficiently (Curtiss and Hirschfelder, 1952). Since then, the numerical analysis of stiff systems has been extensively studied, leading to the development of robust algorithms and stability criteria (Hairer and Wanner, 1996; Shampine and Gear, 1979).

Stiff systems are characterized by the presence of eigenvalues of widely varying magnitudes in the Jacobian matrix of the system, leading to rapid variations in some components of the solution (Gear, 1971). Traditional explicit solvers, such as Runge-Kutta methods, often require prohibitively small step sizes to maintain stability, rendering them inefficient for stiff problems. Instead, implicit schemes, such as backward differentiation formulas (BDFs) and Rosenbrock methods, have been developed to handle stiffness effectively (Cash, 1983; Butcher, 2008). In this paper, we consider the numerical approximation of first order stiff IVPs of the form:

$$y' = f(x, y), \quad y(a) = y_0, \quad a \leq x \leq b \tag{1}$$

One of the key challenges in solving stiff IVPs is choosing appropriate numerical solvers that balance accuracy and efficiency. The A-stability and L-stability properties of numerical methods are crucial in addressing stiffness (Dahlquist, 1963). A-stable methods remain stable regardless of step size, while L-stable methods further dampen unwanted oscillations, making them particularly useful for stiff problems. The development of adaptive step-size control in implicit solvers has significantly improved their performance, with algorithms such as the variable step size BBDF, and variable-order variable step size BDF Methods being widely used (Suleiman et al., 2013; Ibrahim et al., 2008; Zawawi et al., 2021; Abasi et al., 2014).

Several real-world applications highlight the importance of efficient stiff IVP solvers. In chemical kinetics, reactions often occur on vastly different timescales, necessitating implicit solvers for accurate simulation (Verwer et al., 1999). Similarly, in electrical circuit analysis, stiff differential equations arise due to the presence of components with drastically different time constants (Wanner and Chen, 2008). In climate modeling, atmospheric and oceanic interactions lead to stiff systems that require stable numerical integration techniques (Schiesser and Griffiths, 2009).

Despite significant advances in stiff IVP solvers, challenges remain in optimizing computational efficiency, particularly for large-scale systems in high-performance computing environments (Knoll and Keyes, 2004). Recent developments in exponential integrators methods offer promising directions for improving the efficiency of stiff problem solvers (Suleiman et al., 2015; Ibrahim et al., 2007; Alhassan et al., 2024; Alhassan and Musa, 2023; Alhassan et al., 2023; Ijam and Ibrahim, 2019). A famous result due to Dahlquist (1963) has shown that no A-stable linear multistep method (LMM) can have order greater than 2. However, the strategies for improving stability, order of accuracy, and efficiency of explicit and implicit multistep methods have been suggested, which include:

- Using higher derivatives of the solution
- Throwing in additional stages, off-step points, super future points, and the likes, which leads to larger field of general linear methods (Hairer and Wanner, 1996).

In an attempt to overcome the Dahlquist's second barrier, the conventional non-block implicit backward differentiation formula (BDF) was modified to develop a new class of generalized multistep methods

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called implicit extended backward differentiation formulas (EBDFs) for stiff IVPs (Cash, 1980). This was achieved by incorporating a "super future point" into the BDF method, as suggested by (Hairer and Wanner, 1996). The EBDF method is a non-block implicit scheme that approximates only one solution value per step and exhibits A-stable methods of order up to four corresponding to the step number k ranging from 1 to 4 and also $A(\alpha)$ −stable methods of order up to nine corresponding to the step number k ranging from 5 to 8.

However, the performance of EBDF method is seen to be better than that of CBDF. The EBDF method is of the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_k f_{n+k} + \beta_{k+1} f_{n+k+1} \tag{2}$$

The implementation procedures for formula (2) involve predicting the required solution using the conventional BDF and correcting the solution using the EBDF method of higher order. The procedures outlined are as follows:

i. Compute $\bar{y}_{n+k}^{(n)}$ as the solution of conventional k-step BDF
$$y_{n+k} - h\hat{\beta}_k f_{n+k} = -\sum_{j=0}^{k-1} \hat{\alpha}_j y_{n+j} \tag{3}$$

ii. Compute $\bar{y}_{n+k+1}^{(n)}$ as the solution of
$$y_{n+k+1} - h\hat{\beta}_k f_{n+k+1} = -\hat{\alpha}_{k-1}\bar{y}_{n+k}^{(n)} - \sum_{j=0}^{k-2} \hat{\alpha}_j y_{n+j+1} \tag{4}$$

iii. Compute $\bar{f}_{n+k+1} = f\left(x_{n+k+1}, \bar{y}_{n+k+1}^{(n)}\right)$

iv. Compute y_{n+k} from (2) written in the form
$$y_{n+k} - h\beta_k f_{n+k} = -\sum_{j=0}^{k-1} \alpha_j y_{n+j} + h\beta_{k+1}\bar{f}_{n+k+1} \tag{5}$$

In this paper, we extend the idea presented in (2) to the existing 2-point diagonally implicit super class of block backward differentiation formula (2DSBBDF) method of the form (Babangida et al., 2016):

$$\sum_{j=0}^{1+k} \alpha_j y_{n+j-1} = h\beta_k (f_{n+k} - \rho f_{n+k-1}),\ k = 1,2 \tag{6}$$

We propose a new block implicit scheme, denoted as the 2-point diagonally implicit super class of block extended backward differentiation formula (2SDBEBDF), by introducing an additional future point to (6). This yields a formula of the form:

$$\sum_{j=0}^{1+k} \alpha_j y_{n+j-1} = h\beta_k (f_{n+k} - \rho f_{n+k-1}) + h\beta_{k+1} f_{n+k+1} \tag{7}$$

The remainder of this paper is organized as follows. Section 2 provides the derivation of the proposed method, while Section 3 presents the determination of the method's order and error constant. A stability analysis of the method is conducted in Section 4. Implementation details are outlined in Section 5. Numerical results and test problems are presented in section 6, followed by concluding remarks in section 7.

2. Derivation Of The Method

This section presents the mathematical formulation of 2DSBEBDF method by modifying and incorporating the super future point to the existing third order diagonally implicit 2-point super class of block backward differentiation formula (2DSBBDF) for the integration of Stiff IVPs developed in this study, which has been derived using Taylor series and express as (Babangida et al., 2016):

$$\left.\begin{aligned} y_{n+1} &= \frac{1+\rho}{-3+\rho} y_{n-1} - \frac{4}{-3+\rho} y_n + \frac{2}{-3+\rho}\rho h f_n - \frac{2}{-3+\rho} h f_{n+1} \\ y_{n+2} &= -\frac{2+\rho}{2\rho-11} y_{n-1} + \frac{3(2\rho+3)}{2\rho-11} y_n - \frac{3(\rho+6)}{2\rho-11} y_{n+1} - \frac{6}{2\rho-11}\rho h f_{n+1} + \frac{6}{2\rho-11} h f_{n+2} \end{aligned}\right\} \tag{8}$$

These formulae in (8) represent A-stable block implicit method of order 3 that approximates two solution values concurrently per integration step. Therefore, the interpolation points involved for the newly proposed method is as shown below.

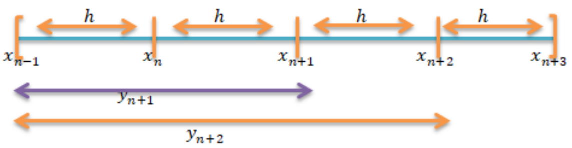

Figure 1: Interpolation Points Involved in the 2DSBEBDF method

Definition 1: The 2-point diagonally implicit super class of block extended backward differentiation formula (2DSBEBDF) is defined by

$$\sum_{j=0}^{1+k} \alpha_j y_{n+j-1} = h\beta_k (f_{n+k} - \rho f_{n+k-1}) + h\beta_{k+1} f_{n+k+1},\ k = 1,2 \tag{9}$$

where $k = 1$ represents the first point, and $k = 2$ corresponds to the second points. The scheme (9) is derived using Taylor's series expansion.

Derivation of the First Point: $k = 1$

To determine the coefficient of the first point, the linear difference operator L_i associated with the first point of (9) is defined by:

$$L_1[y(x_n), h]: \alpha_0 y_{n-1} + \alpha_1 y_n + \alpha_2 y_{n+1} - h\beta_1 f_{n+1} + h\rho\beta_1 f_n - h\beta_2 f_{n+2} = 0, \tag{10}$$

By expanding the corresponding approximate relationship for (10) as a Taylor series about any point x_n and collecting the like terms, we have the following system of equations as:

$$c_0 y(x_n) + c_1 h y'(x_n) + c_2 h^2 y''(x_n) + + c_3 h^3 y'''(x_n) + \cdots \tag{11}$$

where,

$$\left.\begin{aligned} C_0 &= \alpha_0 + \alpha_1 + \alpha_2 = 0 \\ C_1 &= -\alpha_0 + \alpha_2 - \beta_1(1-\rho) - \beta_2 = 0 \\ C_2 &= \frac{1}{2}\alpha_0 + \frac{1}{2}\alpha_2 - \beta_1 - 2\beta_2 = 0 \\ C_3 &= -\frac{1}{6}\alpha_0 + \frac{1}{6}\alpha_2 - \frac{1}{2}\beta_1 - 2\beta_2 = 0 \end{aligned}\right\}, \tag{12}$$

when obtaining the first point, the coefficient α_2 is normalized to one. Solving this system of equations in (12) provides the values for α_j and β_j as:

Table 1: Coefficient of the first point 2DSBEBDF

α_0	α_1	α_2	β_1	β_2
$-\dfrac{8\rho+5}{16\rho-23}$	$-\dfrac{4(2\rho-7)}{16\rho-23}$	1	$-\dfrac{22}{16\rho-23}$	$\dfrac{2(\rho+2)}{16\rho-23}$

Substituting these values in equation (10), we obtain

$$y_{n+1} = \frac{8\rho+5}{16\rho-23} y_{n-1} + \frac{4(2\rho-7)}{16\rho-23} y_n - \frac{22}{16\rho-23} h f_{n+1} + \frac{22}{16\rho-23}\rho h f_n + \frac{2(\rho+2)}{16\rho-23} h f_{n+2}, \tag{13}$$

Derivation of the second point: $k = 2$

Similarly, to determine the coefficient of the second point, the linear difference operator associated with the second point of (9) is defined as:

$$L_2[y(x_n), h]: \alpha_0 y_{n-1} + \alpha_1 y_n + \alpha_2 y_{n+1} + \alpha_3 y_{n+2} - h\beta_2 f_{n+2} + h\rho\beta_2 f_{n+1} - h\beta_3 f_{n+3} = 0, \tag{14}$$

Expanding the corresponding approximate relation for (14) as a Taylor series about any point x_n and grouping the like terms gives:

$$c_0 y(x_n) + c_1 h y'(x_n) + c_2 h^2 y''(x_n) + + c_3 h^3 y'''(x_n) + \cdots \tag{15}$$

where,

$$\left.\begin{aligned} c_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ c_1 &= -\alpha_0 + \alpha_2 + 2\alpha_3 - \beta_2(1-\rho) - \beta_3 = 0 \\ c_2 &= \frac{1}{2}\alpha_0 + \frac{1}{2}\alpha_2 + 2\alpha_3 - \beta_2(2-\rho) - 3\beta_3 = 0 \\ c_3 &= -\frac{1}{6}\alpha_0 + \frac{1}{6}\alpha_2 + \frac{4}{3}\alpha_3 - \beta_2\left(2 - \frac{1}{2}\rho\right) - \frac{9}{2}\beta_3 = 0 \\ c_4 &= \frac{1}{24}\alpha_0 + \frac{1}{24}\alpha_2 + \frac{2}{3}\alpha_3 - \beta_2\left(\frac{4}{3} - \frac{1}{6}\rho\right) - \frac{9}{2}\beta_3 = 0 \end{aligned}\right\}, \tag{16}$$

when obtaining the second point, the coefficient α_3 is normalized to one. Solving this system of equations in (16) provides the values for α_j and β_j as:

Table 2: Coefficient of the second point for 2DSBEBDF

α_0	α_1	α_2	α_3	β_2	β_3
$\dfrac{14\rho+17}{76\rho-197}$	$-\dfrac{9(12\rho+11)}{76\rho-197}$	$\dfrac{9(2\rho+31)}{76\rho-197}$	1	$-\dfrac{150}{76\rho-197}$	$\dfrac{6(\rho+3)}{76\rho-197}$

Substituting these values in equation (14), we obtain

$$y_{n+2} = -\frac{14\rho+17}{76\rho-197} y_{n-1} + \frac{9(12\rho+11)}{76\rho-197} y_n - \frac{9(2\rho+31)}{76\rho-197} y_{n+1} - \frac{150}{76\rho-197} h f_{n+2} + \frac{150}{76\rho-197}\rho h f_{n+1} + \frac{6(\rho+3)}{76\rho-197} h f_{n+3} \tag{17}$$

Therefore, the expression for diagonally implicit 2-point super class of block extended backward differentiation formula (2DSBEBDF) is given as:

$$\left.\begin{aligned} y_{n+1} &= \frac{8\rho+5}{16\rho-23} y_{n-1} + \frac{4(2\rho-7)}{16\rho-23} y_n - \frac{22}{16\rho-23} h f_{n+1} \\ &\quad + \frac{22}{16\rho-23}\rho h f_n + \frac{2(\rho+2)}{16\rho-23} h f_{n+2} \\ y_{n+2} &= -\frac{14\rho+17}{76\rho-197} y_{n-1} + \frac{9(12\rho+11)}{76\rho-197} y_n - \frac{9(2\rho+31)}{76\rho-197} y_{n+1} \\ &\quad - \frac{150}{76\rho-197} h f_{n+2} + \frac{150}{76\rho-197}\rho h f_{n+1} + \frac{6(\rho+3)}{76\rho-197} h f_{n+3} \end{aligned}\right\} \tag{18}$$

To ensure stability, the parameter ρ is constrained within the range $-1 \leq$

$\rho < 1$, allowing for any value within this interval to be utilized. For the purpose of this paper and the numerical implementation of the method (18), the value of ρ has been specifically chosen as $\frac{1}{2}$. Setting $\rho = \frac{1}{2}$ in (18) leads to the following formulae as:

$$\left.\begin{aligned} y_{n+1} &= -\tfrac{3}{5}y_{n-1} + \tfrac{8}{5}y_n + \tfrac{22}{15}hf_{n+1} - \tfrac{11}{15}hf_n - \tfrac{1}{3}hf_{n+2} \\ y_{n+2} &= \tfrac{8}{53}y_{n-1} - \tfrac{51}{53}y_n + \tfrac{96}{53}y_{n+1} + \tfrac{50}{53}hf_{n+2} - \tfrac{25}{53}hf_{n+1} - \tfrac{7}{53}hf_{n+3} \end{aligned}\right\} \tag{19}$$

To achieve optimal accuracy in the analysis of basic properties, the subsequent sections will provide a generalized analysis of our method's basic stability and convergence properties, including order, consistency, zero-stability and A-stability of our proposed method.

This analysis will be presented in terms of the parameter ρ, allowing for a comprehensive understanding of the method's behavior.

3. Derivation Of Order And Error Constant Of The Method

This section presents the order and error constant of the method for different values of ρ corresponding to the formulae in (18). To derive the order of the method, the formulae in (18) can be expressed as:

$$\left.\begin{aligned} &-\tfrac{8\rho+5}{16\rho-23}y_{n-1} - \tfrac{4(2\rho-7)}{16\rho-23}y_n + y_{n+1} \\ &= \tfrac{22}{16\rho-23}\rho hf_n - \tfrac{22}{16\rho-23}hf_{n+1} + \tfrac{2(\rho+2)}{16\rho-23}hf_{n+2} \\ &\tfrac{14\rho+17}{76\rho-197}y_{n-1} - \tfrac{9(12\rho+11)}{76\rho-197}y_n + \tfrac{9(2\rho+31)}{76\rho-197}y_{n+1} + y_{n+2} \\ &= \tfrac{150}{76\rho-197}\rho hf_{n+1} - \tfrac{150}{76\rho-197}hf_{n+2} + \tfrac{6(\rho+3)}{76\rho-197}hf_{n+3} \end{aligned}\right\} \tag{20}$$

The matrix representation associated with (20) is given by

$$\sum_{j=0}^{1} A_j^* Y_{m+j-1} = h\sum_{j=0}^{2} B_j^* F_{m+j-1}, \tag{21}$$

Where A_0^*, A_1^*, B_0^*, B_1^* and B_2^* are square matrices defined by

$$A_0^* = \begin{bmatrix} -\frac{8\rho+5}{16\rho-23} & -\frac{4(2\rho-7)}{16\rho-23} \\ \frac{14\rho+17}{76\rho-197} & -\frac{9(12\rho+11)}{76\rho-197} \end{bmatrix} \quad A_1^* = \begin{bmatrix} 1 & 0 \\ \frac{9(2\rho+31)}{76\rho-197} & 1 \end{bmatrix} \quad B_0^* = \begin{bmatrix} 0 & \frac{22\rho}{16\rho-23} \\ 0 & 0 \end{bmatrix}$$

$$B_1^* = \begin{bmatrix} -\frac{22}{16\rho-23} & \frac{2(\rho+2)}{16\rho-23} \\ \frac{150\rho}{76\rho-197} & -\frac{150}{76\rho-197} \end{bmatrix} \quad B_2^* = \begin{bmatrix} 0 & 0 \\ \frac{6(\rho+3)}{76\rho-197} & 0 \end{bmatrix}$$

and Y_{m-1}, Y_m, F_{m-1}, F_m and F_{m+1} are column vectors defined by

$$Y_{m-1} = \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix},\ Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix},\ F_{m-1} = \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix},\ F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix},\ F_{m+1} = \begin{bmatrix} f_{n+3} \\ f_{n+4} \end{bmatrix}$$

Equation (21) can also be expressed as

$$\begin{bmatrix} -\frac{8\rho+5}{16\rho-23} & -\frac{4(2\rho-7)}{16\rho-23} \\ \frac{14\rho+17}{76\rho-197} & -\frac{9(12\rho+11)}{76\rho-197} \end{bmatrix}\begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \frac{9(2\rho+31)}{76\rho-197} & 1 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = h\begin{bmatrix} 0 & \frac{22\rho}{16\rho-23} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix}$$
$$+h\begin{bmatrix} -\frac{22}{16\rho-23} & \frac{2(\rho+2)}{16\rho-23} \\ \frac{150\rho}{76\rho-197} & -\frac{150}{76\rho-197} \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix} + h\begin{bmatrix} 0 & 0 \\ \frac{6(\rho+3)}{76\rho-197} & 0 \end{bmatrix}\begin{bmatrix} f_{n+3} \\ f_{n+4} \end{bmatrix} \tag{22}$$

Let A_0^*, A_1^*, B_0^*, B_1^* and B_2^* be block matrices defined by

$A_0^* = (A_0 \quad A_1)$, $A_1^* = (A_2 \quad A_3)$, $B_0^* = (B_0 \quad B_1)$, $B_1^* = (B_2 \quad B_3)$ and $B_2^* = (B_4 \quad B_5)$

Where,

$$A_0 = \begin{bmatrix} -\frac{8\rho+5}{16\rho-23} \\ \frac{14\rho+17}{76\rho-197} \end{bmatrix}, A_1 = \begin{bmatrix} -\frac{4(2\rho-7)}{16\rho-23} \\ -\frac{9(12\rho+11)}{76\rho-197} \end{bmatrix}, A_2 = \begin{bmatrix} 1 \\ \frac{9(2\rho+31)}{76\rho-197} \end{bmatrix}, A_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} \frac{22\rho}{16\rho-23} \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} -\frac{22}{16\rho-23} \\ \frac{150\rho}{76\rho-197} \end{bmatrix}, B_3 = \begin{bmatrix} \frac{2(\rho+2)}{16\rho-23} \\ -\frac{150}{76\rho-197} \end{bmatrix}, B_4 = \begin{bmatrix} 0 \\ \frac{6(\rho+3)}{76\rho-197} \end{bmatrix}, B_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Definition 2 (Order): The order of the method and its associated linear difference operator given by

$$L[y(x);h] = \sum_{j=0}^{k=3}[A_j y(x+jh)] - h\sum_{j=0}^{k+1}[B_j y'(x+jh)], \tag{23}$$

is defined as a unique integer p such that C_q, $0(1)p$, and $c_{p+1} \neq 0$ where C_q is a constant column vector defined by

$$\left.\begin{aligned} c_0 &= A_0 + A_1 + A_2 + \cdots + A_k \\ c_1 &= A_1 + 2A_2 + 3A_3 + \cdots + kA_k - \begin{pmatrix} B_0 + B_1 \\ +B_2 + \cdots + B_{k+1} \end{pmatrix} \\ &\cdots\cdots\cdots \\ c_q &= \tfrac{1}{q!}\begin{pmatrix} A_1 + 2^q A_2 + 3^q A_3 \\ + \cdots + k^q A_k \end{pmatrix} - \tfrac{1}{(q-1)!}\begin{pmatrix} B_1 + 2^{q-1}B_2 \\ +3^{q-1}B_3 + \cdots + (k+1)^{q-1}B_{k+1} \end{pmatrix} \end{aligned}\right\} \tag{24}$$

For $q = 0(1)5$ we have

$$c_0 = \sum_{j=0}^{3} A_j = A_0 + A_1 + A_2 + A_3$$

$$= \begin{bmatrix} -\frac{8\rho+5}{16\rho-23} \\ \frac{14\rho+17}{76\rho-197} \end{bmatrix} + \begin{bmatrix} -\frac{4(2\rho-7)}{16\rho-23} \\ -\frac{9(12\rho+11)}{76\rho-197} \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{9(2\rho+31)}{76\rho-197} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_1 = \sum_{j=0}^{3} jA_j - \sum_{j=0}^{4} B_j = (A_1 + 2A_2 + 3A_3) - (B_1 + B_2 + B_3 + B_4)$$

$$= \left[\begin{bmatrix} -\frac{4(2\rho-7)}{16\rho-23} \\ -\frac{9(12\rho+11)}{76\rho-197} \end{bmatrix} + (2)\begin{bmatrix} 1 \\ \frac{9(2\rho+31)}{76\rho-197} \end{bmatrix} + (3)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$
$$-\left[\begin{bmatrix} \frac{22\rho}{16\rho-23} \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{22}{16\rho-23} \\ \frac{150\rho}{76\rho-197} \end{bmatrix} + \begin{bmatrix} \frac{2(\rho+2)}{16\rho-23} \\ -\frac{150}{76\rho-197} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{6(\rho+3)}{76\rho-197} \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_2 = \sum_{j=0}^{3}\frac{(j^2 D_j)}{2!} - \sum_{j=0}^{4}\frac{(jG_j)}{1!}$$

$$= \tfrac{1}{2!}(D_1 + 2^2 D_2 + 3^2 D_3) - \tfrac{1}{1!}(G_1 + 2^1 G_2 + 3^1 G_3 + 4^1 G_4)$$

$$= \frac{1}{2!}\left[\begin{bmatrix} -\frac{4(2\rho-7)}{16\rho-23} \\ -\frac{9(12\rho+11)}{76\rho-197} \end{bmatrix} + (2)^2\begin{bmatrix} 1 \\ \frac{9(2\rho+31)}{76\rho-197} \end{bmatrix} + (3)^2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$
$$-\frac{1}{1!}\left[\begin{bmatrix} \frac{22\rho}{16\rho-23} \\ 0 \end{bmatrix} + (2)\begin{bmatrix} -\frac{22}{16\rho-23} \\ \frac{150\rho}{76\rho-197} \end{bmatrix} + (3)\begin{bmatrix} \frac{2(\rho+2)}{16\rho-23} \\ -\frac{150}{76\rho-197} \end{bmatrix} + (4)\begin{bmatrix} 0 \\ \frac{6(\rho+3)}{76\rho-197} \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_3 = \sum_{j=0}^{3}\frac{(j^3 D_j)}{3!} - \sum_{j=0}^{4}\frac{(j^2 G_j)}{2!} = \frac{1}{3!}(D_1 + 2^3 D_2 + 3^3 D_3)$$
$$-\frac{1}{2!}(G_1 + 2^2 G_2 + 3^2 G_3 + 4^2 G_4)$$

$$= \frac{1}{3!}\left[\begin{bmatrix} -\frac{4(2\rho-7)}{16\rho-23} \\ -\frac{9(12\rho+11)}{76\rho-197} \end{bmatrix} + (2)^3\begin{bmatrix} 1 \\ \frac{9(2\rho+31)}{76\rho-197} \end{bmatrix} + (3)^3\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$
$$-\frac{1}{2!}\left[\begin{bmatrix} \frac{22\rho}{16\rho-23} \\ 0 \end{bmatrix} + (2)^2\begin{bmatrix} -\frac{22}{16\rho-23} \\ \frac{150\rho}{76\rho-197} \end{bmatrix} + (3)^2\begin{bmatrix} \frac{2(\rho+2)}{16\rho-23} \\ -\frac{150}{76\rho-197} \end{bmatrix} + (4)^2\begin{bmatrix} 0 \\ \frac{6(\rho+3)}{76\rho-197} \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_4 = \sum_{j=0}^{3}\frac{(j^4 D_j)}{4!} - \sum_{j=0}^{4}\frac{(j^3 G_j)}{3!} = \frac{1}{4!}(D_1 + 2^4 D_2 + 3^4 D_3)$$

$$-\frac{1}{3!}\left(G_1+2^3G_2+3^3G_3+4^3G_4\right)$$

$$=\frac{1}{4!}\left[\begin{bmatrix}-\dfrac{4(2\rho-7)}{16\rho-23}\\ -\dfrac{9(12\rho+11)}{76\rho-197}\end{bmatrix}+(2)^4\begin{bmatrix}1\\ \dfrac{9(2\rho+31)}{76\rho-197}\end{bmatrix}+(3)^4\begin{bmatrix}0\\1\end{bmatrix}\right]$$

$$-\frac{1}{3!}\left[\begin{bmatrix}\dfrac{22\rho}{16\rho-23}\\0\end{bmatrix}+(2)^3\begin{bmatrix}-\dfrac{22}{16\rho-23}\\ \dfrac{150\rho}{76\rho-197}\end{bmatrix}+ (3)^3\begin{bmatrix}\dfrac{2(\rho+2)}{16\rho-23}\\ -\dfrac{150}{76\rho-197}\end{bmatrix}+(4)^3\begin{bmatrix}0\\ \dfrac{6(\rho+3)}{76\rho-197}\end{bmatrix}\right]=\begin{bmatrix}-\dfrac{1}{6}\dfrac{17+14\rho}{16\rho-23}\\0\end{bmatrix}$$

$$c_5=\sum_{j=0}^{3}\frac{(j^4D_j)}{4!}-\sum_{j=0}^{4}\frac{(j^3G_j)}{3!}=\frac{1}{5!}\left(D_1+2^5D_2+3^5D_3\right)$$

$$-\frac{1}{4!}\left(G_1+2^4G_2+3^4G_3+4^4G_4\right)$$

$$=\frac{1}{5!}\left[\begin{bmatrix}-\dfrac{4(2\rho-7)}{16\rho-23}\\ -\dfrac{9(12\rho+11)}{76\rho-197}\end{bmatrix}+(2)^5\begin{bmatrix}1\\ \dfrac{9(2\rho+31)}{76\rho-197}\end{bmatrix}+(3)^5\begin{bmatrix}0\\1\end{bmatrix}\right]$$

$$-\frac{1}{4!}\left[\begin{bmatrix}\dfrac{22\rho}{16\rho-23}\\0\end{bmatrix}+(2)^4\begin{bmatrix}-\dfrac{22}{16\rho-23}\\ \dfrac{150\rho}{76\rho-197}\end{bmatrix}+ (3)^4\begin{bmatrix}\dfrac{2(\rho+2)}{16\rho-23}\\ -\dfrac{150}{76\rho-197}\end{bmatrix}+(4)^4\begin{bmatrix}0\\ \dfrac{6(\rho+3)}{76\rho-197}\end{bmatrix}\right]=\begin{bmatrix}-\dfrac{1}{15}\dfrac{71+52\rho}{16\rho-23}\\ -\dfrac{1}{10}\dfrac{62\rho+111}{76-197}\end{bmatrix}\neq\begin{bmatrix}0\\0\end{bmatrix}$$

Therefore, the 2DSBEBDF method is of order four with error constant given as

$$c_5=\begin{bmatrix}-\frac{1}{15}\frac{71+52\rho}{16\rho-23}\\ -\frac{1}{10}\frac{62\rho+111}{76-197}\end{bmatrix}$$

Definition 3 (Consistency): a linear Multistep method (LMM) is said to be consistent if it has order $p\geq 1$. It follows that a LMM is consistent if and only if the following conditions are satisfied (Alhassan et al., 2022):

i. $\sum_{j=0}^{k}A_j=0$

ii. $\sum_{j=0}^{k}jA_j=\sum_{j=0}^{k+1}B_j$

Theorem 1: The derived 2-point diagonally implicit super class of block extended backward differentiation formula (2DSBEBDF) method is consistent.

Proof:

To show that the 2DSBEBDF scheme is consistent. It suffices to show that the consistency conditions in definition 3 are satisfied. Let A_0, A_1, A_2, A_3, A_4, A_5, B_0, B_1, B_2, B_3, B_4, B_5 be as previously defined, then

$\sum_{j=0}^{3}A_j=A_0+A_1+A_2+A_3$

$$=\begin{bmatrix}-\frac{8\rho+5}{16\rho-23}\\ \frac{14\rho+17}{76\rho-197}\end{bmatrix}+\begin{bmatrix}-\frac{4(2\rho-7)}{16\rho-23}\\ -\frac{9(12\rho+11)}{76\rho-197}\end{bmatrix}+\begin{bmatrix}1\\ \frac{9(2\rho+31)}{76\rho-197}\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Thus, the first consistency condition in (i) is satisfied.

Similarly,

$\sum_{j=0}^{3}jA_j=A_1+2A_2+3A_3$

$$=\begin{bmatrix}-\frac{4(2\rho-7)}{16\rho-23}\\ -\frac{9(12\rho+11)}{76\rho-197}\end{bmatrix}+(2)\begin{bmatrix}1\\ \frac{9(2\rho+31)}{76\rho-197}\end{bmatrix}+(3)\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}\frac{3(8\rho-6)}{16\rho-23}\\ \frac{12(13\rho-11)}{76\rho-197}\end{bmatrix}$$

$\sum_{j=0}^{4}B_j=B_1+B_2+B_3+B_4$

$$=\begin{bmatrix}\frac{22\rho}{16\rho-23}\\0\end{bmatrix}+\begin{bmatrix}-\frac{22}{16\rho-23}\\ \frac{150\rho}{76\rho-197}\end{bmatrix}+\begin{bmatrix}\frac{2(\rho+2)}{16\rho-23}\\ -\frac{150}{76\rho-197}\end{bmatrix}+\begin{bmatrix}0\\ \frac{6(\rho+3)}{76\rho-197}\end{bmatrix}=\begin{bmatrix}\frac{3(8\rho-6)}{16\rho-23}\\ \frac{12(13\rho-11)}{76\rho-197}\end{bmatrix}$$

Therefore, the second consistency condition in (ii) is satisfied, hence in accordance with definition 3, the 2DSBEBDF method is consistent

4. Stability Analysis Of The Method

In this section, we investigate the stability of the method based on A-stability and zero-stability, using the matrix representation of (12) as

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\end{bmatrix}=\begin{bmatrix}0&0\\-\frac{9(2\rho+31)}{76\rho-197}&0\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\end{bmatrix}+\begin{bmatrix}\frac{8\rho+5}{16\rho-23}&\frac{4(2\rho-7)}{16\rho-23}\\-\frac{14\rho+17}{76\rho-197}&\frac{9(12\rho+11)}{76\rho-197}\end{bmatrix}\begin{bmatrix}y_{n-1}\\y_n\end{bmatrix}+$$
$$h\begin{bmatrix}0&\frac{22\rho}{16\rho-23}\\0&0\end{bmatrix}\begin{bmatrix}f_{n-1}\\f_n\end{bmatrix}+h\begin{bmatrix}-\frac{22}{16\rho-23}&\frac{2(\rho+2)}{16\rho-23}\\ \frac{150\rho}{76\rho-197}&-\frac{150}{76\rho-197}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+2}\end{bmatrix}+$$
$$h\begin{bmatrix}0&0\\ \frac{6(\rho+3)}{76\rho-197}&0\end{bmatrix}\begin{bmatrix}f_{n+3}\\f_{n+4}\end{bmatrix}\qquad(25)$$

which is equivalent to

$$\begin{bmatrix}1&0\\ \frac{9(2\rho+31)}{76\rho-197}&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\end{bmatrix}=\begin{bmatrix}\frac{8\rho+5}{16\rho-23}&\frac{4(2\rho-7)}{16\rho-23}\\-\frac{14\rho+17}{76\rho-197}&\frac{9(12\rho+11)}{76\rho-197}\end{bmatrix}\begin{bmatrix}y_{n-1}\\y_n\end{bmatrix}+$$
$$h\begin{bmatrix}0&\frac{22\rho}{16\rho-23}\\0&0\end{bmatrix}\begin{bmatrix}f_{n-1}\\f_n\end{bmatrix}+h\begin{bmatrix}-\frac{22}{16\rho-23}&\frac{2(\rho+2)}{16\rho-23}\\ \frac{150\rho}{76\rho-197}&-\frac{150}{76\rho-197}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+2}\end{bmatrix}+$$
$$h\begin{bmatrix}0&0\\ \frac{6(\rho+3)}{76\rho-197}&0\end{bmatrix}\begin{bmatrix}f_{n+3}\\f_{n+4}\end{bmatrix}\qquad(26)$$

We define the k-block, r-point method (18) in general matrix form as

$$A_0Y_m=A_1Y_{m-1}+h(B_0F_{m-1}+B_1F_m+B_2F_{m+1})\qquad(27)$$

where,

$$A_0=\begin{bmatrix}1&0\\ \frac{9(2\rho+31)}{76\rho-197}&0\end{bmatrix},\ A_1=\begin{bmatrix}\frac{8\rho+5}{16\rho-23}&\frac{4(2\rho-7)}{16\rho-23}\\-\frac{14\rho+17}{76\rho-197}&\frac{9(12\rho+11)}{76\rho-197}\end{bmatrix},\ B_0=\begin{bmatrix}0&\frac{22\rho}{16\rho-23}\\0&0\end{bmatrix},$$

$$B_1=\begin{bmatrix}-\frac{22}{16\rho-23}&\frac{2(\rho+2)}{16\rho-23}\\ \frac{150\rho}{76\rho-197}&-\frac{150}{76\rho-197}\end{bmatrix},\quad B_2=\begin{bmatrix}0&0\\ \frac{6(\rho+3)}{76\rho-197}&0\end{bmatrix},\quad Y_m=\begin{bmatrix}y_{n+1}\\y_{n+2}\end{bmatrix}=\begin{bmatrix}y_{2m+1}\\y_{2m+2}\end{bmatrix},$$

$$Y_{m-1}=\begin{bmatrix}y_{n-1}\\y_n\end{bmatrix}=\begin{bmatrix}y_{2m-1}\\y_{2m-2}\end{bmatrix}=\begin{bmatrix}y_{2(m-1)+1}\\y_{2(m-1)+2}\end{bmatrix},\ F_m=\begin{bmatrix}f_{n+1}\\f_{n+2}\end{bmatrix}=\begin{bmatrix}f_{2m+1}\\f_{2m+2}\end{bmatrix},$$

$$F_{m-1}=\begin{bmatrix}f_{n-1}\\f_n\end{bmatrix}=\begin{bmatrix}f_{2m-1}\\f_{2m-2}\end{bmatrix}=\begin{bmatrix}f_{2(m-1)+1}\\f_{2(m-1)+2}\end{bmatrix},$$

$$F_{m+1}=\begin{bmatrix}f_{n+3}\\f_{n+4}\end{bmatrix}=\begin{bmatrix}f_{2m+3}\\f_{2m+4}\end{bmatrix}=\begin{bmatrix}f_{2(m+1)+1}\\f_{2(m+1)+2}\end{bmatrix}$$

By substituting the linear test ordinary differential equation $y'=\lambda y$ into (27) and using $h\lambda=\bar{h}$, we obtain:

$$A_0Y_m=A_1Y_{m-1}+\bar{h}(B_0Y_{m-1}+B_1Y_m+B_2Y_{m+1}),\qquad(28)$$

where A_0,A_1,B_0,$B_1$and$B_2$are as previously defined and

$$Y_{m+1}=\begin{bmatrix}y_{n+3}\\y_{n+4}\end{bmatrix}=\begin{bmatrix}y_{2m+3}\\y_{2m+4}\end{bmatrix}=\begin{bmatrix}y_{2(m+1)+1}\\y_{2(m+1)+2}\end{bmatrix}$$

To obtain the characteristics polynomial, the determinant of the following equation is computed in MATLAB 18 environment which is given by:

$$\pi(t,\bar{h})=det\left((A_0-\bar{h}B_1-\bar{h}B_2)t-(A_1+\bar{h}B_0)\right),\qquad(29)$$

Therefore, (29) is evaluated and is equivalent to

$$\begin{aligned}\sigma(t,\bar{h})=-\frac{1}{(16\rho-23)(76\rho-197)}\big(&312\bar{h}^2\rho^2t^2+3432\bar{h}^2\rho^2t\\&+660\bar{h}^2\rho t^2-36\bar{h}\rho^2t^2+396\bar{h}^2\rho t-3228\bar{h}^2t^2\\&+824\bar{h}\rho^2t-4702\bar{h}\rho t^2-1216\rho^2t^2-308\bar{h}\rho^2\\&-6876\rho t\bar{h}+6668\bar{h}t^2+2192\rho^2t+4900\rho t^2\\&-374\bar{h}\rho+2356\bar{h}t-976\rho^2-3824\rho t-4531t^2\\&-1076\rho+4550t-19\big)=0\end{aligned}$$

To demonstrate the zero-stability of our proposed 2DSBEBDF method, we substitute $\bar{h}=0$ into the characteristics polynomial (30), yielding the first characteristics polynomial as:

$$\sigma(t,0) = -\frac{-1216\rho^2t^2+2192\rho^2t+4900\rho t^2-976\rho^2-3824\rho t-4531t^2-1076\rho+4550t-19}{(16\rho-23)(76\rho-197)} = 0 \quad (31)$$

Solving (31) for t, the following roots are obtained:

$$t = 1, t = \frac{976\rho^2+1076\rho+19}{1216\rho^2-4900\rho+4531} \quad (32)$$

Definition 4 (Zero-stability): A linear multistep method (LMM) is said to be zero-stable if no root of the first characteristics polynomial, $\sigma(t)$ has modulus greater than one, and if every root with modulus one is simple (Musa and Alhassan, 2025).

By substituting two distinct values of the free parameter ρ into equation (32), we obtain the roots of the first characteristic polynomial given in equation (32) as:

i. When $\rho = \frac{1}{2}$

$t = 1, t = 0.3358490566$

ii. When $\rho = -\frac{3}{4}$

$t = 1, t = -0.02688413948$

According to Definition 4, the 2DSBEBDF method is zero-stable, as the absolute value of all roots of the first characteristic polynomial is less than or equal to 1, and the root with a modulus of 1 is simple (i.e., unique).

Definition 5 (A-stability): A linear multistep method (LMM) is said to be A-stable if the stability region covers the entire left-hand half-plane (Alhassan et al., 2024).

A method with a region of absolute stability covering the entire negative left-hand complex plane imposes no step-size constraints for stability (Lambert, 1973). However, achieving A-stability severely limits the choice of linear multistep methods (LMMs), due to Dahlquist's second barrier which dictates that A-stable LMMs cannot exceed order 2 (Dahlquist, 1963). This limitation motivates the search for higher-order LMMs with improved stability properties.

To determine the region of absolute stability (RAS) of the method (18) using a locus boundary, the boundary of absolute stability region of the method when $\rho = -3/4$ and $\rho = 1/2$ is determined by substituting $t = e^{i\theta}$ into equation (30). The graphs of the stability regions for the method plotted in MAPLE environment is given below:

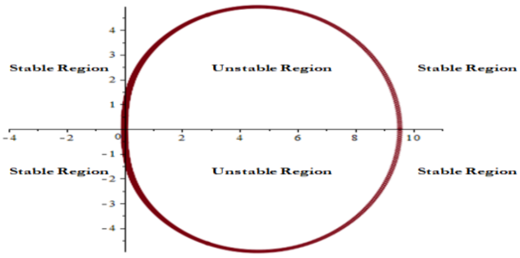

Figure 2: Stability Region for 2DSBEBDF (ρ=-3/4)

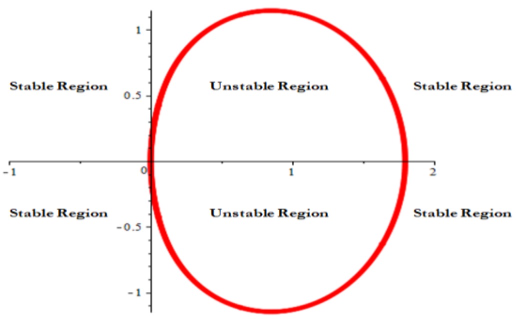

Figure 3: Stability Region for 2DSBEBDF(ρ=1/2)

Thus, the regions of absolute stability (RAS) of the 2DSBEBDF method are the areas outside the circular boundary. Notably, the RAS covers the entire left-half of the complex plane, indicating that the 2DSBEBDF method is A-stable. Having fulfilled this A-stability criterion, the method is well-suited for numerically integrating stiff problems.

5. Implementation Of The Method

The Newton's iteration is applied for the implantation of the 2DSBEBDF method. The description of the iteration is given below; we first start by defining the error.

Definition 6 (Absolute Error): Let y_i and $y(x_i)$ be the approximate and theoretical solutions of the differential equation (1) respectively. Then the absolute error is defined and given by

$$(error_i)_t = \left|(y_i)_t - \left(y(x_i)\right)_t\right| \quad (33)$$

The maximum error is defined by

$$MAXE = \underbrace{max}_{1\le i\le T}\left(\underbrace{max(error_i)_t}_{1\le i\le N}\right) \quad (34)$$

where T is the total number of steps and N is the number of equations

Define

$$\left.\begin{aligned} F_1 &= y_{n+1} + \frac{22}{16\rho-23}hf_{n+1} - \frac{22}{16\rho-23}h\rho f_n \\ &\quad - \frac{2(\rho+2)}{16\rho-23}hf_{n+2} - \tau_1 \\ F_2 &= y_{n+2} + \frac{9(2\rho+31)}{76\rho-197}y_{n+1} + \frac{150}{76\rho-197}hf_{n+2} \\ &\quad - \frac{150}{76\rho-197}h\rho f_{n+1} - \frac{6(\rho+3)}{76\rho-197}hf_{n+3} - \tau_2 \end{aligned}\right\} \quad (35)$$

Where

$$\left.\begin{aligned} \tau_1 &= \frac{8\rho+5}{16\rho-23}y_{n-1} + \frac{4(2\rho-7)}{16\rho-23}y_n \\ \tau_2 &= -\frac{14\rho+17}{76\rho-197}y_{n-1} + \frac{9(12\rho+11)}{76\rho-197}y_n \end{aligned}\right\}, \quad (36)$$

are the back values. Let $y_{n+j}^{(i+1)}$, $j = 1,2$ denote the $(i+1)^{th}$ iterative values of y_{n+j} and define

$$e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)}, j = 1,2 \quad (37)$$

The Newton's iteration for the 2DSBEBDF scheme takes the form:

$$y_{n+j}^{(i+1)} = y_{n+j}^{(i)} - \left(F'_j\left(y_{n+j}^{(i)}\right)\right)^{-1}\left(F_j\left(y_{n+j}^{(i)}\right)\right), j = 1,2 \quad (38)$$

This equation (38) takes the form:

$$\left(F'_j\left(y_{n+j}^{(i)}\right)\right)e_{n+j}^{(i+1)} = -\left(F_j\left(y_{n+j}^{(i)}\right)\right), j = 1,2 \quad (39)$$

The matrix representation of (39) is equivalently written as

$$\begin{bmatrix} 1 + \frac{22h}{16\rho-23}\frac{\partial f_{n+1}}{\partial y_{n+1}} & -\frac{2(\rho+2)h}{16\rho-23}\frac{\partial f_{n+2}}{\partial y_{n+2}} \\ \frac{9(2\rho+31)}{76\rho-197} - \frac{150\rho h}{76\rho-197}\frac{\partial f_{n+1}}{\partial y_{n+1}} & 1 + \frac{150h}{76\rho-197}\frac{\partial f_{n+2}}{\partial y_{n+2}} \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} =$$

$$\begin{bmatrix} -1 & 0 \\ -\frac{9(2\rho+31)}{76\rho-197} & -1 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} + \begin{bmatrix} -\frac{22}{16\rho-23} & \frac{2(\rho+2)}{16\rho-23} \\ \frac{150\rho}{76\rho-197} & -\frac{150}{76\rho-197} \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix} +$$

$$h\begin{bmatrix} 0 & 0 \\ \frac{6(\rho+3)}{76\rho-197} & 0 \end{bmatrix}\begin{bmatrix} f_{n+3} \\ f_{n+4} \end{bmatrix} + h\begin{bmatrix} 0 & \frac{22\rho}{16\rho-23} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} +$$

$$\begin{bmatrix} \frac{8\rho+5}{16\rho-23} & \frac{4(2\rho-7)}{16\rho-23} \\ -\frac{14\rho+17}{76\rho-197} & \frac{9(12\rho+11)}{76\rho-197} \end{bmatrix}\begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} \quad (40)$$

6. Test Problems And Numerical Results

This section utilizes C programming language to test the developed method on stiff systems of ordinary differential equations (ODEs), assessing its efficiency and reliability. These types of problems are prevalent in engineering and physical sciences, particularly in areas such as reaction kinetics, string vibrations, electrical circuits, and so on

Problem 1: This system of stiff oscillatory problem is considered in (Lambert, 1991; Aminikhah and Hemmantnezhad, 2011):

$y_1' = -2y_1 + y_2 + 2\sin x,$ $\quad y_1(0) = 2, 0 \le x \le 10,$

$y_2' = 998y_1 - 999y_2 + 999(\cos x - \sin x),$ $\quad y_2(0) = 3.$

Whose eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -1000$ and its corresponding exact solution is given by:

$$y_1 = 2e^{-x} + \sin x,$$
$$y_2 = 2e^{-10x} + \cos x$$

Problem 2: This is the linear stiff IVP:

$$y_1 = \frac{1}{199}e^{-200x} + \frac{200}{199}e^{-x},$$
$$y_2 = -\frac{200}{199}e^{-x} + \frac{200}{199}e^{-200x}$$

The eigenvalues of the differential equations are -1 and -200.

Source: Artificial Problem.

Problem 3: Consider the following first order linear stiff IVP of the form:

$$y_1' = -15y_1 - 14y_2, \qquad y_1(0) = 1, \quad 0 \le x \le 10$$
$$y_2' = -14y_1 - 15 \qquad y_2, y_2(0) = 0,$$

The exact solution is given by:

$$y_1 = \frac{1}{2}e^{-29x} + \frac{1}{2}e^{-x},$$
$$y_2 = \frac{1}{2}e^{-29x} - \frac{1}{2}e^{-2x},$$

The eigenvalues of the differential equations are -1 and -29.

Source: Artificial Problem.

Problem 4: This is a physical stiff problem taken from (Musa et al., 2012):

$$y_1' = -\frac{3}{100}y_1, \qquad y_1(0) = 50, 0 \le x \le 20$$
$$y_2' = \frac{3}{100}y_1 - \frac{3}{50}y_2, \qquad y_2(0) = 0$$

The exact solution is given by:

$$y_1 = 50e^{-\frac{3}{100}x},$$
$$y_2 = 50e^{-\frac{3}{50}}\left(-1 + e^{\frac{3}{100}x}\right)$$

The eigenvalues of the differential equations are $-\frac{6}{100}$, and $-\frac{3}{100}$.

The numerical results for the stiff problems given are presented in Table 3-5. The problems are solved using our proposed 2DSBEBDF method when $\rho = 1/2$ and the 2-point diagonally implicit block backward differentiation formula (2DBBDF) developed by (Zawawi et al., 2012). For easy referencing the 2DBBDF is expressed as:

$$\left.\begin{aligned} y_{n+1} &= -\frac{1}{3}y_{n-1} + \frac{4}{3}y_n + \frac{2}{3}hf_{n+1} \\ y_{n+2} &= \frac{2}{11}y_{n-1} - \frac{9}{11}y_n + \frac{18}{11}y_{n+1} + \frac{6}{11}hf_{n+2} \end{aligned}\right\} \tag{41}$$

This scheme is shown to be A-stable, consistent and convergent. The following notations are used in the tables:

H: Step Size

MAXE: Maximum Absolute Error

TS: Total Number of Step

CPU TIME: Computation Time in seconds

MTD: Methods Used.

2DBBDF: 2-point Diagonally Implicit Block Backward Differentiation Formula (Zawawi et al., 2012)

2DSBEBDF: 2-point Diagonally Implicit Super Class of Block Extended Backward Differentiation Formula (our proposed method).

Table 3: Numerical Result for Problem 1

H	MTD	TS	MAXE	CPU TIME
10^{-2}	2DBBDF 2DSBEBDF	500 500	1.70236E+093 7.70428E-002	3.34300E-004 4.31200E-005
10^{-3}	2DBBDF 2DSBEBDF	5000 5000	2.08995E+101 3.58395E-006	3.42100E-003 4.67400E-004
10^{-4}	2DBBDF 2DSBEBDF	50000 50000	1.50076E-004 3.59824E-008	3.75200E-002 4.14400E-003
10^{-5}	2DBBDF 2DSBEBDF	500000 500000	1.50065E-005 3.59982E-010	4.30600E-001 4.61800E-002
10^{-6}	2DBBDF 2DSBEBDF	5000000 5000000	1.50066E-006 3.59979E-012	4.71100E+001 5.25600E-001

Table 4: Numerical Result for Problem 2

H	MTD	TS	MAXE	CPU TIME
10^{-2}	2DBBDF 2DSBEBDF	100 100	1.42035E+007 1.02086E+000	3.63200E-003 2.31000E-005
10^{-3}	2DBBDF 2DSBEBDF	1000 1000	7.57316E-002 3.68866E-002	3.54800E-002 2.55600E-004
10^{-4}	2DBBDF 2DSBEBDF	10000 10000	1.40727E-002 6.77281E-004	3.88200E-002 2.81500E-003
10^{-5}	2DBBDF 2DSBEBDF	100000 100000	1.47164E-003 7.18833E-006	3.20600E-001 3.37200E-002
10^{-6}	2DBBDF 2DSBEBDF	1000000 1000000	1.47809E-004 7.23122E-008	4.51100E-001 3.83900E-001

Table 5: Numerical Result for Problem 3

H	MTD	TS	MAXE	CPU TIME
10^{-2}	2DBBDF 2DSBEBDF	500 500	3.00703E-002 2.81166E-002	4.02800E-003 2.42800E-004
10^{-3}	2DBBDF 2DSBEBDF	5000 5000	9.95422E-003 6.88539E-004	4.29300E-003 2.78600E-004
10^{-4}	2DBBDF 2DSBEBDF	50000 50000	1.06267E-003 7.50580E-006	4.66900E-002 3.10800E-003

Table 5(Cont.): Numerical Result for Problem 3

10^{-5}	2DBBDF 2DSBEBDF	500000 500000	1.06943E-004 7.57075E-008	4.90600E-002 4.52600E-002
10^{-6}	2DBBDF 2DSBEBDF	5000000 5000000	1.07011E-005 7.57727E-010	5.74400E-001 4.99400E-001

Table 6: Numerical Result for Problem 4

H	MTD	TS	MAXE	CPU TIME
10^{-2}	2DBBDF 2DSBEBDF	1000 1000	1.35080E-002 2.42438E-002	2.82600E-003 3.14200E-004
10^{-3}	2DBBDF 2DSBEBDF	10000 10000	1.35298E-003 2.42994E-007	2.38300E-002 3.66700E-003
10^{-4}	2DBBDF 2DSBEBDF	100000 100000	1.35320E-004 2.42994E-009	2.26100E-002 3.29100E-003
10^{-5}	2DBBDF 2DSBEBDF	1000000 1000000	1.35323E-005 2.43048E-011	2.12700E-001 3.09600E-002
10^{-6}	2DBBDF 2DSBEBDF	10000000 10000000	1.35290E-006 2.39784E-013	2.37400E-001 2.31800E-001

In order to visually demonstrate the efficancy of our method, graphs depicting the relationship between $\log_{10}(MAXE)$ and H for the tested problems are generated. Below are the graphs illustrating the scaled maximum error for each individual problem.

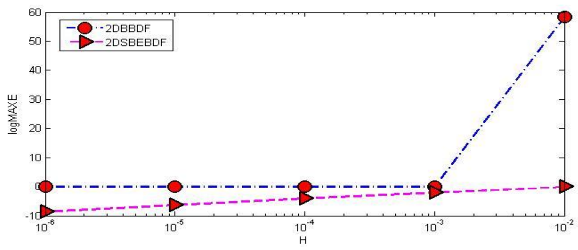

Figure 4: Graph of $\log(MAXE)$ against H for Problem 1

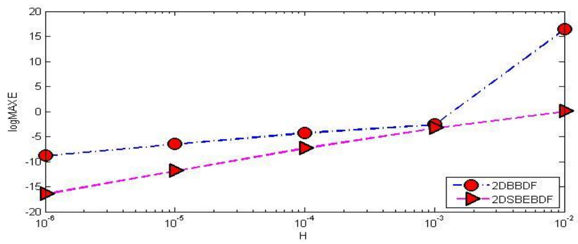

Figure 5: Graph of $\log(MAXE)$ against H for Problem 2

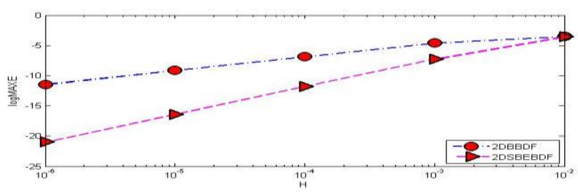

Figure 6: Graph of $\log(MAXE)$ against H for Problem 3

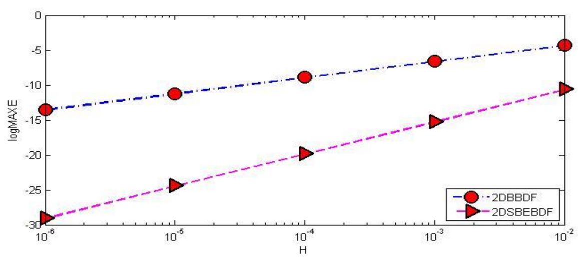

Figure 7: Graph of $\log(MAXE)$ against H for Problem 4

7. Discussion On The Result

The provided tables present the numerical results obtained for various selected test problem using two different numerical methods, including 2-point Diagonally Implicit Block Backward Differentiation Formula (2DBBDF), and 2-point Diagonally Implicit Superclass Block Extended Backward Differentiation Formula (2DSBEBDF). The results are analyzed based on the step size (H), total number of steps (TS), maximum error (MAXE), and CPU time.

For each test problem, as the step size decreases, the total number of steps increases significantly, reflecting a finer discretization of the domain. Correspondingly, the maximum error decreases, indicating higher accuracy with smaller step sizes. Notably, the 2DSBEBDF method generally exhibits lower maximum errors compared to the 2DBBDF method across different step sizes, suggesting superior accuracy.

However, it's important to consider the computational efficiency of the methods, as reflected in the CPU time. As expected, the CPU time increases with decreasing step size, indicating higher computational costs for finer discretization. Additionally, for each problem tested, the 2DSBEBDF method tends to have lower CPU times compared to the 2DBBDF method.

Overall, the results demonstrate the trade-off between accuracy and computational cost. While the 2DSBEBDF method generally offers superior accuracy, it may require slightly less computational resources compared to the 2DBBDF method. These findings provide valuable insights for selecting an appropriate numerical method based on the desired balance between accuracy and computational efficiency for specific problem instances.

8. Conclusion

The diagonally implicit 2-point super class of block extended backward differentiation formula (2SBEBDF), designed to efficiently handle stiff ODEs is developed. The method extends the concept of introducing an additional super future point to the existing 2-point super class of block backward differentiation formula, resulting in higher-order A-stable and more accurate block scheme. The derivation process, order determination, and stability analysis of the 2SDBEBDF method is presented. The paper establishes that the 2DSBEBDF method is of fourth order with specific error constant. The stability analysis explores both zero and A-stability, confirming that the method is zero-stable, and A-stable, making it suitable for solving first-order stiff initial value problems. Implementation details in Dev C++ compiler environment using Newton's iteration is provided, and the methods are tested on various stiff ODEs. The numerical simulation of results demonstrates the effectiveness and efficiency of the 2DSBEBDF method, outperforming existing 2-point diagonally implicit block backward differentiation formulae (2DBBDF) algorithms in terms of accuracy and computational cost.

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Author Contribution

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Conflicts Of Interest

The authors affirm that they have no conflicts of interest related to the publication of this paper.

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