ZIBELINE INTERNATIONAL PUBLISHING

Matriks Sains Matematik (MSMK)

DOI : http://doi.org/10.26480/msmk.01.2018.13.17

ISSN: 2521-0831 (Print)
ISSN: 2521-084X (Online)

CODEN : MSMADH

CrossMark

EXISTENCE RESULTS TO A CLASS OF HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS

Zakir Ullah*, Amjad Ali, Rahmat Ali Khan and Muhammad Iqbal

Department of Mathematics, University of Malakand, Dir (L), Khyber Pakhtunkhwa, Pakistan.
****Corresponding Author Email: zakir.uom332@gmail.com***

ARTICLE DETAILS

Article History:

Received 12 November 2017
Accepted 09 December 2017
Available online 18 January 2018

ABSTRACT

This article is devoted to the study of existence results to a class of boundary value problems for hybrid fractional differential equations. A couple of hybrid fixed point theorems for the sum of three operators are used for proving the main results. Examples illustrating the results are also presented.

KEYWORDS

Hybrid fractional differential equations, Boundary value problems, Lebesgue dominated convergence theorem, Fixed point theorem.

1. INTRODUCTION

FODEs have been got proper attention from many researchers because of their usability to model complex phenomena of real world process. Due to the large numbers of applications of FODEs in sciences and engineering, plenty of research papers have been written in this area. As a result, the theory of FODEs has emerged as an important area of investigation in recent years [1, 2]. Valuable contribution has been done dealing with the qualitative theory and numerical analysis of solutions to initial and boundary value problems (BVPs) of nonlinear FODEs. The researchers have given much attention on qualitative theory of solutions for mentioned FODEs, and references therein. Since BVPs arise in various disciplines of physics, engineering as any physical differential equation will have them [3-11].

From applications point of view, here we refer some famous BVPs of differential equations which are the wave equation, like the computation of the normal modes, the Sturm-Liouville problems and Dirichlet problem, etc. For usability purposes, a BVP should be well posed which implies that a unique solution exists corresponding to the input which depends continuously on the input [12-16]. In thermal sciences BVPs have significant applications, for instance to find the temperature at all points of an iron bar with one end kept at lowest energy level and the other end at the freezing point of water. Due to these importance applications researchers studied BVPs of both classical and arbitrary order differential equations from different aspects. One of the important aspect which has been greatly developed and well explored by different researchers is known as existence theory. The respective aspects have been explored for BVPs of FODEs, see for some detail [17-32].

The natural extension of traditional differential equations is known as fractional differential equations (FDEs). FDEs has gained considerable attention of researchers, due to its large number of applications in diverse discipline of engineering and science, such as physics, chemistry, aerodynamics, electrodynamics of complex medium, economics, control theory, signal and image processing, biophysics, blood flow phenomena, etc [33-36]. Mostly significant phenomena in fluid flow, fluid dynamic like diffusion, traffic model, electromagnetic, solid mechanics, statistical mechanics, colored noise, polarization, bioengineering, electrochemical processes and modeling of frequency dependent damping behavior of viscoelastic flow are well described by fractional differential equations [28-32].

One of the important area of FDEs is Hybrid fractional differential equations (HFDEs). All physical phenomena are non-homogenous in nature and take place in form of fractional differential equations, which are well described by HFDEs [37]. The most significant feather, that attracted the consideration of researchers is to investigates the conditions under the system of HFDEs is positive solutions. For the aforesaid purpose, Dhage, Lakshmikantham and Krasnoselskii are extensively studied the system of HFDEs [38-40]. In a researcher the author's established the conditions for existence of ordinary hybrid differential equation with linear perturbations of first type given as

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), t \in [0,1], \\ x(t_0) = x_0, \end{cases} \tag{1.1}$$

where $f \in C(J \times R, R_{\backslash\{0\}})$ and $g \in C(J \times R, R)$. Furthermore, a group researchers, extend the above result of hybrid differential equations to fractional order differential equation involving Riemann-Liouville differential operators [41].

$$\begin{cases} D_{0+}^{\alpha}\left(\dfrac{x(t)}{f(t,x(t)}\right) = g(t,x(t)), \quad t \in [0,T], \\ x(0) = 0, \end{cases}$$

where $0 < \alpha < 1$, $f \in C(J \times R, R_{\backslash\{0\}})$ and $g \in C(J \times R, R)$. They develop the sufficient condition forexistence and uniqueness of solution for the aforesaid class of hybrid fractional differential equation. A group researcher, generalized the above results to the following hybrid fractional differential equations with boundary conditions involving Caputo's derivative [42].

$$\begin{cases} D_{0+}^{\alpha}\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad t\in[0,T], \\ a\dfrac{x(0)}{f(0,x(0))} + b\dfrac{x(T)}{f(T,x(T))} = c, \end{cases}$$

where $0 < \alpha < 1$, a,b,c are real constants with $a + b \neq 0$ and $f \in C(J \times R, R_{\setminus\{0\}})$ $g \in C(J \times R, R)$.

In this article, we extend the aforesaid hybrid fractional differential equation to boundary condition involving ordinary derivatives given as

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\right) = h(t,x(t)), \ t\in[0,1], \quad \alpha\in(1,2], \\ \left[\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=0} = 0, \ \left[\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=1} = 0 \end{cases} \tag{1.2}$$

where $g \in C(J \times R, R)$ and $f, h \in C(J \times R, R)$. With the help of theory develop by a researcher, we establish the sufficient conditions under which the considered problem of hybrid fractional differential equations has at least one solution [37]. At the end, we give an example for illustrative purposes.

2. PRELIMINARIES

In this section, we recall some definitions and results of fractional calculus and hybrid fixed point theory, that are necessary for further investigation [28, 29, 37].

Definition 2.1. The Riemann-Liouville fractional derivative of order α > 0 of a function f : (0,∞) → R is given by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_0^t (t-s)^{n-\alpha-1} f(s)\,ds, n-1<\alpha<n,$$

where $n = [\alpha] + 1, [\alpha]$ represents the integer part of a real number α , provided that the right-hand side is point-wise defined on $(0,\infty)$, where Γ is the gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-s} s^{\alpha-1} ds$

Definition 2.2. The Riemann-Liouville fractional integral of order α > 0 of a continuous function f : (0,∞) → R is given by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,ds$$

provided that the right-hand side is point-wise defined on (0,∞) .

Definition 2.3. A map θ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that θ : P → [0,∞) is continuous and

θ(tx+(1−t)y) ≥ tθ (x) + (1− t)θ (y) ,

for all x, y ∈ P and 0 ≤ t ≤1.

The next two lemmas play an important role for obtaining the equivalent integral equation of BVP (1.2).

Lemma 2.4. [1]. If we assume x ∈C (0,1) ∩ L (0,1) , then the fractional differential equation

$D^{\alpha} u(t) = 0$

of order α > 0 has a unique solution of the form

$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \ldots + C_N t^{\alpha-N}$, $C_i \in R$, $i = 1, 2,..., n$ and $n-1<\alpha<n$.

The following law of composition can be easily deduced from Lemma 2.4.

Lemma 2.5. Assume that x ∈C(0,1) ∩ L(0,1) , with a fractional derivative of order α that belongs to C(0,1) ∩ L(0,1) , then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = x(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \ldots + C_N t^{\alpha-N}, \ C_i \in R \text{ and } i = 1,2,\ldots,N .$$

Lemma 2.6. [1] For x ∈C(0,T) ∩ L(0,T) , the solution of fractional differential equation

$$D^{\alpha} x(t) = y(t), \ n-1<\alpha<n$$

is given by

$$x(t) = I^{\alpha} y(t) + \sum_{i=0}^{n-1} C_i t^i .$$

Let $E = C(J, R)$ be the space of continuous real-valued functions defined on $J = [0,T]$. Define a norm $\|\bullet\|$ and a multiplication in E by $\|x\| = \sup_{t\in J} |x(t)|$ and $(xy)(t) = x(t)y(t)$, for all $t \in J$. Clearly, E is a Banach algebra with respect to the above supremum norm and the multiplication defined on it.

3. HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS

In this section, we consider the Boundary Value Problem (1.2). The following hybrid fixed point theorem for three operators in a Banach algebra E , due to Dhage, will be used to prove the existence result for the Boundary Value Problem (1.2) [37].

Lemma 3.1. Let S be a nonempty, closed convex and bounded subset of a Banach algebra E and let A,C : E → E and B : S → E be three operators satisfying:

a. A and C are Lipschitzian with Lipschitz constants φ_1 and φ_2 , respectively
b. B is completely continuous
c. If $x = AxBy + Cx$, then $x \in S$ for all $y \in S$
d. If $\varphi_1 M + \varphi_2 < 1$ where $M = \|B(S)\|$, then the operator equation $x =$ AxBy + Cx has a solution.

Lemma 3.2. Let z ∈C([0,1], R) , then the solution of BVP of hybrid differential equation of fractional order

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\right) = z(t), \ t\in[0,1], \quad \alpha\in(1,2], \\ \left[\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=0} = 0, \ \left[\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=1} = 0 \end{cases} \tag{3.1}$$

is given by

$$x(t) = f(t,x(t)) + g(t,x(t))\int_0^1 G(t,s)z(s)ds, \tag{3.2}$$

where $\lambda_1 = g(0, x(0)) \neq 0$ and G(t, s) is the Green's function defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} (t-s)^{\alpha-1} - t(1-s)^{\alpha-1}, \ 0\le s\le t\le 1, \\ -t(1-s)^{\alpha-1}, \ 0\le t\le s\le 1. \end{cases}$$

Proof. Applying the Riemann-liouville fraction integral operator of order α to both sides of (3.1) and using Lemma (2.6), we have

$$I^{\alpha}\left[D^{\alpha}\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right)\right] = C_0 + C_1 t + I^{\alpha} y(t) \tag{3.3}$$

In view of boundary conditions

$\left[\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=0} = 0, \ \left[\dfrac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=1} = 0$, (3.3) takes the form

$$x(t) = f(t,x(t)) + g(t,x(t))\int_0^1 G(t,s)z(s)ds,$$

where G(t, s) is the Green's function as given in (3.2).

Thanks to Lemma (3.2), the proposed problem is equivalent to the following integral equation

$$x(t) = f(t,x(t)) + g(t,x(t))\int_0^1 G(t,s)h(s,x(s))ds, \ t\in[0,1] .$$

Here we remarked that

$$\hat{G} = \max_{t\in[0,1]} \int_0^1 |G(t,s)| ds \le \frac{1}{\Gamma(\alpha+1)}.$$

Theorem 3.3. Assume that the functions $g : J \times R \to R_{\setminus\{0\}}$ and $h, f : J \times R \to R$ with f (0, x(0)) = h(0, x(0)) = 0 are continuous and let the following hypothesis hold. (H1). There exist two positive functions θ and ϖ with bound ||θ|| and ||ϖ|| respectively, such that

$$| f(t, x(t)) - f(t, y(t) | \le \theta(t) | x(t) - y(t) |$$

and

$$| g(t, x(t)) - h(t, y(t) | \le \varpi(t) | x(t) - y(t) |$$

for t ∈ J and x, y ∈ R .

(H_2). There exist a function p ∈ C (J, R^+) and a continuous non-decreasing function Ψ :[0,∞) → (0,∞) such that

$$| h(t, x(t)) | \le p(t)\Psi(| x |), (t, x) \in J \times R \quad (3.5)$$

(H_3). There exists a number r > 0 such that

$$r \ge \frac{f_0 + \frac{g_0\|p\|\Psi(r)}{\Gamma(\alpha+1)}}{1-(\|\theta\| + \frac{\|\varphi\|\|p\|\Psi(r)}{\Gamma(\alpha+1)})} \quad (3.6)$$

where $f_0 = \sup_{t\in J} | f(t,0) |$ and $g_0 = \sup_{t\in J} | g(t,0) |$ and

$$\|\theta\| + \frac{\|\varphi\|\|p\|\Psi(r)}{\Gamma(\alpha+1)} < 1 \quad (3.7)$$

Then the Problem (1.2) has at least one solution on J .

Proof. Set E = C(J, R) and define a subset S of E as

$$S = \{x \in E : \|x\| \le r\},$$

where r satisfies Inequality (3.6). Clearly, S is closed, convex, and bounded subset of the Banach space E . Due to the integral Equation (3.2), now we define respectively the three operators A: E → E , C : E → E and B : S → E by

$$Ax(t) = f(t, x(t)),\ t \in J \quad (3.8)$$

$$Cx(t) = g(t, x(t)),\ t \in J \quad (3.9)$$

and

$$Bx(t) = \int_0^1 G(t,s)h(s, x(s))ds,\ t \in J.$$

Then the integral Equation (3.2) can be written in the operator form as

$$x(t) = Ax(t) + Bx(t)Cx(t),\ t \in J \quad (3.10)$$

We shall show that the operators A , B and C satisfy all the conditions of Lemma (3.1). This will be achieved in the following series of steps.

Step 1. We first show that A and C are Lipschitzian on E . Let x, y ∈ E , then by (H1), for t ∈ J , we have

Taking maximum over [0,1] , we get

$$\|Ax - Ay\| \le \|\theta\| \|x - y\| \le \|\theta\| r,$$

where $r = \|x - y\|$, which implies

$$\|Ax - Ay\| \le \|\theta\| r \text{ for all } x, y \in E.$$

Therefore A is a Lipschitzian on E with Lipschitz constant $\|\theta\|$. Now for $C : E \to E, x, y \in E$, we

have

$$| Cx(t) - Cy(t) | = | g(t, x(t) - g(t, y(t) | \le \varpi | x(t) - y(t) |$$

Taking maximum over [0,1], implies

$$\|Cx - Cy\| \le \|\varpi\|\|x - y\| \le \|\phi_2\| r,$$

where $r = \|x - y\|$, which implies

$$\|Cx - Cy\| \le \|\varpi\| r \text{ for all } x, y \in E.$$

Hence, $C : E \to E$ is a Lipschitzian on E with Lipschitz constant $\|\varpi\|$.

Step 2. The operator $B : S \to E$ is completely continuous on S. We first show that the operator B is continuous on E . Let {x_n} be a sequence in S converging to a point x ∈ S . Then by the Lebesgue dominated convergence theorem for all t ∈ J , we obtain

$$\begin{aligned}\lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty}\int_0^1 G(t,s)h(s, x_n(s))ds \\ &= \int_0^1 G(t,s)\lim_{n\to\infty} h(s, x_n(s))ds \\ &= \int_0^1 G(t,s)h(s, x(s))ds = Bx(t).\end{aligned}$$

Hence, $\lim_{n\to\infty} Bx_n(t) = Bx(t)$. So B is continuous on S.

Next we will show that the set B(S) uniformly bounded in S . For any x ∈ S , we have

$$\begin{aligned}| Bx(t) | &\le \int_0 | G(t,s) | | h(s, x(s)) | ds \\ &\le \int_0^1 | G(1,s) || p(s) | \Psi(| x(s) |)ds \\ &\le \frac{\|p\|\Psi(r)}{\Gamma(\alpha+1)} := K,\end{aligned} \quad (3.11)$$

for t ∈ J . Therefore, Bx ≤ K which shows that B is uniformly bounded on S . Now, we will show that B(S) is an equi-continuous set in E . If τ ,t ∈ J with τ < t and x ∈ S , then we have

$$\begin{aligned}| Bx(t) - Bx(\tau) | &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} | h(s, x(s)) | ds \\ &- \int_0^\tau \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} | h(s, x(s)) | ds \\ &+ (t-\tau)\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} | h(s, x(s)) | ds \\ &\le \frac{\|p\|\Psi(r)}{\Gamma(\alpha+1)}[t^\alpha - \tau^\alpha + t - \tau].\end{aligned} \quad (3.12)$$

Now t →τ the right-hand side of the above inequality tends to zero. Therefore, | Bx(t) − Bx(τ) |→ 0 as t →τ .

It follows from the Arzelá-Ascoli theorem that B is a completely continuous operator on S .

Step 3. The hypothesis (c) of Lemma (3.1) is satisfied. Let x ∈ E and y ∈ S be arbitrary elements such that x = AxBx + Cx . Then we have

$$\begin{aligned}| x(t) | &= | Ax(t) + Bx(t)Cx(t) | \\ &\le | Ax(t) | + | Bx(t)Cx(t) |\end{aligned}$$

$$|x(t)| \le |f(t,x(t))| + \left| g(t,x(t)) \int_0^1 G(t,s) h(s,x(s)) ds \right|$$
$$\le |f(t,x(t)) - f(t,0)| + |f(t,0)| + [|g(t,x(t))$$
$$- g(t,0)| + |g(t,0)|] \int_0^1 |G(t,s)| |h(s,x(s))| ds$$
$$\le |\theta(t)| |x(t)| + |f(t,0)|$$
$$+ [|\varpi(t)| |x(t)| + |g(t,0)|] \int_0^1 |G(t,s)| p(s) |\Psi(x(s)) ds.$$

From which we have $\|x\| \le \|\theta\| r + f_0 + [\|\varpi\| r + g_0] \dfrac{\|p\| \Psi(r)}{\Gamma(\alpha+1)} \le r.$

Which yields that

$$\|x\| \le r \tag{3.13}$$

Step 4. Finally, we show that As $\|\theta\| + \frac{\|\varpi\| \|p\| \Psi(r)}{\Gamma(\alpha+1)} < 1$, that is, (d) of Lemma

(3.1) holds.

$$\mathrm{N} = \|B(S)\|$$
$$= \sup_{x \in S}\{\sup_{x \in J} |Bx(t)|\}$$
$$\le \frac{\|p\| \Psi(r)}{\Gamma(\alpha+1)},$$

and applying (H3), we get $\|\theta\| + \frac{\|\varpi\| \|p\| \Psi(r)}{\Gamma(\alpha+1)} < 1$. Thus, all the conditions of

Lemma (3.1) are satisfied and hence the operator equation x = AxBx + Cx has a solution in S . In consequence, the considered problem has a solution on J .

4. EXAMPLE

We provide the following boundary value problem of fractional hybrid differential equations to demonstrate our main result.

Example 4.1

$$\begin{cases} D^{1.5}\left[\dfrac{x(t) - \frac{\exp(-t)\sin|x(t)|}{10}}{\frac{\exp(-t)\cos|x(t)|}{40}}\right] = \dfrac{t}{4}\sin|x(t)|, \ t \in [0,1], \\ \left[\dfrac{\frac{\exp(-t)\sin|x(t)|}{10}}{\frac{\exp(-t)\cos|x(t)|}{40}}\right]_{t=0} = 0, \ \left[\dfrac{\frac{\exp(-t)\sin|x(t)|}{10}}{\frac{\exp(-t)\cos|x(t)|}{40}}\right]_{t=1} = 0, \end{cases}$$

from (4.1), we see that α = 1.5 and

$$|f(t,x) - f(t,y)| \le \frac{\exp(-t)}{10} |x - y|, \ |g(t,x) - g(t,y)| \le \frac{\exp(-t)}{40} |x - y|$$

and

$$|h(t,x(t))| \le p(t)\Psi(|x|), \text{ where } p(t) = \frac{t}{4}, \ \Psi(x) = \sin|x|.$$

Hence, we have θ = $\frac{\exp(-t)}{}$, ϖ - $\frac{\exp(-t)}{40}$ which gives on taking supremum over [0,1], ||θ|| = 1 , ||ϖ ||= $\frac{1}{40}$, Ψ(x) = sin | x | . Hence, all the conditions of Theorem (3.3) are satisfied along with condition that $\|\theta\| + \frac{\|\varpi\| \|p\| \Psi(r)}{\Gamma(2.5)} < 1$ for all t ∈[0,1] .

Thus, the given Problem (4.1) has at least one solution.

5. CONCLUSION

By using famous hybrid fixed point theorem due to Dhage, we have developed some adequate conditions for the existence of at least one solution to a boundary value problem of hybrid fractional differential equations. The respective results have been verified by providing a suitable example.

COMPETING INTEREST

It is declared that no competing interest exists regarding this manuscript.

REFERENCES

[1] Podlubny, I. 1999. Fractional Differential Equation Academic Press, New York.

[2] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J. 2006. Theory and Applications of Fractional Differential Equations North-Holland Mathematics studies. Elsevier Science B.V., Amsterdam, 204.

[3] Benchohra, B., Graef, J.R., Hamani, S. 2008. Existence results for boundary value problem with nonlinear fractional differential equation. Applicable Analysis, 87 (2), 851-863.

[4] Zhang, S.Q. 2000. The existence of a positive solution for a nonlinear fractional differential equation. Journal of Mathematical Analysis and Applications, 252 (2), 804-812.

[5] Belarbi, A., Benchohra, M., Ouahab, A. 2006. Existence result for fractional differential equation of fractional order. Applicable Analysis, 85 (6), 1459-1470.

[6] Bai, Z., Lu, H. 2005. Positive solutions for boundary value problem of nonlinear fractional differential equation. Journal of Mathematical Analysis and Applications, 311 (2), 495 -505.

[7] Li, C.F., Luo, X.N., Zhou, Y. 2010. Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations. Computers & Mathematics with Applications, 59 (2), 1363 -1375.

[8] Babakhani, A., Gejji, V.D. 2003. Existence of positive solutions of nonlinear fractional differential equations. Journal of Mathematical Analysis and Applications, 278 (3), 434 -442.

[9] Zhang, S.Q. 2003. Existence of positive solution for some class of a nonlinear fractional differential equations. Journal of Mathematical Analysis and Applications, 278 (3), 136 -148.

[10] Bai, Z., Wang, Y., Ge, W. 2004. Triple positive solutions for a class of two-point boundary-value problems Electron. Journal of Differential Equations, 6 (4), 1-8.

[11] Ahmad, B., Nieto, J.J. 2010. Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory, Topological Meth. Nonlinear Analysis, 35 (1), 295-304.

[12] Shah, K., Khan, R.A. 2016. Multiple positive solutions to a coupled system of nonlinear fractional differential equations. SpringerPlus, 5 (1), 1–20.

[13] Shah, K., Zeb, S., Khan, R.A. 2015. Existence and uniqueness of solutions for fractional order m-point boundary value problems. Fractional Differential Calculus, 5 (2), 171-181.

[14] Shah, K., Ali, A., Khan, R.A. 2016. Degree theory and existence of positive solutions to coupled systems of multi-point boundary value problems. Boundary Value Problems, 43.

[15] Shah, K., Khalil, H., Khan, R.A. 2015. Investigation of positive solution to a coupled system of impulsive boundary value problems for nonlinear fractional order differential equations, Chaos. Solitons and Fractals, 77 (1), 240-246.

[16] Khan, R.A., Shah, K. 2015. Existence and uniqueness of solutions to fractional order multi point boundary value problems. Communications in Applied Analysis, 19 (1), 515-526.

[17] Lijun, G., Wang, D., Wang, G. 2015. Further results on exponential stability for impulsive switched nonlinear time-delay systems with delayed impulse effects. Applied Mathematics and Computation, 268 (5), 186-200.

[18] Stamova, I. 2015. Mittag-Leffler stability of impulsive differential equations of fractional order. Quarterly of Applied Mathematics, 73 (3), 525-535.

[19] Trigeassou, J.C., Maamri, N., Sabatier, J., Oustaloup, A. 2011. A Lyapunov approach to the stability of fractional differential equations, Signal Processing, 91 (3), 437-445.

[20] Ulam, S.M. 1940. Problems in Modern Mathematics, John Wiley and sons, New York, USA.

[21] Ulam, S.M. 1960. A Collection of Mathematical Problems, Interscience, New York.

[22] Obloza, M. 1993. Hyers stability of the linear differential equation, Rocznik Nauk-Dydakt. Prace Mat., 13 (1), 259-270.

[23] Wang, J., Li, X. 2015. Ulam Hyers stability of fractional Langevin equations. Applied Mathematics and Computation, 258 (1), 72-83.

[24] Stamova, I. 2015. Mittag-Leffler stability of impulsive differential equations of fractional order. Quarterly of Applied Mathematics, 73 (3), 525-535.

[25] Kumama, P., Ali, A., Shah, K., Khan, R.A. 2017. Existence results and Hyers-Ulam stability to a class of nonlinear arbitrary order differential equations. Journal of Nonlinear Sciences and Applications, 10 (7), 2986-2997.

[26] Wang, J., Lv, L., Zhou, Y. 2011. Ulam stability and data dependence for fractional differential equations with Caputo derivative. Electronic Journal of Qualitative Theory of Differential Equations, 63 (1), 1-10.

[27] Haq, F., Shah, K., Rahman, G., Shahzad, M. 2017. Hyers–Ulam stability to a class of fractional differential equations with boundary conditions. International Journal of Applied and Computational Mathematics, 1-13.

[28] Caputo, M. 1967. Linear models of dissipation whose Q is almost frequency independent. International Journal of Geographical Information Science, 13 (5), 529-539.

[29] Yang, Y., Ma, Y., Wang, L. 2015. Legendre polynomials operational matrix method for solving fractional partial differential equations with variable coefficients. Mathematical Problems in Engineering.

[30] Mainardi, F. 1997. Fractals and Fractional Calculus Continuum Mechanics" Springer Verlag.

[31] Rostamy, D., Karimi, K., Mohamadi, E. 2013. Solving fractional partial differential equations by an efficient new basis. International Journal of Applied and Computational Mathematics, 5 (1), 6-12.

[32] Tarasov, V.E. 2009. Fractional integro differentional equations for electromagnetic waves in dielectric media. Theoretical and Mathematical Physics, 158 (3), 355-359.

[33] Baillie, R.T. 1996. Long memory processes and fractional integration in econometrics. Journal of Economics, 73, 5-59.

[34] Magin, R.L. 2004. Fractional calculus in bioengineering. Critical Reviews in Biomedical Engineering, 32 (2), 1-104.

[35] Magin, R.L. 2004. Fractional calculus in bioengineering-part 2. Critical Reviews in Biomedical Engineering, 32 (2), 105-193.

[36] Magin, R.L. 2004. Fractional calculus in bioengineering-part 3. Critical Reviews in Biomedical Engineering, 32 (3-4), 194-377.

[37] Dhage, B.C., Lakshmikantham, V. 2010. Basic results on hybrid differential equations. Nonlinear Analysis, 4 (1), 414-424.

[38] Hilal, K., Kajouni, A. 2015. Boundary value problem for hybrid differential equations with fractional order. Advances in Difference Equations, 183 (1), 1-19.

[39] Herzallah, M.A.E., Baleanu, D. 2014. On Fractional Order Hybrid Differential Equations. Abstract and Applied Analysis, Article ID 389386,7 pages http://dx.doi.org/10.1155/2014/389386

[40] Lu, H., Sun, S., Yang, D., Teng, H. 2013. Theory of fractional hybrid differential equations with linear perturbations of second type. Boundary Value Problems, 23 (3), 1-16.

[41] Zhao, Y., Sun, S., Han, Z., Li, Q. 2011. Theory of fractional hybrid differential equations. Computers & Mathematics with Applications, 62 (1), 1312-1324.

[42] Hilal, K., Kajouni, A. 2015. Boundary value problem for hybrid differential equations with fractional order. Advances in Difference Equations, 183 (1), 1-19.