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*RESEARCH ARTICLE*

# UNIQUENESS OF PERIODIC SOLUTIONS OF A CERTAIN ULTRA-HYPERBOLIC BOUNDARY VALUE PROBLEM

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## ABSTRACT

In this paper, uniqueness of periodic solutions of a certain ultra-hyperbolic boundary value problem was investigated using energy method and superposition principle. Periodic solutions and test for periodicity were obtained using separation of variable method and Fourier series. Through the exploit of energy conservation and initial conditions, uniqueness of periodic solutions were established. Application of our results can be seen in wave propagation and vibration in complex system with periodic forcing. Numerical simulations were used to demonstrate the behavior of the solutions extending known results in literature.

## KEYWORDS

Uniqueness, Ultra-hyperbolic equation, Periodic solutions

## 1. INTRODUCTION

In this paper, we consider uniqueness of periodic solutions of non-homogenous ultra-hyperbolic boundary value problem of the form

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial z^2}-\frac{\partial^2 u}{\partial t^2}=f(x,y,z,t) \qquad (1)$$

With the following boundary and initial conditions

$$\begin{array}{llll} u(0,y,z,t)=0 & u(L_x,y,z,t)=0 & 0\le x\le L_x & t\ge 0 \\ u(x,0,z,t)=0 & u(x,L_y,z,t)=0 & 0\le y\le L_y & \\ u(x,y,0,t)=0 & u(x,y,L_z,t)=0 & 0\le z\le L_z & \\ u(x,y,z,0)=f(x,y,z) & & \frac{\partial u}{\partial t}(x,y,z,0)=g(x,y,z) & \end{array}$$

$L_x$ represent the length of the domain in the $x$ direction

$L_y$ represent the length of the domain in the $y$ direction

$L_z$ represent the length of the domain in the $z$ direction

$u$ is the unknown scalar function and $x,y,z,t$ are the independent variables. Equation (1) is a time-dependent wave model that model how disturbance evolve over time in a medium that supports oscillations or signals.

Ultra-hyperbolic equation is a partial differential equation for an unknown scalar function $u$ of $2n$ variables $x_1,\dots,x_n,\ y_1,\dots,y_n$ of the form

$$\frac{\partial^2 u}{\partial x_1^2}+\dots+\frac{\partial^2 u}{\partial x_n^2}-\frac{\partial^2 u}{\partial y_1^2}-\dots-\frac{\partial^2 u}{\partial y_n^2}=0 \qquad (2)$$

see (Courant and Hebert, 1962). The equation resembles the classical wave equation which has led to number of developments due to its characteristics (Frit, 1938). The applications of ultra-hyperbolic equation can be seen in the modelling of space time (Wang et al., 2022). Designing of harbours and dams, earthquake analysis (electrodynamic wave propagation) and design of antennas (electromagnetic wave propagation) (Paul Groenenboom, 1983). The equation can also be used in the study of symmetric spaces and elliptic differential operators (Zhang et al., 2022). Due to the characteristics of the ultra-hyperbolic boundary value problem, many authors have studied ultra-hyperbolic equation producing sound results. For instance see (Walter and Steven, 2008 ; Helgason, 1959 ; Owens, 1960 ; Golgeleyen, 2023) and there references therein.

Uniqueness is a property of a partial differential equation that explain under what condition a problem have at most one solution. A solution to a partial differential might exist but not unique. It is the uniqueness of solution that imply the existence but the converse is not true. Several methods for investigating uniqueness of ultra-hyperbolic equation exist in literature. Energy method which is a powerful tool for this investigation was introduced in (Lee, 2025). The main idea in energy method is to obtain an estimate using energy like quantities. Energy method was used to prove uniqueness for ultra-hyperbolic equation with variable coefficient (Murray, 1977). Other methods of proving uniqueness include analytical formulation, carleman estimate, variational method and pointwise carleman type inequality (Wang et al., 2022 ; Jena, 2024 ; Steglinski, 2021; Golgeleyen and Yamamato, 2020). However to the best of our knowledge, the search for uniqueness of periodic solution of ultra-hyperbolic boundary value problem of four independent variables using energy method and superposition principle is still yet not covered.

Motivated by the above literature, the objective of this paper is to investigate uniqueness of periodic solutions of (1) using energy method and superposition principle. The periodic solution and test for periodicity will be established using separation of variable and Fourier series. In section 2, we introduce some theorems which will help us to obtain our main results. The analysis will be shown in section 3 while the paper is concluded in section 5 with a numerical simulations in section 4 to demonstrate the behavior of the system.

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## 2. PERLIMINARIES

### 2.1 Definition 2.1 (Separation of variable method) (Arfken, 1985)

Separation of variables is a method of solving ordinary and partial differentiation. For a partial differential equation in a function $\phi(x, y, \dots)$ and variables $x, y, \dots$, separation of variables can be applied by making a substitution of the form

$$\phi(x, y, \dots) = X(x)Y(y)\dots, \tag{3}$$

breaking the resulting equation into a set of independent ordinary differential equations, solving these for $X(x), Y(y), \dots$, and then plugging them back into the original equation. Separation of variables was used by L'Hospital in 1970. It is useful in solving equations arising in Mathematical Physics such as Laplace's equation, the Helmholtz differential equation and the Schrodinger equation.

### 2.2 Definition 2.2 (Periodic Solution)

A solution $u(x, y, z, t)$ to a partial differential equation with four independent variables is said to be periodic in one or more variables if there exist $T > 0, T_x > 0, T_y > 0$ and $T_z > 0$ such that

$$u(x, y, z, t + T) = u(x, y, z, t)$$

$$u(x + T_x, y, z, t) = u(x, y, z, t)$$

$$u(x, y + T_y, z, t) = u(x, y, z, t)$$

$$u(x, y, z + T_z, t) = u(x, y, z, t)$$

for $(x, y, z, t) \in \mathbb{R}$. $T, T_x, T_y$ and $T_z$ are the periods.

### 2.3 Definition 2.3 (Fourier series)

This is a technique used to convert a complex wave function into a single sine or cosine function. Let $f(x)$ be a function in the interval $[-L, L]$, then Fourier series formula is given by

$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$ where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$,

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \;\; n > 0,\;\; b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \;\; n > 0$ are Fourier coefficients of $f$.

### 2.4 Definition

For a PDE involving an unknown function $u$ of four independent variables, say $x_1, x_2, x_3, x_4$, a solution is a function $u(x_1, x_2, x_3, x_4)$ defined on a specific domain that makes the equation a true statement over that domain.

### 2.4 Theorem 2.5 (Superposition Principle) (Chasnov, 2022)

The principle of superposition state that if $u_1$ and $u_2$ are solutions of a linear PDEs then the linear combination $\alpha u_1 + \beta u_2$ is a solution where $\alpha$ and $\beta$ are arbitrary constants.

Consider the ultra-hyperbolic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = 0 \tag{4}$$

Suppose that $u = u_1(x, y, z, t\quad)$ and $u = u_2(x, y, z, t)$ are solutions to equation (4). We consider a linear combination of $u_1$ and $u_2$ by letting $u = \alpha u_1 + \beta u_2$ where $\alpha$ and $\beta$ are constants. The principle of superposition states that $u$ is a solution of equation (4). To prove this, we compute

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2(\alpha u_1 + \beta u_2)}{\partial x^2} + \frac{\partial^2(\alpha u_1 + \beta u_2)}{\partial y^2} - \frac{\partial^2(\alpha u_1 + \beta u_2)}{\partial z^2} - \frac{\partial^2(\alpha u_1 + \beta u_2)}{\partial t^2} \tag{5}$$

$$= \frac{\partial^2 \alpha u_1}{\partial x^2} + \frac{\partial^2 \beta u_2}{\partial x^2} + \frac{\partial^2 \alpha u_1}{\partial y^2} + \frac{\partial^2 \beta u_2}{\partial y^2} - \frac{\partial^2 \alpha u_1}{\partial z^2} - \frac{\partial^2 \beta u_2}{\partial z^2} - \frac{\partial^2 \alpha u_1}{\partial t^2} - \frac{\partial^2 \beta u_2}{\partial t^2} \tag{6}$$

$$= \alpha\frac{\partial^2 u_1}{\partial x^2} + \beta\frac{\partial^2 u_2}{\partial x^2} + \alpha\frac{\partial^2 u_1}{\partial y^2} + \beta\frac{\partial^2 u_2}{\partial y^2} - \alpha\frac{\partial^2 u_1}{\partial z^2} - \beta\frac{\partial^2 u_2}{\partial z^2} - \alpha\frac{\partial^2 u_1}{\partial t^2} - \beta\frac{\partial^2 u_2}{\partial t^2} \tag{7}$$

$$= \alpha\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} - \frac{\partial^2 u_1}{\partial z^2} - \frac{\partial^2 u_1}{\partial t^2}\right) + \beta\left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} - \frac{\partial^2 u_2}{\partial z^2} - \frac{\partial^2 u_2}{\partial t^2}\right) \tag{8}$$

$$= \alpha \times 0 + \beta \times 0 = 0$$

Since $u_1$ and $u_2$ were assumed to be solution of equation (4). We have therefore shown that any linear combination of solution to homogenous ultra-hyperbolic equation is also a solution.

### 2.5 Theorem

If $u_1$ and $u_2$ are solutions of a linear inhomogeneous equation, then $u_1 - u_2$ is also a solution of the corresponding homogenous equation.

Proof: Let the linear inhomogeneous equation be written in the form $L(u) = f$ and the corresponding homogenous equation be of the form $L(u) = 0$. Since $u_1$ and $u_2$ are solutions of inhomogeneous equation we have $L(u_1) = f$ and $L(u_2) = f$. Subtracting the two gives $L(u_1) - L(u_2) = f - f = 0$. Since $L$ is linear, we have $L(u_1 - u_2) = 0$. This shows that $u_1 - u_2$ satisfies the homogenous equation.

### 2.7 Definition 2.7 (Unit normal)

Unit normal is a vector with a magnitude of one that is perpendicular to the surface or a curve at a given point. Given a surface or curve defined by a function $f(x, y, z)$, the unit normal vector $\bar{n}$ at a point $(x_0, y_0, z_0)$ is defined as $\bar{n} = \frac{\nabla f(x_0, y_0, z_0)}{\|\nabla f(x_0, y_0, z_0\|}$ where $\nabla f$ is the gradient of $f$ and $\|\nabla f\|$ denote the Euclidean norm.

### 2.8 Definition 2.8 (Energy Method)

Energy method is a powerful tool for analyzing and solving partial differential equations (PDEs) which are ubiquitous in various fields of Science and Engineering. The method is based on the concept of energy estimate which involves deriving bounds on the solution of a PDE using energy-like quantities. These energy estimates can be used to prove the existence, uniqueness and stability of solutions to PDEs (Lee, 2025).

## 3. MAIN RESULTS

3.1 Periodic Solutions of Ultra-hyperbolic equation

To established periodic solutions of equation (1.1) we set $f(x, y, z, t) = 0$ which gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = 0\ , \tag{9}$$

with the following boundary and initial conditions

$u(0, y, z, t) = 0 \qquad u(L_x, y, z, t) = 0 \qquad 0 \le x \le L_x \quad t \ge 0$

$u(x, 0, z, t) = 0 \qquad u(x, L_y, z, t) = 0 \qquad 0 \le y \le L_y$

$u(x, y, 0, t) = 0 \qquad u(x, y, L_z, t) = 0 \qquad 0 \le z \le L_z$

$u(x, y, z, 0) = f(x, y, z) \qquad \frac{\partial u}{\partial t}(x, y, z, 0) = g(x, y, z)$

We assume a solution of the form

$$u(x, y, z, t) = X(x)Y(x)Z(z)T(t) \tag{10}$$

Substituting equation (9) in equation (10) gives

$$\frac{\partial^2(X(x)Y(x)Z(z)T(t))}{\partial x^2} + \frac{\partial^2(X(x)Y(x)Z(z)T(t))}{\partial y^2} - \frac{\partial^2(X(x)Y(x)Z(z)T(t))}{\partial z^2} - \frac{\partial^2(X(x)Y(x)Z(z)T(t))}{\partial t^2} = 0 \tag{11}$$

$$YZT\frac{d^2X}{dx^2} + XZT\frac{d^2Y}{dy^2} - XYT\frac{d^2Z}{dz^2} - XYZ\frac{d^2T}{dt^2} = 0 \tag{12}$$

Dividing by $XYZT$ we have

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{1}{Z}\frac{d^2Z}{dz^2} - \frac{1}{T}\frac{d^2T}{dt^2} = 0 \tag{13}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{1}{T}\frac{d^2T}{dt^2} \tag{14}$$

Since the LHS is a function of $x$ and $y$ and RHS is a function of $z$ and $t$, we set equation (14) equal to $k$ where $k$ is the separation constant.

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{1}{T}\frac{d^2T}{dt^2} = k \tag{15}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = k \text{ and } \frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{1}{T}\frac{d^2T}{dt^2} = k \tag{16}$$

Considering $\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = k$ we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = k - \frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda \qquad (1$$

Where $\lambda$ is another separation constant. Hence we have

$\frac{1}{X}\frac{d^2X}{dx^2} = \lambda$ which gives $X''(x) - \lambda X(x) = 0$. Simplifying $k - \frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda$ gives $Y''(y) - vY(y) = 0$ where $v = k - \lambda$. For $\frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{1}{T}\frac{d^2T}{dt^2} = k$ we have $\frac{1}{Z}\frac{d^2Z}{dz^2} = k - \frac{1}{T}\frac{d^2T}{dt^2} = \tau$ where $\tau$ is another constant. Simplifying further gives $Z''(z) - \tau Z(z) = 0$ and $T''(t) - wT(t) = 0$ where $w = k - \tau$. The system of equations is given by

$$X''(x) - \lambda X(x) = 0 \tag{18}$$

$$Y''(y) - vY(y) = 0 \tag{19}$$

$$Z''(z) - \tau Z(z) = 0 \tag{20}$$

$$T''(t) - wT(t) = 0 \tag{21}$$

Case 1: Considering for $\lambda > 0, v > 0, \tau > 0$ and $w > 0$ we have

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x} \tag{22}$$

$$Y(y) = c_3 e^{\sqrt{v}y} + c_4 e^{-\sqrt{v}y} \tag{23}$$

$$Z(z) = c_5 e^{\sqrt{\tau}z} + c_6 e^{-\sqrt{\tau}z} \tag{24}$$

$$T(t) = c_7 e^{\sqrt{w}t} + c_8 e^{-\sqrt{w}t} \tag{25}$$

Where $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ are constants. Hence the general solution for case 1 is given by

$$u(x,y,z,t) = (c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x})(c_3 e^{\sqrt{v}y} + c_4 e^{-\sqrt{v}y}\quad)\quad(c_5 e^{\sqrt{\tau}z} + c_6 e^{-\sqrt{\tau}z})(c_7 e^{\sqrt{w}t} + c_8 e^{-\sqrt{w}t})$$

Applying the boundary conditions we have

$$u(0.y,z,t) = 0 = (c_1 + c_2)\,(c_3 e^{\sqrt{v}y} + c_4 e^{-\sqrt{v}y}\quad)\quad(c_5 e^{\sqrt{\tau}z} + c_6 e^{-\sqrt{\tau}z})(c_7 e^{\sqrt{w}t} + c_8 e^{-\sqrt{w}t})$$

Which implies that

$$c_1 + c_2 = 0 \tag{26}$$

Also $u(L_x, y, z, t) = 0$ gives

$$c_1 e^{\sqrt{\lambda}L_x} + c_2 e^{-\sqrt{\lambda}L_x} = 0 \tag{27}$$

$$u(x,0,z,t) = 0 = c_3 + c_4(c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x})(c_5 e^{\sqrt{\tau}z} + c_6 e^{-\sqrt{\tau}z})(c_7 e^{\sqrt{w}t} + c_8 e^{-\sqrt{w}t})$$

implies that

$$c_3 + c_4 = 0 \tag{28}$$

And $u(x, L_y, z, t) = 0$ gives

$$c_3 e^{\sqrt{v}L_y} + c_2 e^{-\sqrt{v}L_y} = 0 \tag{29}$$

$$u(x,y,0,t) = 0 = (c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x})(c_3 e^{\sqrt{v}y} + c_4 e^{-\sqrt{v}y})(c_5 + c_6)(c_7 e^{\sqrt{w}t} + c_8 e^{-\sqrt{w}t})$$

implies that

$$c_5 + c_6 = 0 \tag{30}$$

$$u(x,y,L_z,t) = (c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x})(c_3 e^{\sqrt{v}y} + c_4 e^{-\sqrt{v}y})(c_5 e^{\sqrt{\tau}L_z} + c_6 e^{-\sqrt{\tau}L_z})(c_7 e^{\sqrt{w}t} + c_8 e^{-\sqrt{w}t})$$

$u(x, y, L_z, t) = 0$ gives

$$(c_5 e^{\sqrt{\tau}L_z} + c_6 e^{-\sqrt{\tau}L_z}) = 0 \tag{31}$$

Equations (3.18) and (3.19) possess a non-trivial solution if and only if

$$\begin{vmatrix} 1 & 1 \\ e^{\sqrt{\lambda}L_x} & e^{-\sqrt{\lambda}L_x} \end{vmatrix} = 0$$

$$e^{-\sqrt{\lambda}L_x} - e^{\sqrt{\lambda}L_x} = 0 \tag{32}$$

Dividing equation (4.1.31) by $e^{-\sqrt{\lambda}L_x}$ we have

$$1 - e^{2\sqrt{\lambda}L_x} = 0 \tag{33}$$

Equation (33) implies that $e^{2\sqrt{\lambda}L_x} = 1$. Further simplification gives $\sqrt{\lambda}L_x = 0$. This implies that $\sqrt{\lambda} = 0$ since $L_x \neq 0$. The same result goes for equations (28) and (29). This results is against the assumption for Case 1. Hence the solution is not acceptable.

Case II: for $\lambda = 0, v = 0, \tau = 0$ and $w = 0$ we have

$$\frac{d^2X}{dx^2} = 0 \tag{34}$$

$$\frac{d^2Y}{dy^2} = 0 \tag{35}$$

$$\frac{d^2Z}{dz^2} = 0 \tag{36}$$

$$\frac{d^2T}{dt^2} = 0 \tag{37}$$

Solutions for equations (34), (35), (36) and (37) is given by

$$X = Ax + B \tag{38}$$

$$Y = Cy + D \tag{39}$$

$$Z = Ez + F \tag{40}$$

$$T = Gt + H \tag{41}$$

Where B, D, F and $H$ are constants. Therefore the required solution is

$$u(x,y,z,t) = (Ax + B)(Cy + D)(Ez + F)(Gt + H) \tag{42}$$

Using the boundary conditions we have

$u(0.y,z,t) = 0 = B(Cy + D)(Ez + F)(Gt + H)$. This implies that $B = 0$

$u(L_x, y, z, t) = 0 = AL_x(Cy + D)(Ez + F)(Gt + H)$. This implies that $A = 0$

$u(x, 0, z, t) = 0 = D(Ax + B)(Ez + F)(Gt + H)$. This implies that $D = 0$

$u(x, L_y, z, t) = 0 = CL_y(Ax + B)(Ez + F)(Gt + H)$. This implies that $C = 0$

$u(x, y, 0, t) = 0 = F(Ax + B)(Cy + D)(Gt + H)$. This implies that $F = 0$

$u(x, y, L_z, t) = 0 = EL_z(Ax + B)(Cy + D)(Gt + H)$. This implies $E = 0$

Hence only trivial solution is possible.

Case III: for $\lambda < 0: \lambda = -\alpha^2, \alpha > 0$, $v < 0: v = -\beta^2, \beta > 0$, $\tau < 0: \tau = -\gamma^2, \gamma > 0$ and

$w < 0: w = -\eta^2, \eta > 0$. The differential equations are

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0 \tag{43}$$

$$\frac{d^2Y}{dy^2} + \beta^2 Y = 0 \tag{44}$$

$$\frac{d^2Z}{dz^2} + \gamma^2 Z = 0 \tag{45}$$

$$\frac{d^2T}{dt^2} + \eta^2 T = 0 \tag{46}$$

Auxiliary equation for equation (43) is given by

$$m^2 + \alpha^2 = 0 \tag{47}$$

$m = \pm\sqrt{-\alpha^2},\ m = \pm i\alpha$. Hence the solution of equation of (43) is given by

$$X(x) = c_1 cos\alpha x + c_2 sin\alpha x \tag{48}$$

Similarly for equations (44), (45) and (46), the solutions are

$$Y(y) = c_3 cos\beta y + c_4 sin\beta y \tag{49}$$

$$Z(z) = c_5 cos\gamma z + c_6 sin\gamma z \tag{50}$$

$$T(x) = c_7 cos\eta t + c_8 sin\eta t \tag{51}$$

Hence the general solution is given by

$$u(x,y,z,t) = (c_1 cos\alpha x + c_2 sin\alpha x)(c_3 cos\beta y + c_4 sin\beta y)(c_5 cos\gamma z + c_6 sin\gamma z)$$

$$(c_7 cos\eta t + c_8 sin\eta t) \tag{52}$$

Using the boundary conditions we have that for

$u(0,y,z,t) = 0,\qquad c_1 = 0$

$u(L_x,y,z,t) = 0,\qquad sin\alpha L_x = 0$

$u(x,0,z,t) = 0,\qquad c_3 = 0$

$u(x,L_y,z,t) = 0,\qquad sin\beta L_y = 0$

$u(x,y,0,t) = 0,\qquad c_5 = 0$

$u(x,y,L_z,t) = 0,\qquad sin\gamma L_z = 0$

For $sin\alpha L_x = 0,\ \alpha_n = \frac{n\pi}{L_x},\ n = 1,2,\ldots,$ are the eigenvalues. The solution is

$$u_n(x,t) = sin\frac{n\pi x}{L_x}\left(A_n cos\frac{n\pi t}{L_x} + B_n sin\frac{n\pi t}{L_x}\right), n = 1,2,\ \ldots, \tag{53}$$

For $sin\beta L_y = 0,\ \beta_m = \frac{m\pi}{L_y},\ m = 1,2,\ldots,$ are the eigenvalues. The solution is

$$u_m(y,t) = sin\frac{m\pi y}{L_y}\left(C_m cos\frac{m\pi t}{L_y} + D_m sin\frac{m\pi t}{L_y}\right), m = 1,2,\ \ldots, \tag{54}$$

For $sin\gamma L_z = 0,\ \gamma_g = \frac{g\pi}{L_z}, g = 1,2,\ldots,$ are the eigenvalues. The solution is

$$u_g(z,t) = sin\frac{g\pi z}{L_z}\left(E_g cos\frac{g\pi t}{L_z} + F_g sin\frac{g\pi t}{L_z}\right), g = 1,2,\ \ldots, \tag{55}$$

Combining equations (3.45), (3.46) and (3.47) we have

$$u_p(x,y,z,t) = sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}\left(A_n cos\frac{n\pi t}{L_x} + B_n sin\frac{n\pi t}{L_x}\right)\left(C_m cos\frac{m\pi t}{L_y} + D_m sin\frac{m\pi t}{L_y}\right)$$

$$\left(E_g cos\frac{g\pi t}{L_z} + F_g sin\frac{g\pi t}{L_z}\right) \tag{56}$$

Where $p = nmg$ and $A_n, B_n, C_m, D_m, E_g, F_g$ are constant coefficients

Using the superposition principle we have

$$u(x,y,z,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{g=1}^{\infty} sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}\left(A_n cos\frac{n\pi t}{L_x} + B_n sin\frac{n\pi t}{L_x}\right)$$

$$\left(C_m cos\frac{m\pi t}{L_y} + D_m sin\frac{m\pi t}{L_y}\right)\left(E_g cos\frac{g\pi t}{L_z} + F_g sin\frac{g\pi t}{L_z}\right) \tag{57}$$

Applying the initial conditions gives

$$u(x,y,z,0) = f(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{g=1}^{\infty} sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}(A_n)(C_m)(E_g)$$

$$u(x,y,z,0) = f(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{\alpha=1}^{\infty} sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}\quad J_q \tag{58}$$

Where $J_q = (A_n)(C_m)(E_g)$

$J_q$ is the half range Fourier sine series for $A_n$, $C_m$ and $E_g$ where

$$A_n = \frac{2}{L_x}\int_0^{L_x} f(x,y,z) sin\frac{n\pi x}{L_x} dx \tag{59}$$

$$C_m = \frac{2}{L_y}\int_0^{L_y} f(x,y,z) sin\frac{m\pi y}{L_y} dy \tag{60}$$

$$E_g = \frac{2}{L_z}\int_0^{L_z} f(x,y,z) sin\frac{g\pi z}{L_z} dz \tag{61}$$

Combining equations (59), (60) and (61) we have

$$G_w = \frac{8}{L_x L_y L_z}\int_0^{L_x}\int_0^{L_y}\int_0^{L_z} f(x,y,z) sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y}\ sin\frac{g\pi z}{L_z} dxdydz \tag{62}$$

$$\frac{\partial u}{\partial t}(x,y,z,t) = \left(A_n cos\frac{n\pi t}{L_x} + B_n sin\frac{n\pi t}{L_x}\right)\left(\frac{-m\pi}{L_y} C_m sin\frac{m\pi t}{L_y} + \frac{m\pi}{L_y} D_m cos\frac{m\pi t}{L_y}\right)$$

$$\left(\frac{-g\pi}{L_z} E_g sin\frac{g\pi t}{L_z} + \frac{g\pi}{L_z} F_g cos\frac{g\pi t}{L_z}\right) + \left(C_m cos\frac{m\pi t}{L_y} + D_m sin\frac{m\pi t}{L_y}\right)\left(\frac{-g\pi}{L_z} E_g sin\frac{g\pi t}{L_z} + \frac{g\pi}{L_z} F_g cos\frac{g\pi t}{L_z}\right)$$

$$\left(\frac{-n\pi}{L_x} A_n sin\frac{n\pi t}{L_x} + \frac{n\pi}{L_x} B_n cos\frac{n\pi t}{L_x}\right) + \left(E_g cos\frac{g\pi t}{L_z} + F_g sin\frac{g\pi t}{L_z}\right)\left(\frac{-n\pi}{L_x} A_n sin\frac{n\pi t}{L_x} + \frac{n\pi}{L_x} B_n cos\frac{n\pi t}{L_x}\right)$$

$$\left(\frac{-m\pi}{L_y} C_m sin\frac{m\pi t}{L_y} + \frac{m\pi}{L_y} D_m cos\frac{m\pi t}{L_y}\right) \tag{63}$$

$$\frac{\partial u}{\partial t}(x,y,z,0) = A_n D_m F_g \frac{m\pi}{L_y}\frac{g\pi}{L_z} + C_m F_g B_n \frac{g\pi}{L_z}\frac{n\pi}{L_x} + E_g B_n D_m \frac{n\pi}{L_x}\frac{m\pi}{L_y} \tag{64}$$

$$\frac{\partial u}{\partial t}(x,y,z,0) = g(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{\alpha=1}^{\infty} sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}$$

$$\left(A_n D_m F_g \frac{m\pi}{L_y}\frac{g\pi}{L_z} + C_m F_g B_n \frac{g\pi}{L_z}\frac{n\pi}{L_x} + E_g B_n D_m \frac{n\pi}{L_x}\frac{m\pi}{L_y}\right) \tag{65}$$

Hence the required solution is equation (57). Equation (57) is periodic in nature since the solution contain sinusoidal functions. The constant term and periodic term in (57) shows that the solution is periodic which is unstable in the sense that all solution starting sufficiently close to it depart from it with increasing time.

### 3.2 Test for Periodicity of Ultrahyperbolic Equation using Fourier Series Method

We first establish the period for each variable. The formula for period of a periodic function given by $\frac{2\pi}{n}$ where $n$ is the coefficient of $x$ and $2\pi$ is the fundamental period.

The term $sin\frac{n\pi x}{L_x}$ is periodic in $x$ with period $T_x = \frac{2\pi}{\frac{n\pi}{L_x}} = \frac{2L_x}{n}$

The term $cos\frac{n\pi t}{L_x}$ is periodic in $t$ with period $T = \frac{2L_x}{n}$

The term $sin\frac{m\pi y}{L_y}$ is periodic in $y$ with period $T_y = \frac{2L_y}{m}$

The term $sin\frac{g\pi z}{L_z}$ is periodic in $z$ with period $T_z = \frac{2L_z}{g}$

Next is to establish Fourier Series coefficient for each variable.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L_x}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L_x}\right) \tag{66}$$

For $f(x) = sin\frac{n\pi x}{L_x}$, Let $\frac{n\pi}{L_x} = c$, we have

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{-L}^{L} sincxdx \tag{67}$$

$$= \frac{1}{L}\left[\frac{-coscx}{c}\right]_{-L}^{L} = 0 \text{ for } c \neq 0 \tag{68}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} sincxcoscxdx \tag{69}$$

$\sin(c + c)\,x = sincxcoscx + coscxsincx$

$sin2cx = 2sincxcoscx$. Hence $sincxcoscx = \frac{sin2cx}{2}$

$$\frac{1}{L}\int_{-L}^{L}\frac{sin2cx}{2}dx = \frac{1}{2L}\left[\frac{-cos2cx}{2c}\right]_{-L}^{L} = 0 \tag{70}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} sincxsincxdx = \frac{1}{L}\quad\int_{-L}^{L} sin^2 cxdx \tag{71}$$

$$= \frac{1}{L}\int_{-L}^{L} \frac{1-cos2cx}{2}dx = \frac{1}{L}\left[\frac{x}{2} - \frac{sin2cx}{4c}\right]_{-L}^{L} \quad (72)$$

$$= 1 - \frac{sin2cL}{4c} + \frac{sin2cL}{4c} = 1 \quad (73)$$

Therefore the Fourier series for variable $x$ in given by

$$f(x) = \sum_{n=1}^{\infty} sin\frac{n\pi x}{L_x} \qquad n = 1,2,\ldots \quad (74)$$

Similarly for the function $f(y) = sin\frac{m\pi y}{L_y}$.Then the Fourier series is given by

$$f(y) = \sum_{m=1}^{\infty} sin\frac{m\pi y}{L_y} \quad m = 1,2,\ldots \quad (75)$$

For the function $f(z) = sin\frac{g\pi z}{L_z}$.. Then the Fourier series is given by

$$f(z) = \sum_{\alpha=1}^{\infty} sin\frac{g\pi z}{L_z} \quad g = 1,2,\ldots \quad (76)$$

For the function $f(t) = cos\frac{n\pi t}{L_x}$, the Fourier series is gven by

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L_x}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L_x}\right) \quad (77)$$

Let $\frac{n\pi}{L_x} = c$, then the coefficients are given by

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt = \frac{1}{L}\int_{-L}^{L} cosctdt \quad (78)$$

$$= \frac{1}{L}\left[\frac{sinct}{c}\right]_{-L}^{L} = \frac{1}{L}\left[\frac{sincL}{c} - \frac{sinc(-L)}{c}\right] = 0 \quad (79)$$

This is because $sincL = 0$

$$a_n = \frac{1}{L}\int_{-L}^{L} cosctcosctdt = \frac{1}{L}\int_{-L}^{L} cos^2ctdt \quad (80)$$

$$= \frac{1}{L}\int_{-L}^{L} \frac{cos2ct+1}{2}dt \quad \text{since } cos2A = 2cos^2A - 1$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \frac{cos2ct}{2}dt + \frac{1}{L}\int_{-L}^{L} \frac{1}{2}dt \quad (81)$$

$$= \frac{1}{L}\left[\frac{sin2ct}{4c}\right]_{-L}^{L} + \frac{1}{L}\left[\frac{t}{2}\right]_{-L}^{L} \quad (82)$$

$$= \frac{1}{L}\left[\frac{sin2cL}{4c} - \frac{sin2c(-L)}{4c}\right] + \frac{1}{L}\left[\frac{L}{2} + \frac{L}{2}\right] = 1 \quad (83)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} cosctsinctdt \quad (84)$$

$$sin2ct = sinctcosct + cosctsinct$$

$$sin2ct = 2sinctcosct$$

$$cosctsinct = \frac{sin2ct}{2}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \frac{sin2ct}{2}dt = \frac{1}{2L}\left[\frac{-cos2ct}{2c}\right]_{-L}^{L} \quad (85)$$

$$= -\frac{cos2cL}{4Lc} + \frac{cos2c(-L)}{4Lc} = 0 \quad (86)$$

Therefore the Fourier series is given by

$$f(t) = \sum_{n=1}^{\infty} cos\frac{n\pi t}{L_x} \quad \text{for } n = 1,2.3, \ . \ . \ . \quad (87)$$

Since each variable has a Fourier series representation at a particular frequency and the Fourier coefficient are not all zeros, then the solution of ultrahyperbolic equation exhibits periodicity at that frequency.

To show that the solution of ultra-hyperbolic equation for four independent is periodic we established that

1. $u(x + T_x, y, z, t) = u(x, y, z, t)$

$$u(x + T_x, y, z, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{g=1}^{\infty} sin\frac{n\pi(x+T_x)}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}\left(A_n cos\frac{n\pi t}{L_x} + B_n sin\frac{n\pi t}{L_x}\right)$$
$$\left(C_m cos\frac{m\pi t}{L_y} + D_m sin\frac{m\pi t}{L_y}\right)\left(E_g cos\frac{g\pi t}{L_z} + F_g sin\frac{g\pi t}{L_z}\right)$$

Considering $sin\frac{n\pi(x+T_x)}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}$ we have (88)

$$sin\frac{n\pi(x+T_x)}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} = \sin\left(\frac{n\pi x}{L_x} + \frac{n\pi T_x}{L_x}\right) sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} \quad (89)$$

Note that $\sin(A + B) = sinAcosB + cosAsinB$. Let $\frac{n\pi x}{L_x} = A$ and $\frac{n\pi T_x}{L_x} = B$

$$sin\left(\frac{n\pi x}{L_x} + \frac{n\pi T_x}{L_x}\right) = sin\frac{n\pi x}{L_x} cos\frac{n\pi T_x}{L_x} + cos\frac{n\pi x}{L_x} sin\frac{n\pi T_x}{L_x} \quad (90)$$

Equation (3.81) becomes

$$\sin\left(\frac{n\pi x}{L_x} + \frac{n\pi T_x}{L_x}\right) sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} = \left(sin\frac{n\pi x}{L_x} cos\frac{n\pi T_x}{L_x} + cos\frac{n\pi x}{L_x} sin\frac{n\pi T_x}{L_x}\right) sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}$$

$$= sin\frac{n\pi x}{L_x} cos\frac{n\pi T_x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} + cos\frac{n\pi x}{L_x} sin\frac{n\pi T_x}{L_x} \ sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} \quad (91)$$

Since $T_x = \frac{2L_x}{n}$, $cos\frac{n\pi T_x}{L_x} = \cos 2\pi = 1$. Also $sin\frac{n\pi T_x}{L_x} = \sin 2\pi = 0$. Then we have

$$\sin\left(\frac{n\pi x}{L_x} + \frac{n\pi T_x}{L_x}\right) sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} = sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} \quad (92)$$

$$u(x + T_x, y, z, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{g=1}^{\infty} sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}\left(A_n cos\frac{n\pi t}{L_x} + B_n sin\frac{n\pi t}{L_x}\right)$$
$$\left(C_m cos\frac{m\pi t}{L_y} + D_m sin\frac{m\pi t}{L_y}\right)\left(E_g cos\frac{g\pi t}{L_z} + F_g sin\frac{g\pi t}{L_z}\right) = u(x, y, z, t) \quad (93)$$

The same results is true for $y, z$ and $t$. Hence we conclude that the solution of ultra-hyperbolic equation is periodic.

### 3.3 Uniqueness of solution to Ultra-hyperbolic equation

Consider the non-homogenous ultra-hyperbolic equation of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = f(x, y, z, t) \quad (94)$$

With conditions $0 < x < L_x, \quad 0 < y < L_y, 0 < z < L_z, t > 0$

$u(0, y, z, t) = a(t) \qquad u(L_x, y, z, t) = b(t) \qquad t \geq 0$

$u(x, 0, z, t) = c(t) \qquad u(x, L_y, z, t) = d(t)$

$u(x, y, 0, t) = e(t) \qquad u(x, y, L_z, t) = h(t)$

$u(x, y, z, 0) = f(x, y, z) \qquad u_t(x, y, z, 0) = g(x, y, z)$

Let $u_1, u_2$ be two solutions of the problem (94). By the superposition principle, the function $r = u_1 - u_2$ is a solution of the problem

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} - \frac{\partial^2 r}{\partial z^2} - \frac{\partial^2 r}{\partial t^2} = 0 \quad (95)$$

With conditions $0 < x < L_x, \quad 0 < y < L_y, 0 < z < L_z, t > 0$

$r(0, y, z, t) = 0 \qquad r(L_x, y, z, t) = 0$

$r(x, 0, z, t) = 0 \qquad r(x, L_y, z, t) = 0$

$r(x, y, 0, t) = 0 \qquad r(x, y, L_z, t) = 0$

$r(x, y, z, 0) = 0 \qquad r_t(x, y, z, 0) = 0$

In order to show that energy is conserved, we assume that the total energy of the system is given by

$$E(t) = \int_{\Omega}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 - \left(\frac{\partial r}{\partial z}\right)^2 - \left(\frac{\partial r}{\partial t}\right)^2\right] dxdydzdt \quad (96)$$

Where $\Omega$ is the domain of integration and $r$ is the solution to the ultra-hyperbolic equation.

Let us assume that the ultra-hyperbolic equation

$$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}-\frac{\partial^2 r}{\partial z^2}-\frac{\partial^2 r}{\partial t^2}=0 \tag{97}$$

Multiplying (97) by $\frac{\partial r}{\partial t}$ and integrating over the domain gives

$$\int_\Omega \left[\frac{\partial r}{\partial t}\left(\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}-\frac{\partial^2 r}{\partial z^2}-\frac{\partial^2 r}{\partial t^2}\right)\right] dxdydzdt=0 \tag{98}$$

Using the integration by parts we have

$$\int_\Omega \frac{\partial r}{\partial t}\left(\frac{\partial^2 r}{\partial x^2}\right) dxdydzdt=\int_\Omega \frac{\partial r}{\partial t}\frac{\partial r}{\partial x}n_x dxdydzdt-\int_\Omega \frac{\partial^2 r}{\partial t\partial x}\frac{\partial r}{\partial x}dxdydzdt \tag{99}$$

$$=\left[\frac{\partial r}{\partial t}(L_x,t)\left(\frac{\partial r}{\partial x}(L_x,t)-\frac{\partial r}{\partial t}(0,t)\frac{\partial r}{\partial x}(0,t)\right]-\int_\Omega \left(\frac{\partial^2 r}{\partial t\partial x}\frac{\partial r}{\partial x}\right)dxdydzdt \tag{100}$$

Where $n_x$ is the $x$ component of the outward unit normal to the boundary $\partial\Omega$

Applying the boundary conditions we have

$$\int_\Omega \frac{\partial r}{\partial t}\left(\frac{\partial^2 r}{\partial x^2}\right) dxdydzdt=-\int_\Omega \left(\frac{\partial^2 r}{\partial t\partial x}\frac{\partial r}{\partial x}\right)dxdydzdt \tag{101}$$

$$\int_\Omega \frac{\partial r}{\partial t}\left(\frac{\partial^2 r}{\partial y^2}\right) dxdydzdt=\int_\Omega \frac{\partial r}{\partial t}\frac{\partial r}{\partial y}n_y dxdydzdt-\int_\Omega \frac{\partial^2 r}{\partial t\partial y}\frac{\partial r}{\partial y}dxdydzdt \tag{102}$$

$$=\left[\frac{\partial r}{\partial t}(L_y,t)\left(\frac{\partial r}{\partial y}(L_y,t)-\frac{\partial r}{\partial t}(0,t)\frac{\partial r}{\partial y}(0,t)\right]-\int_\Omega \left(\frac{\partial^2 r}{\partial t\partial y}\frac{\partial r}{\partial y}\right)dxdydzdt \tag{103}$$

Where $n_y$ is the $y$ component of the outward unit normal to the boundary $\partial\Omega$

Applying the boundary conditions we have

$$\int_\Omega \frac{\partial r}{\partial t}\left(\frac{\partial^2 r}{\partial y^2}\right) dxdydzdt=-\int_\Omega \left(\frac{\partial^2 r}{\partial t\partial y}\frac{\partial r}{\partial y}\right)dxdydzdt \tag{104}$$

$$\int_\Omega \frac{\partial r}{\partial t}\left(\frac{\partial^2 r}{\partial z^2}\right) dxdydzdt=\int_\Omega \frac{\partial r}{\partial t}\frac{\partial r}{\partial z}n_z dxdydzdt-\int_\Omega \frac{\partial^2 r}{\partial t\partial z}\frac{\partial r}{\partial z}dxdydzdt \tag{105}$$

$$=\left[\frac{\partial r}{\partial t}(L_z,t)\left(\frac{\partial r}{\partial z}(L_z,t)-\frac{\partial r}{\partial t}(0,t)\frac{\partial r}{\partial y}(0,t)\right]-\int_\Omega \left(\frac{\partial^2 r}{\partial t\partial z}\frac{\partial r}{\partial z}\right)dxdydzdt \tag{106}$$

Where $n_z$ is the $z$ component of the outward unit normal to the boundary $\partial\Omega$

Applying the boundary conditions we have

$$\int_\Omega \frac{\partial r}{\partial t}\left(\frac{\partial^2 r}{\partial z^2}\right) dxdydzdt=-\int_\Omega \left(\frac{\partial^2 r}{\partial t\partial z}\frac{\partial r}{\partial z}\right)dxdydzdt \tag{107}$$

$$\int_\Omega \frac{\partial r}{\partial t}\left(\frac{\partial^2 r}{\partial t^2}\right) dxdydzdt=\int_\Omega \frac{1}{2}\frac{d}{dt}\left(\frac{\partial r}{\partial x}\right)^2 dxdydzdt \tag{108}$$

Substituting equations (101), (104), (107), (108) in equation (98) gives

$$-\int_\Omega \left(\frac{\partial^2 r}{\partial t\partial x}\frac{\partial r}{\partial x}\right)dxdydzdt-\int_\Omega \left(\frac{\partial^2 r}{\partial t\partial y}\frac{\partial r}{\partial y}\right)dxdydzdt+\int_\Omega \left(\frac{\partial^2 r}{\partial t\partial z}\frac{\partial r}{\partial z}\right)dxdydzdt= \int_\Omega \frac{1}{2}\frac{d}{dt}\left(\frac{\partial r}{\partial t}\right)^2 dxdydzdt \tag{109}$$

Multiplying the LHS of equation (109) by negative sign gives

$$\int_\Omega \left(\frac{\partial^2 r}{\partial t\partial x}\frac{\partial r}{\partial x}\right)dxdydzdt+\int_\Omega \left(\frac{\partial^2 r}{\partial t\partial y}\frac{\partial r}{\partial y}\right)dxdydzdt-\int_\Omega \left(\frac{\partial^2 r}{\partial t\partial z}\frac{\partial r}{\partial z}\right)dxdydzdt- \int_\Omega \frac{1}{2}\frac{d}{dt}\left(\frac{\partial r}{\partial t}\right)^2 dxdydzdt=0 \tag{110}$$

$$\int_\Omega \frac{1}{2}\frac{d}{dt}\left(\frac{\partial r}{\partial x}\right)^2 dxdydzdt+\int_\Omega \frac{1}{2}\frac{d}{dt}\left(\frac{\partial r}{\partial y}\right)^2 dxdydzdt-\int_\Omega \frac{1}{2}\frac{d}{dt}\left(\frac{\partial r}{\partial z}\right)^2 dxdydzdt -\int_\Omega \frac{1}{2}\frac{d}{dt}\left(\frac{\partial r}{\partial t}\right)^2 dxdydzdt=0 \tag{111}$$

Multiplying equation (111) by 2 and simplifying further gives

$$\frac{d}{dt}\int_\Omega \left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2-\left(\frac{\partial r}{\partial z}\right)^2-\left(\frac{\partial r}{\partial t}\right)^2\right]dxdydzdt=0 \tag{112}$$

Comparing equation (96) with equation (112) we have

$$\frac{d}{dt}E(t)=0 \tag{113}$$

Integrating equation (113) gives

$$E(t)=c \tag{114}$$

Where $c$ is a constant

Equation (114) shows that energy is conserved.

Since $r(x,y,z,0)=0$. It follows that $r_x(x,y,z,0)=0$, $r_y(x,y,z,0)=0$, $r_z(x,y,z,0)=0$ and $r_t(x,y,z,0)=0$. Hence we conclude that energy at $t=0$ is zero. Thus $E(t)=0$.

Using equation (97) we have

$$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{\partial^2 r}{\partial z^2}+\frac{\partial^2 r}{\partial t^2} \tag{115}$$

Let $x=z$ and let $y=t$ then $\frac{\partial^2 r}{\partial z^2}\geq 0$ and $\frac{\partial^2 r}{\partial t^2}\geq 0$ since the square of any derivative is non- negative. Setting equation (115) equal to zero gives

$$\frac{\partial^2 r}{\partial z^2}+\frac{\partial^2 r}{\partial t^2}=0 \tag{116}$$

$\frac{\partial^2 r}{\partial z^2}+\frac{\partial^2 r}{\partial t^2}=0$ implies that $\frac{\partial^2 r}{\partial z^2}=0$ and $\frac{\partial^2 r}{\partial t^2}=0$. This implies that $\frac{\partial r}{\partial z}=0$ and $\frac{\partial r}{\partial t}=0$. Therefore $\frac{\partial r}{\partial z}=\frac{\partial r}{\partial t}=0$. Integrating $\frac{\partial r}{\partial z}=0$ gives $r(x,y,z,t)=$ constant.

Applying the initial conditions $r(x,y,z,0)=0$, we have that $r(x,y,z,t)=0$

Hence $0=u_1-u_2$ implies that $u_1=u_2$. This establishes the uniqueness of periodic solution to ultra-hyperbolic boundary value problem of four independent variables.

## 4. NUMERICAL SIMULATIONS OF OUR RESULTS

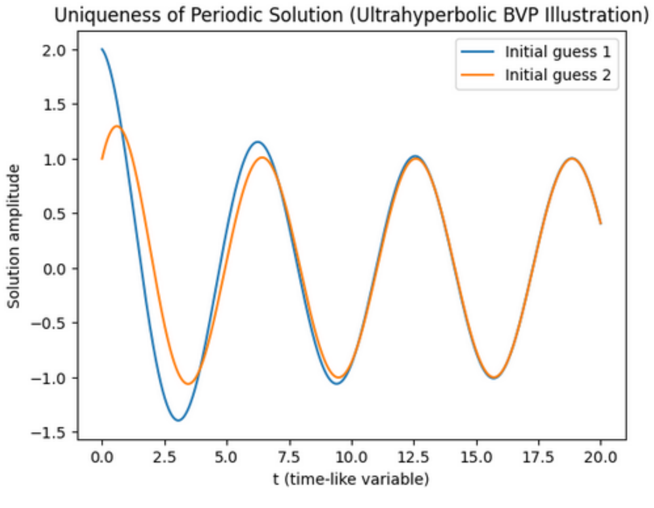

**Figure 1:** Trajectory illustrating uniqueness of periodic solution of ultra-hyperbolic boundary value problem.

## 5. CONCLUSIONS

From our results, energy method and superposition principle are very effective in investigating uniqueness of periodic solution ultra-hyperbolic boundary value problem. The initial conditions was effective in obtaining the zero solution of the unknown function. The advantages of these methods are that it is applicable to various PDEs including nonlinear and non-autonomous system. It is also used to break down complex system into simpler components. Hence, we conclude that uniqueness of periodic solutions of the model is possible with the help of the boundary and initial conditions. The numerical behavior is explained as follows:

Figure 1 describes the periodic solution of ultra-hyperbolic boundary value problem. The blue curve and orange curve correspond to two different solutions. As time increases, both solutions collapse onto the same periodic curve. This behavior shows that the periodic solution is unique.

## COMPETING INTERESTS

The authors declared that they have no competing interests.

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