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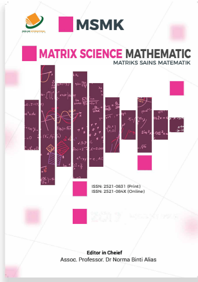

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ANALYTIC SOLUTION OF FRACTIONAL JEFFREY FLUID INDUCED BY ABRUPT MOTION OF THE PLATE

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ABSTRACT

This paper presents some new exact solutions corresponding to unsteady fractional Jeffrey fluid produced by a flat plate between two side walls perpendicular to the plate. The fractional calculus approach in the governing equations is used. The exact solutions are established by means of the Fourier sine transform and discrete Laplace transform. The series solution of velocity field and the associated shear stress in terms of Fox H-functions, satisfying all imposed initial and boundary conditions, have been obtained.

KEYWORDS

Fractional Jeffrey fluid, Fractional derivatives, Fox H-functions, Laplace transform

1. INTRODUCTION

Considerable progress has been made in studying flows of non-Newtonian fluids throughout the last few decades. Due to their viscoelastic nature, a non-Newtonian fluid, such as oils, paints, ketchup, liquid polymers, and asphalt exhibit some remarkable phenomena. Amplifying interest of many researchers has shown that these flows are imperative in industry, manufacturing of food and paper, polymer processing and technology. Dissimilar to the Newtonian fluid, the flows of non-Newtonian fluids cannot be explained by a single constitutive model. In general, the rheological properties of fluids are specified by their so-called constitutive equations. Exact recent solutions for constitutive equations of viscoelastic fluids are given by researchers [1-7]. Amongst non-Newtonian fluids the Jeffrey model is considered to be one of the simplest type of model which best explain the rheological effects of viscoelastic fluids.

The Jeffrey model is a relatively simple linear model using the time derivatives instead of convective derivatives. Recently, the fractional calculus approach has proved to be an important tool for considering behaviors of such types of fluids [8, 9]. Many researchers investigated different problems using the fractional derivative technique for such fluids. In their work, integer order time derivatives in the constitutive models for generalized Jeffrey fluids were replaced by the Riemann-Liouville fractional derivatives. A lot of work has been done on fractional derivatives during the last few years. A researcher proved that fractional derivative models of viscoelastic type fluids were in harmony with the molecular theory and attains the fractional differential equation of order ½ [10]. A scientist developed the fractional derivative method into rheology to investigate various problems [11]. Li and Jiang employ the fractional calculus to examine the behavior of sesbania gum and Xanthan gum in their experiments and attain adequate results [12]. Moreover, here we mention some more contributions which regards with the generalized viscoelastic type fluids [13-19].

Researchers show less attention for the flows of Jeffrey fluids in which the fractional calculus approach has been used. To the best of our knowledge, no investigation is available in the literature regarding generalized Jeffrey fluids which have been set into motion by the impulsive motion of the plate. In this paper we establish exact solutions for the velocity field and the as-sociated shear stress corresponding to the unsteady flow of an incompressible generalized Jeffrey fluid between two side walls perpendicular to the plate. The obtained solutions, expressed under series form in terms of Fox H-functions, are established by means of Fourier sine and Laplace transforms [20,21].

2. GOVERNING EQUATIONS

For an incompressible and unsteady generalized Jeffrey fluid the Cauchy stress tensor is defined as

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S}, \qquad (1+\lambda)\boldsymbol{S} = \mu\left(\boldsymbol{A} + \theta^{\beta}\left(\frac{D^{\beta}}{Dt^{\beta}} + (\boldsymbol{V}.\nabla)\boldsymbol{A}\right)\right), \tag{1}$$

where **S** is the extra stress tensor **I** is the indeterminate spherical stress, μ is the dynamic viscosity, $\mathbf{A} = \mathbf{L} + \mathbf{L}^{\mathrm{T}}$ is the first Rivlin-Ericksen tensor, **L** is the velocity gradient, λ and θ are relaxation and retardation times, β is the fractional calculus parameter such that $0 \leq \beta < 1$, $\mathbf{D}^{\beta}{}_{t}$ is the fractional differentiation operator of order β based on the

Riemann-Liouville definition

$$D_t^{\beta}[f(t)] = \frac{1}{\Gamma(1-p)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^p}\,d\tau, \quad 0 < p < 1, \tag{2}$$

where $\Gamma(.)$ stands for gamma function. Model for ordinary Jeffrey fluid can be obtained by letting $\beta = 1$. For the following problem we consider the velocity field and an extra stress of the form

$$\boldsymbol{V} = \boldsymbol{V}(y,z,t) = u(y,z,t)\boldsymbol{i}, \quad \boldsymbol{S} = \boldsymbol{S}(y,z,t), \tag{3}$$

where $\boldsymbol{u}$ is the velocity and $\boldsymbol{i}$ is the unit vector along the $\boldsymbol{x}$-direction. The continuity equation for such flows is automatically satisfied. We take the extra stress $\boldsymbol{S}$ independent of $\boldsymbol{x}$ as the velocity field is independent of $\boldsymbol{x}$. Also, at $t = 0$ the fluid being at rest is given by

$$S(y,z,0)=\mathbf{0}, \tag{4}$$

Therefore, from Eqs. (1) and (2) it results that $S_{yy} = S_{yz} = S_{zz} = \mathbf{0}$ and the relevant equations

$$(1+\lambda)\tau_1 = \mu\left(1+\theta^{\beta}\frac{D^{\beta}}{Dt^{\beta}}\right)\partial_y u(y,z,t), \tag{5}$$

$$(1+\lambda)\tau_2 = \mu\left(1+\theta^{\beta}\frac{D^{\beta}}{Dt^{\beta}}\right)\partial_z u(y,z,t), \tag{5a}$$

where $\tau_1 = S_{xy}$ and $\tau_2 = S_{xz}$ are the tangential stresses. In the absence of body forces the balance of linear momentum becomes

$$\partial_y\tau_1+\partial_z\tau_2-\partial_x p=\rho\partial_t u, \quad \partial_y p=\partial_z p=0, \tag{6}$$

here $\partial_x p$ is the pressure gradient and ρ represents the density of the fluid. Eliminating the shear stresses τ_1 and τ_2 between Eqs. (5) and (6) and neglecting the pressure gradient, the governing equation reduces to the following form

$$(1+\lambda)\frac{\partial}{\partial t}u(y,z,t)=v\left(1+\theta^{\beta}D_t^{\beta}\right)\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u(y,z,t), \tag{7}$$

where v represents the kinematic viscosity.

3. STATEMENT OF THE PROBLEM

We take an unsteady generalized Jeffrey fluid saturating the space above a flat plate which is perpendicular to the y-axis and lies between two side walls perpendicular to the plate. At first the fluid as well as the plane wall is at rest and at time t = 0, the fluid is set into flow by translating the bottom wall in its own plane, with a time dependent velocity V. Its velocity is of the form of Eq. (3) and the governing equation is given by Eq. (7). The associated initial and boundary conditions are

$$\begin{aligned} &u(y,z,0)=\partial_t u(y,z,0)=0; \quad y>0, 0\le z\le h\\ &u(0,z,t)=V; \quad t>0, 0\le z\le h\\ &u(y,0,t)=u(y,h,t)=0; \quad y,t>0. \end{aligned} \tag{8}$$

The distance between the two side walls is represented by h. Moreover, the natural conditions

$$u(y,z,t),\ \partial_{y}u\ (y,z,t)\to 0 \text{ as } y\to\infty, -h\le z\le h, t>0$$

have to be also satisfied. They are consequences of the fact that the fluid will be at rest at infinity and there is no shear along y-axis.

$$u(y,z,t),\ \partial y u\ (y,z,t)\ 0\to \quad y\to\infty, -h\le z\le h, t>0 \tag{9}$$

have to be also satisfied. They are consequences of the fact that the fluid will be at rest at infinity and there is no shear along y-axis.

4. CALCULATION OF THE VELOCITY FIELD

First, we multiply both sides of Eq. (7) by $\left(\frac{n\pi z}{h}\right)$, and then integrate the obtained resultfrom 0 to h with respect to z, we get the following differential equation

$$(1+\lambda)\frac{\partial u_n(y,n,t)}{\partial t}=v\left(1+\theta^{\beta}D_t^{\beta}\right)\frac{\partial^2}{\partial y^2}u_n(y,n,t)-v\left(\frac{n\pi}{h}\right)^2\left(1+\theta^{\beta}D_t^{\beta}\right)u_n(y,n,t) \tag{10}$$

Applying the Laplace transform to Eq. (10), the image function $\bar{u}_n(y,n,s)$ of $u_n(y,n,t)$ is given by

$$\frac{\partial^2}{\partial y^2}\bar{u}_n(y,n,s)-\left[\xi^2+\frac{s(1+\lambda)}{v(1+\theta^{\beta}s^{\beta})}\right]\bar{u}_n(y,n,s)=0, \tag{11}$$

$$\bar{u}_n(0,n,s)=\frac{V}{s},$$

$$\bar{u}_n(0,n,s)\to 0 \ as\ y\to\infty,$$

where $\xi=\frac{n\pi}{h}$. The solution of above differential equation is in the following form

$$\bar{u}_n=\frac{V}{s}exp\left[-y\sqrt{\xi^2+\frac{s(1+\lambda)}{v(1+\theta^{\beta}s^{\beta})}}\right]. \tag{12}$$

We will apply the inverse Laplace transform technique to obtain analytic solution for the velocity field but to avoid difficult calculations of residues and contour integrals; first we express Eq. (12) in series form as

$$\bar{u}_n(y,n,s)=V\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{j+n+p+q}y^j\xi^{j-2n}v^{-n}}{j!\,n!\,q!\,p!\,\Gamma(n)\Gamma(-n)}\times\frac{\lambda^{n-q}\theta^{-n-p+\beta}\Gamma(p+n)\Gamma(q-n)\Gamma\left(n-\frac{j}{2}\right)}{\Gamma\left(\frac{j}{2}\right)s^{-n+\beta(p+n)+1}}. \tag{13}$$

We apply the inverse Laplace transform to Eq. (13), to obtain

$$u_n(y,n,s)=V\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{j+n+p+q}y^j\xi^{j-2n}v^{-n}}{j!\,n!\,q!\,p!\,\Gamma(n)\Gamma(-n)\Gamma\left(\frac{j}{2}\right)}\times\frac{\lambda^{n-q}\theta^{-n-p+\beta}\Gamma(p+n)\Gamma(q-n)\Gamma\left(n-\frac{j}{2}\right)t^{-n+\beta(p+n)}}{\Gamma(-n+\beta(p+n)+1)}. \tag{14}$$

Taking the inverse finite Fourier sine transform to get the analytic solution of the velocity field

$$\begin{aligned} u(y,z,t)&=\frac{2}{h}\sum_{m=1}^{\infty}\sin\left(\frac{m\pi z}{h}\right)u_n\\ &=\frac{2V}{h}\sum_{m=1}^{\infty}\sin\left(\frac{m\pi z}{h}\right)\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{j+n+p+q}}{j!\,n!\,q!\,p!}\\ &\times\frac{y^j\xi^{j-2n}v^{-n}\Gamma(p+n)\Gamma(q-n)\Gamma\left(n-\frac{j}{2}\right)t^{-n+\beta(p+n)}}{\Gamma(n)\Gamma(-n)\Gamma\left(\frac{j}{2}\right)\lambda^{q-n}\theta^{n+p-\beta}\Gamma(-n+\beta(p+n)+1)}. \end{aligned} \tag{15}$$

To write Eq. (15) in a more compact form, we use the Fox H-function,

$$\begin{aligned} u(y,z,t)&=\frac{2V}{h}\sum_{m=1}^{\infty}\sin\left(\frac{m\pi z}{h}\right)\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^{j+n+p+q}y^j\xi^{j-2n}v^{-n}}{j!\,n!\,q!\,p!\,\lambda^{q-n}\theta^{n-\beta}t^{n-\beta n}}\\ &\times H_{3,5}^{1,3}\left\{\frac{t^{\beta}}{\theta}\left|\begin{matrix}(1-n,1),(1-q+n,0),\left(1-n+\frac{j}{2},0\right).\\(0,1),(1-n,0),(1+n,0),\left(1-j/2,0\right),(n-\beta n,\beta).\end{matrix}\right.\right\}. \end{aligned} \tag{16}$$

To obtain (16), the following Fox H-function property is used:

$$H_{p,q+1}^{1,p}\left\{-X\left|\begin{matrix}(1-a_1,A_1),\dots,(1-a_p,A_p)\\(0,1)(1-b_1,B_1),\dots,(1-b_q,B_q)\end{matrix}\right.\right\}=\sum_{k=1}^{\infty}\frac{\Gamma(a_1+A_1K)\dots\Gamma(a_p+A_pK)}{k!\,\Gamma(b_1+B_1K)\dots\Gamma(b_p+B_pK)}X^k. \tag{17}$$

5. CALCULATION OF THE SHEAR STRESS

To get the shear stress first we apply Laplace transform to Eqs. (5) and (5a), to obtain

$$(1+\lambda)\bar{\tau_1}=\mu\left(1+\theta^{\beta}s^{\beta}\right)\partial_y\bar{u}(y,z,s), \tag{18}$$

$$(1+\lambda)\bar{\tau_2}=\mu\left(1+\theta^{\beta}s^{\beta}\right)\partial_z\bar{u}(y,z,s). \tag{19}$$

Taking inverse Fourier transform of Eq. (12) to get $\bar{\mathbf{u}}(\mathbf{y},\mathbf{z},\mathbf{s})$ and then putting it into Eq. (18), we obtain

$$\bar{\tau_1} = \frac{2V\xi\mu\left(1+\theta^\beta s^\beta\right)}{h(1+\lambda)}\sum_{n=1}^{\infty} sin\left(\frac{n\pi z}{h}\right)$$

$$\times \mathbf{exp}\left[-\boldsymbol{y}\sqrt{\xi^2+\frac{\boldsymbol{s}(\mathbf{1}+\boldsymbol{\lambda})}{\boldsymbol{v}(\mathbf{1}+\boldsymbol{\theta^\beta s^\beta})}}\right]\left[\sqrt{\mathbf{1}+\frac{\boldsymbol{s}(\mathbf{1}+\boldsymbol{\lambda})}{\boldsymbol{\xi^2 v}(\mathbf{1}+\boldsymbol{\theta^\beta s^\beta})}}\right]. \qquad (20)$$

We express Eq. (20) in series form in order to obtain a more suitable form of $\boldsymbol{\tau_1}$

$$\bar{\tau_1}\ (x,y,t)$$

$$=\frac{2V\rho v}{h}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{h}\right)\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\sum_{w=0}^{\infty}\sum_{y=0}^{\infty}\sum_{z=0}^{\infty}\sum_{x=0}^{\infty}\sum_{m=0}^{\infty}\theta^{-n-p+w+x+y+z+\beta}$$

$$\times\frac{y^j(-1)^{j+n+p+q+w+y+z+x+m}\ \lambda^{n-q}s^{n-\beta(p+n-w-y)-m-1/2}}{\xi^{-j+2n+2m}v^{n-m}j!\,n!\,q!\,p!\,w!\,y!\,z!\,x!\,m!}$$

$$\times\frac{\Gamma(p+n)\Gamma(q-n)\Gamma(n-j/2)\Gamma\left(w-\frac{1}{2}\right)\Gamma\left(x+\frac{1}{2}\right)\Gamma\left(m-\frac{1}{2}\right)\Gamma(y-m)\Gamma(z+m)}{\Gamma(n)\Gamma(-n)\Gamma(1/2)\Gamma(j/2)\Gamma(m)\Gamma(-m)\Gamma(1/2)\Gamma(-1/2)}. \quad (21)$$

Taking the inverse Laplace transform of (21), we obtain

$$\tau_1\ (x,y,t)$$

$$=\frac{2V\rho v}{h}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{h}\right)\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\sum_{w=0}^{\infty}\sum_{y=0}^{\infty}\sum_{z=0}^{\infty}\sum_{x=0}^{\infty}\sum_{m=0}^{\infty}\theta^{-n-p+w+x+y+z+\beta}$$

$$\times\frac{y^j(-1)^{j+n+p+q+w+y+z+x+m}\ \lambda^{n-q}t^{-n+\beta(p+n-w-y)+m+1/2}}{\xi^{-j+2n+2m}v^{n-m}j!\,n!\,q!\,p!\,w!\,y!\,z!\,x!\,m!}$$

$$\frac{\boldsymbol{\Gamma(p+n)\Gamma(q-n)\Gamma(n-j/2)\Gamma\left(w-\frac{1}{2}\right)\Gamma\left(x+\frac{1}{2}\right)\Gamma\left(m-\frac{1}{2}\right)\Gamma(y-m)\Gamma(z+m)}}{\boldsymbol{\Gamma(j/2)\Gamma(n)\Gamma(n)\Gamma(-n)\Gamma(1/2)\Gamma(m)\Gamma(-m)\Gamma(1/2)\Gamma(-1/2)\Gamma(-n+\beta n+1/2)}}. \quad (22)$$

Finally, using the Fox H-function we obtain the stress field as

$$\tau_1(y,z,t)$$

$$=\frac{2Vv}{h}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{h}\right)\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\sum_{w=0}^{\infty}\sum_{y=0}^{\infty}\sum_{z=0}^{\infty}\sum_{x=0}^{\infty}\boldsymbol{\xi^{j-2n}}$$

$$\times\frac{\boldsymbol{y^j(-1)^{j+n+p+q+w+y+z+x}\ \lambda^{n-q}t^{-n+\beta(p+n-w-y)+1/2}}}{\boldsymbol{\theta^{n+p-w-x-y-z-\beta}v^n j!\,n!\,q!\,p!\,w!\,y!\,z!\,x!}}$$

$$\times H^{1,8}_{8,10}\left\{\frac{-\xi^2 t}{v}\left|\begin{matrix}(1-p+n,0),\ldots,(1-z,1)\\ \left(1-{}^{j}/_{2},0\right),\ldots,(1+n-\beta n,\beta)\end{matrix}\right.\right\}. \qquad (23)$$

In the similar fashion we can find $\boldsymbol{\tau_2}\ (\boldsymbol{x,y,t})$ from Eqs. (16) and (19).

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